Minimum Cost Flow by Successive Shortest Paths

- Initialize to the 0 flow
- Repeat
  - Send flow along a shortest path in $G_f$

Comments:
- Correctly computes a minimum-cost flow
- Not polynomial time.
- Simple bound of $O(nmCU)$ time.
**Pseudoflow**

**Pseudoflow:** A pseudoflow is a function on the edges of a graph satisfying

\[ 0 \leq f(v, w) \leq u(v, w) \ \forall (v, w) \in E \]

- Given a pseudoflow \( f \), we define the “excess” at \( v \) as

\[ e(v) = b(v) + \sum_{w \in V} f(w, v) - \sum_{w \in V} f(v, w). \]

- If \( e(v) = 0 \ \forall v \in V \), then a pseudoflow is a flow.
- We define reduced cost optimality of a pseudoflow \( f \) as

\[ \exists \pi \text{ s.t. } c^\pi(v, w) \geq 0 \ \forall (v, w) \in G_f \]

**Strategy:** Maintain an \( f \) and \( \pi \) such that \( f \) is a pseudoflow satisfying reduced cost optimality. Work to make \( f \) a flow. When \( f \) is a flow, you know it is optimal.
How do you initialize?

- You can assume that $c(v, w) \geq 0 \forall (v, w) \in E$. Then the 0-flow satisfies reduced cost optimality.
- But what if the assumption doesn’t hold?
How do you initialize?

- You can assume that \( c(v, w) \geq 0 \forall (v, w) \in E \). Then the 0-flow satisfies reduced cost optimality.

- But what if the assumption doesn’t hold?

- Set \( f(v, w) = u(v, w) \) for all edges with \( c(v, w) < 0 \).

- Now, all edges in \( G_f \), satisfy \( c^\pi(v, w) \geq 0 \).

- Update \( e(v) \) accordingly.
Successive Shortest Paths for Minimum Cost Flow

Successive Shortest Path

1. $f = 0; \Pi = 0$
2. $e(v) = b(v) \forall v \in V$
3. Initialize $E = \{v: e(v) > 0\}$ and $D = \{v: e(v) < 0\}$
4. while $E \neq 0$
5. Pick a node $k \in E$ and $\ell \in D$
6. Compute $d(v)$, shortest path distances from $k$ in $G_f$ w.r.t. edge distances $c^\pi$.
7. Let $P$ be a shortest path from $k$ to $\ell$.
8. Set $\pi = \pi - d$
9. Let $\delta = \min\{e(k), -e(\ell), \min\{u_f(v, w) : (v, w) \in P\}\}$
10. Send $\delta$ units of flow on the path $P$
11. Update $f, G_f, E, D$ and $c^\pi$. 
Correctness of successive shortest path algorithm

Lemma: Let $f$ be a pseudoflow satisfying reduced cost optimality with respect to $\pi$. Let $d(v)$ be the shortest path distance from some node $s$ to $v$ in $G_f$ with respect to $c^\pi$. Then

- $f$ satisfies reduced cost optimality with respect to $\pi' = \pi - d$.
- $c^{\pi'}(v, w) = 0$ if $(v, w)$ is on a shortest path from $s$ to some other node.
Correctness of successive shortest path algorithm

**Corollary:** After each iteration of the successive shortest paths algorithm, $f$ satisfies reduced cost optimality.

But still not necessarily polynomial.
Use Capacity Scaling on top of shortest path algorithm

Def:

\[ G_f(\Delta) = \{(v, w) \in G_f : u_f(v, w) \geq \Delta\} \]
Capacity Scaling Algorithm for Minimum Cost Flow

Successive Shortest Path

1. $f = 0; \ \pi = 0$
2. $e(v) = b(v) \ \forall v \in V$
3. $\Delta = 2^{\lceil U \rceil}$
4. while $\Delta \geq 1$
   - ($\Delta$ scaling phase)
5. for every edge $(v, w) \in G_f$
6.   if $u_f(v, w) \geq \Delta$ and $c^\pi(v, w) < 0$
7.     Send $u_f(v, w)$ units of flow on $(v, w)$; update $f$, $e$
8. $S(\Delta) = \{ v \in V : e(v) \geq \Delta \}$
9. $T(\Delta) = \{ v \in V : e(v) \leq -\Delta \}$
10. while $S(\Delta) \neq 0$ and $T(\Delta) \neq 0$
11.   Pick a node $k \in S(\Delta)$ and $\ell \in T(\Delta)$
12.   Compute $d(v)$, shortest path distances from $k$ in $G_f(\Delta)$
13.   w.r.t. edge distances $c^\pi$.
14.   Let $P$ be a shortest path from $k$ to $\ell$.
15.   Set $\pi = \pi - d$
16.   Let $\delta = \min\{e(k), -e(\ell), \min\{u_f(v, w) : (v, w) \in P\}\}$
17.   Send $\delta$ units of flow on the path $P$
18.   Update $f$, $G_f(\Delta)$, $S(\Delta)$, $T(\Delta)$ and $c^\pi$.
19. $\Delta = \Delta / 2$
Analysis of Running Time