# All Pairs Shortest Paths

- **Input: weighted, directed graph $G = (V, E)$, with weight function $w : E \rightarrow$ R.**
- **The weight of path $p =< v_0, v_1, \ldots, v_k >$ is the sum of the weights of its constituent edges:**

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i) \ .$$

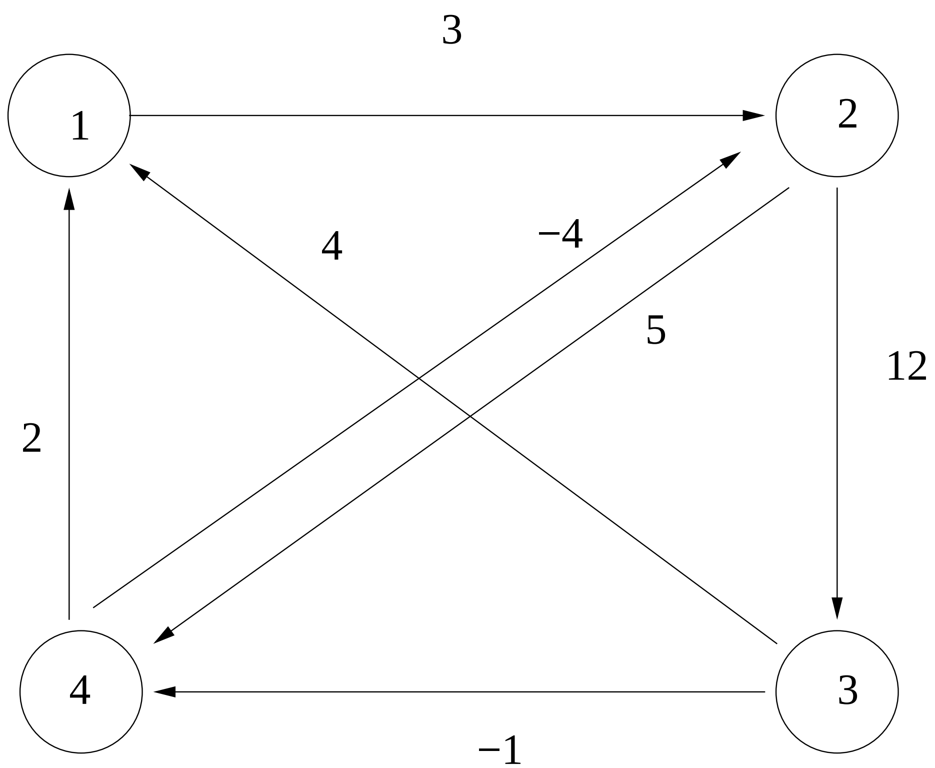
- **The shortest-path weight from $u$ to $v$ is**

$$\delta(u,v) = \begin{cases} \min\{w(p)\} & \text{if there is a path } p \text{ from } u \text{ to } v \ , \\ \infty & \text{otherwise .} \end{cases}$$

- **A shortest path from vertex $u$ to vertex $v$ is then defined as any path $p$ with weight $w(p) = \delta(u, v)$.**

**All Pairs Shortest Paths: Compute $d(u, v)$ the shortest path distance from $u$ to $v$ for all pairs of vertices $u$ and $v$.**

# Example




**Solution**

$$\begin{pmatrix} 0 & 3 & 15 & 8 \\ 7 & 0 & 12 & 5 \\ 1 & 4 & 0 & -1 \\ 2 & -4 & 8 & 0 \end{pmatrix}$$

# Approach 1

**Run Single source shortest paths $V$ times**

- $O(V^2E)$ **for general graphs**
- $O(VE + V^2 \log V)$ **for graphs with non-negative edge weights**

**Other approaches : Share information between the various computations**

# Floyd-Warshall, Dynamic Programming

- **Let $d_{ij}^{(k)}$ be the weight of a shortest path from vertex $i$ to vertex $j$ for which all intermediate vertices are in the set $\{1, 2, \ldots, k\}$.**
- **When $k = 0$, a path from vertex $i$ to vertex $j$ with no intermediate vertex numbered higher than $0$ has no intermediate vertices at all, hence $d_{ij}^{(0)} = w_{ij}$.**

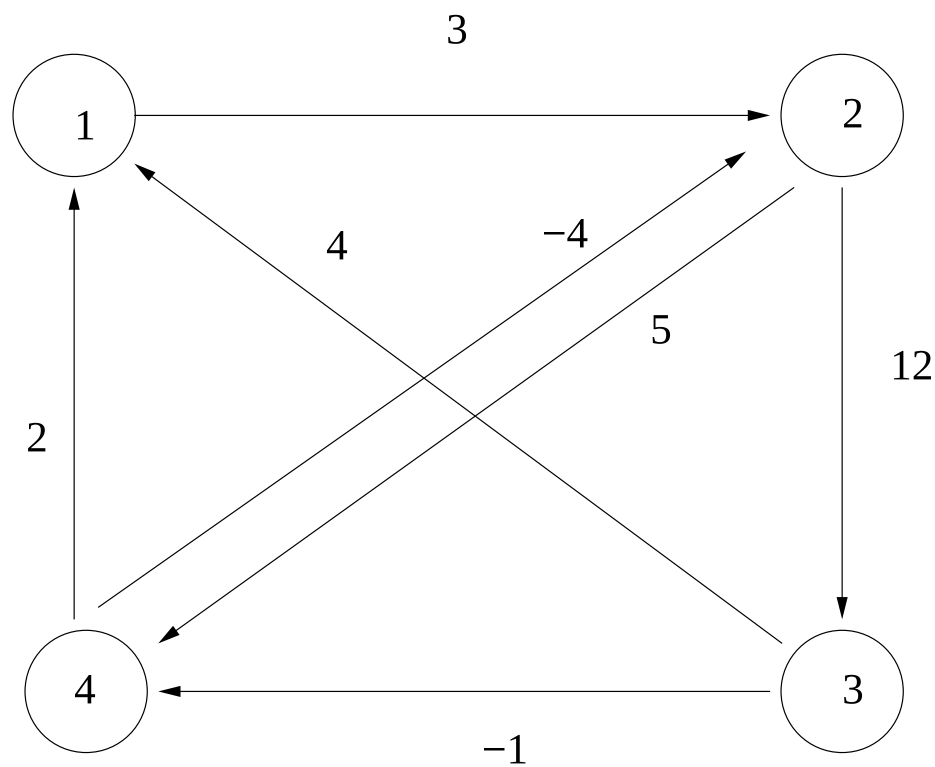
$$d_{ij}^{(k)} = \begin{cases} w_{ij} & \textbf{if } k = 0 \ , \\ \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right) & \textbf{if } k \geq 1 \ . \end{cases} \qquad (1)$$

**Floyd-Warshall**$(W)$

$n \leftarrow rows[W]$<br>
$D^{(0)} \leftarrow W$<br>
**for** $k \leftarrow 1$ **to** $n$<br>
    **do for** $i \leftarrow 1$ **to** $n$<br>
        **do for** $j \leftarrow 1$ **to** $n$<br>
            **do** $d_{ij}^{(k)} \leftarrow \min\left(d_{ij}^{(k-1)}, d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\right)$<br>
**return** $D^{(n)}$

**Running time** $O(V^3)$

# Example




$$D^0 = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 12 & 5 \\ 4 & 0 & 0 & -1 \\ 2 & -4 & 0 & 0 \end{pmatrix} D^1 = \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 12 & 5 \\ 4 & 7 & 0 & -1 \\ 2 & -4 & 0 & 0 \end{pmatrix} D^2 = \begin{pmatrix} 0 & 3 & 15 & 8 \\ 0 & 0 & 12 & 5 \\ 4 & 7 & 0 & -1 \\ 2 & -4 & 8 & 0 \end{pmatrix}$$

$$D^3 = \begin{pmatrix} 0 & 3 & 15 & 8 \\ 16 & 0 & 12 & 5 \\ 4 & 7 & 0 & -1 \\ 2 & -4 & 8 & 0 \end{pmatrix} D^4 = \begin{pmatrix} 0 & 3 & 15 & 8 \\ 7 & 0 & 12 & 5 \\ 1 & -5 & 0 & -1 \\ 2 & -4 & 8 & 0 \end{pmatrix}$$

# Another Algorithm

**RESET ALL DEFINITIONS OF D.**

- **Let** $w_{ij}$ **be the length of edge** $ij$
- **Let** $w_{ii} = 0$
- **Let** $d_{ij}^{m}$ **be the shortest path from** $i$ **to** $j$ **using** $m$ **or fewer edges**

$$d_{ij}^{1} = w_{ij}$$

$$d_{ij}^{m} = \min\{d_{ij}^{m-1}, \min_{1 \leq k \leq n, k \neq j} d_{ik}^{m-1} + w_{kj}\}$$

**Combining these two, we get**

$$d_{ij}^{m} = \min_{1 \leq k \leq n} \{d_{ik}^{m-1} + w_{kj}\}$$

**This would give an** $O(V^4)$ **algorithm**

# Using matrix multiplication analogy

**Note the similarity of**

$$d_{ij}^m = \min_{1 \le k \le n} \{d_{ik}^{m-1} + w_{kj}\}$$

**with matrix multiplication:**

$$c_{ij} = \mathbf{sum}_{1 \le k \le n} \{a_{ik} \cdot k_{kj}\}$$

**Make the following substitutions (which have the right algebraic properties:**

$$\begin{array}{rcl} \mathbf{sum} & \to & \min \\ a_{ij} & \to & d_{ik}^{m-1} \\ \cdot & \to & + \\ b_{kj} & \to & w_{ij} \\ c & \to & d^m \end{array}$$

**Using this matrix multiplication terminology, we have**

$$\begin{array}{lll} D^1 & & = W \\ D^2 & = D^1 \cdot W & = W^2 \\ D^3 & = D^2 \cdot W & = W^3 \\ \dots & \dots & \dots \\ D^m & = D^{m-1} W & = W^m \end{array}$$

**But we can execute $W^m$ be repeated squaring and get $O(V^3 \log V)$ time.**