Successive Shortest Paths for Minimum Cost Flow

Assume no neg. cost cycles in input graph.

Successive Shortest Path

```
f = 0;  Π = 0
e(v) = b(v) ∀v ∈ V
Initialize E = {v : e(v) > 0} and D = {v : e(v) < 0}
while E ≠ 0
      Pick a node k ∈ E and ℓ ∈ D
      Compute d(v), shortest path distances from k in G_f
        w.r.t. edge distances c^π.
      Let P be a shortest path from k to ℓ.
      Set π = π − d
      Let δ = min{e(k), −e(ℓ), min{u_f(v,w) : (v,w) ∈ P}}
      Send δ units of flow on the path P
      Update f, G_f, E, D and c^π.
```

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Correctness of successive shortest path algorithm

Lemma: Let f be a pseudoflow satisfying reduced cost optimality with respect to π. Let $d(v)$ be the shortest path distance from some node s to v in G_f with respect to c^π. Then

- f satisfies reduced cost optimality with respect to $\pi' = \pi - d$.
- $c^{\pi'}(v,w) = 0$ if (v,w) is on a shortest path from s to some other node.

Pf $\quad c^\pi(v,w) \geq 0, \quad \pi'(v) = \pi(v) - d(v), \quad d(w) \overset{=}{\leq} d(v) + c^\pi(v,w)$

$$c^{\pi'}(v,w) = c(v,w) - \pi'(v) + \pi'(w)$$
$$= c(v,w) - \pi(v) + d(v) + \pi(w) - d(w)$$
$$= c^\pi(v,w) + d(v) - d(w) \overset{=}{\geq} 0.$$

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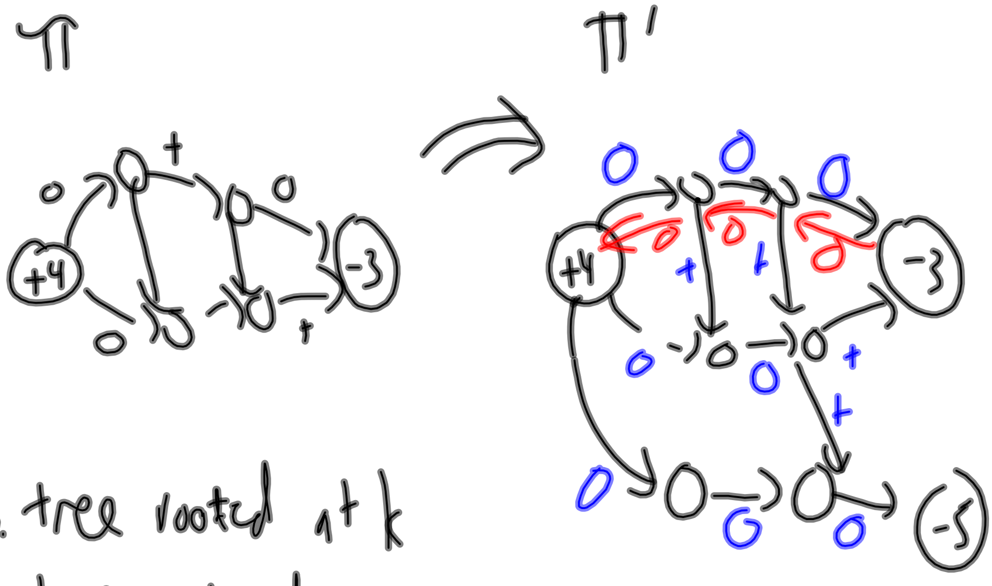

s.p. tree rooted at k
w/ all 0-cost edges
$(c^{\pi'})$

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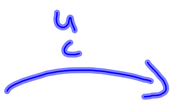
Correctness of successive shortest path algorithm

Corollary: After each iteration of the successive shortest paths algorithm, f satisfies reduced cost optimality.

But still not necessarily polynomial.



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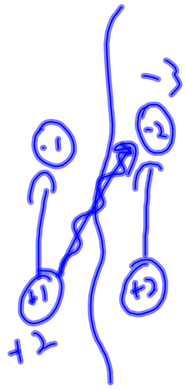
Successive Shortest Path

$f = 0$; $\Pi = 0$
$e(v) = b(v)\ \forall v \in V$
Initialize $E = \{v : e(v) > 0\}$ and $D = \{v : e(v) < 0\}$
while $E \neq 0$
    Pick a node $k \in E$ and $\ell \in D$
    Compute $d(v)$, shortest path distances from k in G_f w.r.t. edge distances c^π.
    Let P be a shortest path from k to ℓ.
    Set $\pi = \pi - d$
    Let $\delta = \min\{e(k), -e(\ell), \min\{u_f(v,w) : (v,w) \in P\}\}$
    Send δ units of flow on the path P
    Update f, G_f, E, D and c^π.



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Corollary: After each iteration of the successive shortest paths algorithm, f satisfies reduced cost optimality.

But still not necessarily polynomial.

Send flow on 0-cost arcs, new residual edges are 0-cost ∴ r.c.o. is maintained.

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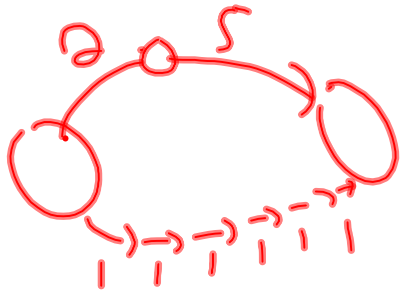

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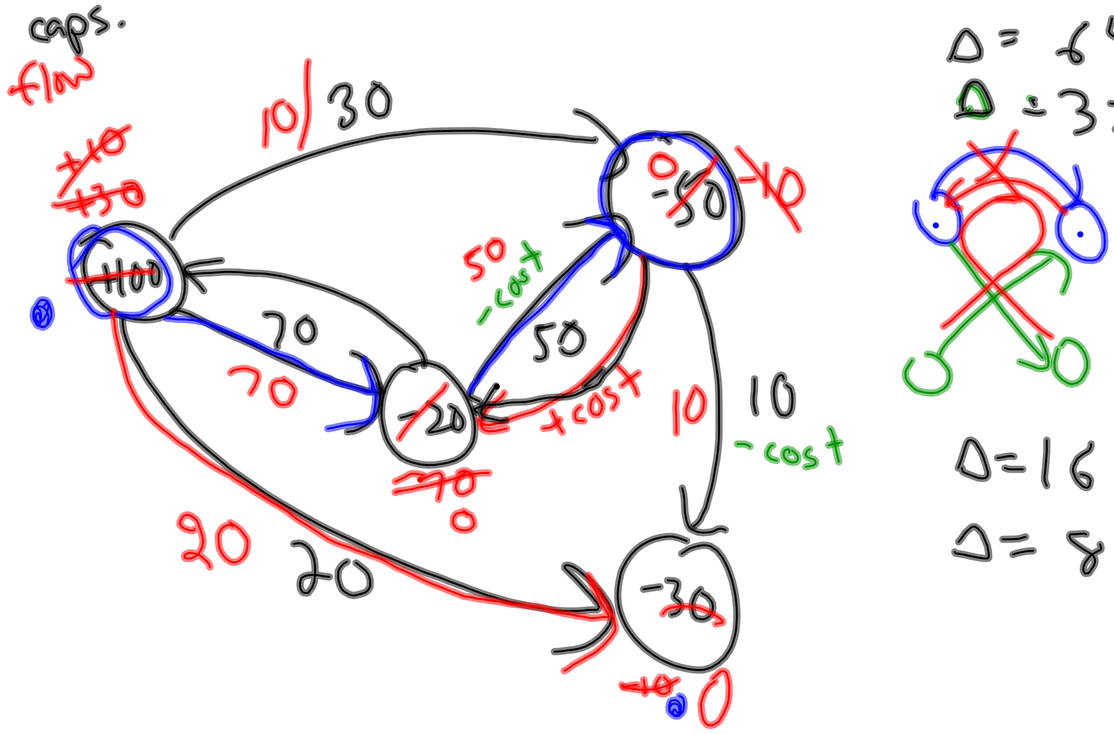

Δ= 64
Δ=32
Δ=16
Δ= 8

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Capacity Scaling Algorithm for Minimum Cost Flow

r.c.o. $G_f(\Delta)$

Successive Shortest Path

```
f = 0;  π = 0
e(v) = b(v) ∀v ∈ V
Δ = 2^⌊U⌋
while Δ ≥ 1
    (Δ scaling phase )
    for every edge (v, w) ∈ G_f
        if u_f(v, w) ≥ Δ and c^π(v, w) < 0
            Send u_f(v, w) units of flow on (v, w); update f, e
    S(Δ) = {v ∈ V : e(v) ≥ Δ}
    T(Δ) = {v ∈ V : e(v) ≤ −Δ}
    while S(Δ) ≠ 0 and T(Δ) ≠ 0
        Pick a node k ∈ S(Δ) and ℓ ∈ T(Δ)
        Compute d(v), shortest path distances from k in G_f(Δ)
          w.r.t. edge distances c^π.
        Let P be a shortest path from k to ℓ.
        Set π = π − d
        Let δ = min{e(k), −e(ℓ), min{u_f(v, w) : (v, w) ∈ P}}
        Send δ units of flow on the path P
```

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$$c^{\pi}(v,w) \geq 0 \quad \forall (v,w) \in G_f(\Delta)$$

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lg U iterations

$$|S(\Delta)| = \sum_{v \in S(\Delta)} e(v)$$

$$|T(\Delta)| = -\sum_{v \in T(\Delta)} e(v)$$

track $\min\left(|S(\Delta)|, |T(\Delta)| \right)$

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