Matchings

Definition: Give an undirected graph G, a matching M is a subset of the edges $E \subseteq M$ such that each vertex $v \in V$ is incident to at most one edge from M.

Variants of Matching

- Graph can be bipartite or general
- Graph can be weighted or unweighted

Terms

- A matching M such that, for all edges $e \notin M$, $M \cup \{e\}$ is not a matching, is called **maximal**.
- A maximum cardinality matching is called **maximum**.
- A matching of size $|V|/2$ is called **perfect**.
- The weight of a matching M is $w(M) = \Sigma_{e \in M} w(e)$.
- A maximum weight matching is the matching of maximum weight.
- All variants polynomial time, bipartite matching seems “easier”.

Matchings in Bipartite Graphs

- **Can solve via max flow. Ford Fulkerson is $O(m|f|) = O(nm)$.**
- **Will develop from first principles to understand terminology and to see improvements.**

Definitions

- **An alternating path with respect to a matching M is a path in which edges alternate between those in M and those not in M .**
- **A matched vertex is one incident to an edge in M**
- **An free vertex is a vertex that is not matched**
- **An augmenting path is an alternating path that starts and ends with a free vertex.**
- **A shortest augmenting path is an augmenting path of shortest length.**
- **Symmetric Difference of 2 sets: $A \oplus B = (A \cup B) - (A \cap B)$**

Augmenting Paths

Facts

- Let M be a matching and P be an augmenting path relative to M. Then $M \oplus P$ is a matching and $|M \oplus P| = |M| + 1$.
- Let M be a matching and $P_1, P_2, \ldots P_k$ be k vertex-disjoing augmenting path relative to M. Then $M \oplus (P_1 \cup P_2 \cup \ldots \cup P_k)$ is a matching and $|M \oplus (P_1 \cup P_2 \cup \ldots \cup P_k)| = |M| + k$.

Hopkroft-Karp Algorithm

- $M = \emptyset$
- Repeat Until $P = \emptyset$
  - Let $P = (P_1 \cup P_2 \cup \ldots \cup P_k)$ be a maximal set of vertex disjoint shortest augmenting paths with respect to M.
  - $M = M \oplus (P_1 \cup P_2 \cup \ldots \cup P_k)$

Facts

- **You can find a maximal set of vertex-disjoint augmenting paths in $O(m)$ time via breadth-first search.**
- **Given 2 matchings M and M', let $G' = (v, M \oplus M')$, then**
  - **For all vertices, the degree of v in G' is at most 2.**
  - **G' is a set of alternating paths and alternating cycles.**
  - **If $|M| < |M'|$, then G' has at least $|M'| - |M|$ vertex disjoint augmenting paths with respect to M**

Main Lemma

Lemma **Let ℓ be the length of a shortest augmenting path with respect to M. Let $P_1, \ldots, P_k$ be a maximal set of vertex disjiont shortest augmenting paths. Let $M' = M \oplus (P_1 \cup \ldots P_k)$. Let P be a shortest augmenting path with respect to M'. Then $|P| > \ell$.**

Lemma

Lemma **If the shortest augmeting path with respect to M has ℓ edges, and M' is a maximum matching, then**

$$|M'| \leq |M| + n/\ell$$

.

Theorem **Hopkroft-Karp is an $O(\sqrt{n}m)$ time algorithm for bipartite matching.**

Assignment Problem

Minimum weighted perfect bipartite matching in a complete graph.

$$\min \sum_{(v,w)\in E} c(v,w)x(v,w)$$

$$\textbf{s.t.}$$

$$\sum_{v\in V} x(v,w) = 1 \qquad \forall w \in V$$

$$\sum_{w\in V} x(v,w) = 1 \qquad \forall v \in V$$

$$x(v,w) \geq 0 \quad \forall (v,w) \in E$$

Dual:

$$\max \sum_{v\in V} \pi(v)$$

$$\textbf{s.t.}$$

$$\pi(v) + \pi(w) \leq c(v,w) \;\; \forall (v,w) \in E$$

Dual is a shortest path problem.

Stable Matching

- **Given a set of men X and women W.**
- **Each man ranks the women, and each woman ranks the men.**
- **Let $r(m, w)$ be the rank assigned from man m to woman w. Define $r(w, m)$ similarly.**
- **Given a matching M, a pair (m, w) is unstable if m and w each prefer each other to their current matched partner.**
- **A stable matching is a matching with no unstable pairs.**

Results on Stable Matching

Theorem For any set of preferences, a stable marriage exists and can be found in $O(n^2)$ time.

Algorithm

- Proceed in Rounds
- In each round, any unmatched man proposes to his highest ranked woman who has not yet rejected him.
- Woman accept any proposal if it is preferred to their current matching (and prefer anyone to being unmatched).

Properties of Algorithm

- **Once a woman is matched, she stays matched.**
- **The partner of a woman only improves over time**
- **Once a woman rejects a man, she would always reject him in the future.**
- **A woman always accepts her first proposal.**

Conclusion: **Every woman is eventually matched.**

Claims:

- **The matching is stable.**
- **Each man is matched to the highest ranked woman he could match in any stable marriage.**
- **Each woman is matched to the lowest ranked man she could match in any stable marriage.**