# Shortest Paths

- **Input: weighted, directed graph** $G = (V, E)$**, with weight function** $w : E \rightarrow$ **R.**
- **The** weight **of path** $p =< v_0, v_1, \ldots, v_k >$ **is the sum of the weights of its constituent edges:**

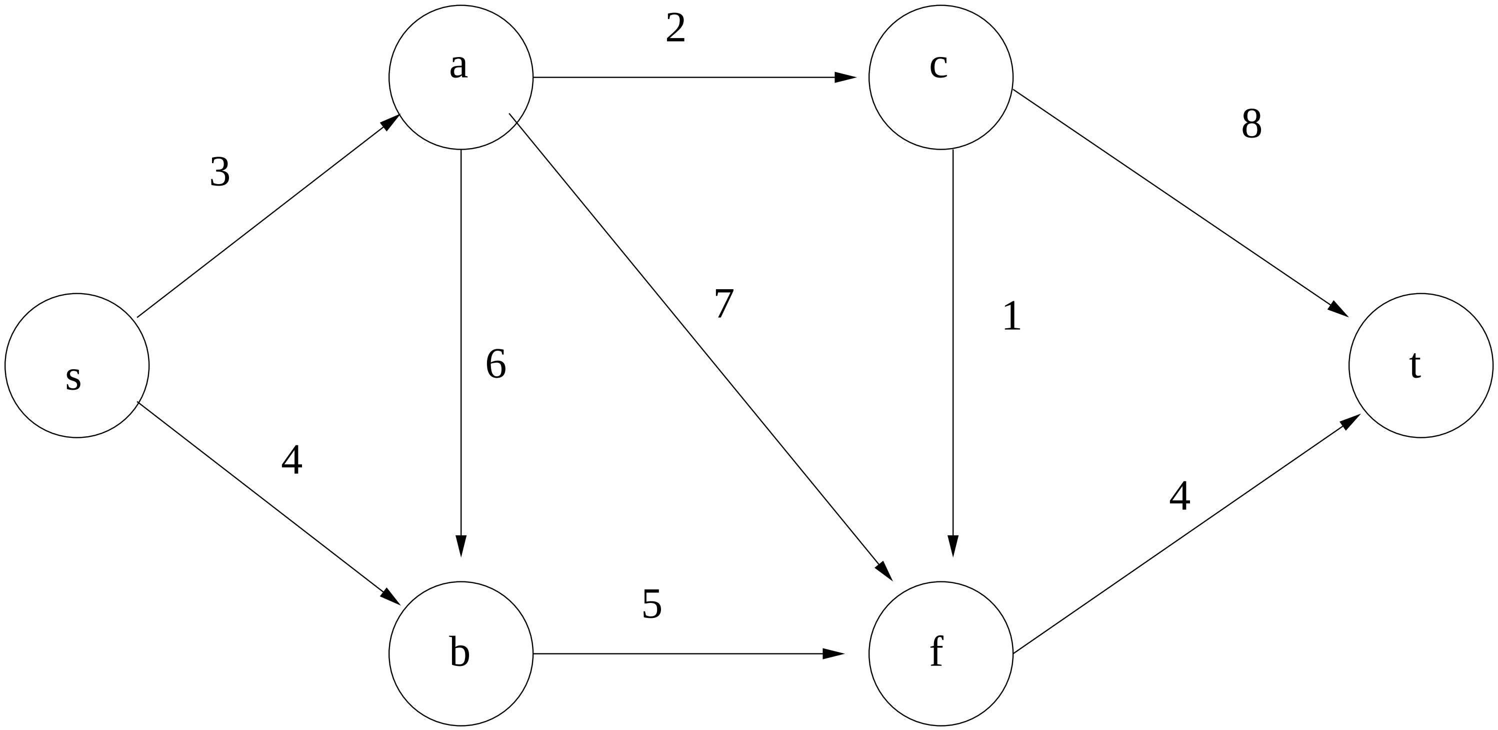
$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i) \ .$$

- **The** shortest-path weight **from** $u$ **to** $v$ **is**

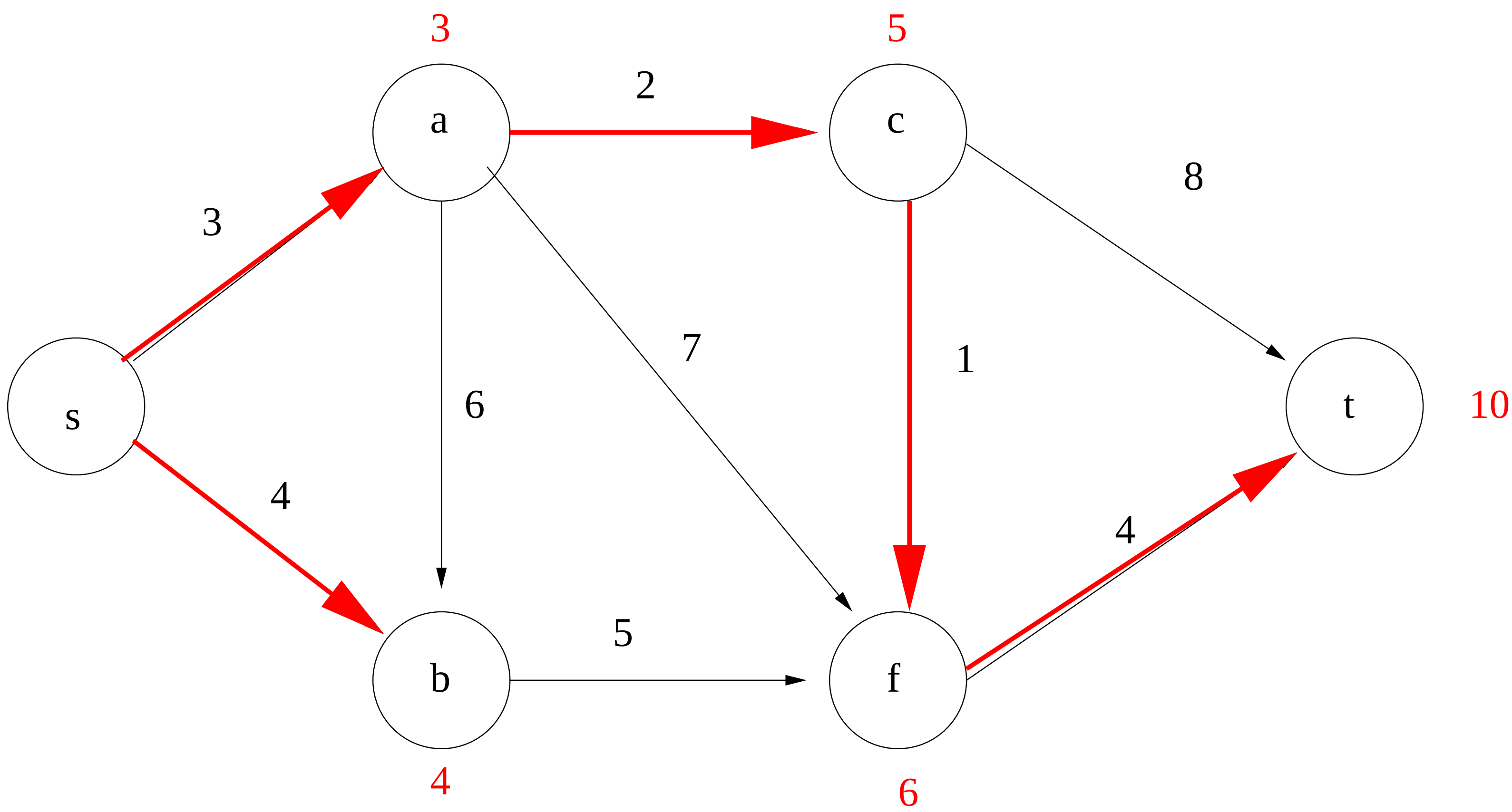
$$\delta(u, v) = \begin{cases} \min\{w(p)\} & \textbf{if there is a path } p \textbf{ from } u \textbf{ to } v \ , \\ \infty & \textbf{otherwise} \ . \end{cases}$$

- **A** shortest path **from vertex** $u$ **to vertex** $v$ **is then defined as any path** $p$ **with weight** $w(p) = \delta(u, v)$**.**

# Example



# Solution



# Shortest Paths

## Shortest Path Variants

- **Single Source-Single Sink**
- **Single Source (all destinations from a source $s$)**
- **All Pairs**

## Defs:

- **Let $\delta(v)$ be the real shortest path distance from $s$ to $v$**
- **Let $d(v)$ be a value computed by an algorithm**

## Edge Weights

- **All non-negative**
- **Arbitrary**

**Note:** **Must have no negative cost cycles**

# Single Source Shortest Paths

**Key Property: Subpaths of shortest paths are shortest paths** Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \rightarrow \mathbf{R}$, let $p =< v_1, v_2, \ldots, v_k >$ be a shortest path from vertex $v_1$ to vertex $v_k$ and, for any $i$ and $j$ such that $1 \leq i \leq j \leq k$, let $p_{ij} =< v_i, v_{i+1}, \ldots, v_j >$ be the subpath of $p$ from vertex $v_i$ to vertex $v_j$. Then, $p_{ij}$ is a shortest path from $v_i$ to $v_j$.

**Note: this is optimal substructure**

**Corollary 1** For all edges $(u, v) \in E$,

$$\delta(v) \leq \delta(u) + w(u, v)$$

**Corollary 2** Shortest paths follow a tree of edges for which

$$\delta(v) = \delta(u) + w(u, v)$$

More precisely, any edge in a shortest path must satisfy

$$\delta(v) = \delta(u) + w(u, v)$$

# Relaxation

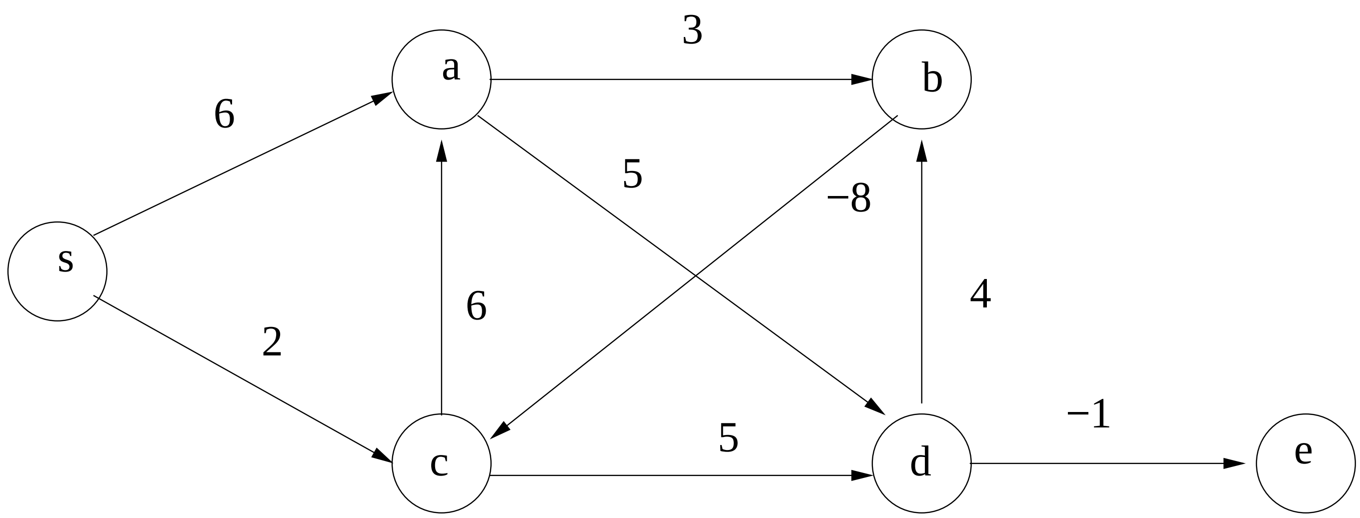
**Relax**$(u, v, w)$

**if** $d[v] > d[u] + w(u, v)$
  **then** $d[v] \leftarrow d[u] + w(u, v)$
    $\pi[v] \leftarrow u$ **(keep track of actual path)**

**Lemma:** **Assume that we initialize all** $d(v)$ **to** $\infty$**,** $d(s) = 0$ **and execute a series of Relax operations. Then for all** $v$**,** $d(v) \geq \delta(v)$**.**

**Lemma:** **Let** $P = e_1, \ldots, e_k$ **be a shortest path from** $s$ **to** $v$**. After initialization, suppose that we relax the edges of** $P$ **in order (but not necessarily consecutively). Then** $d(v) = \delta(v)$**.**

# Example



# Algorithms

**Goal of an algorithm:** **Relax the edges in a shortest path in order (but not necessarily consecutively).**

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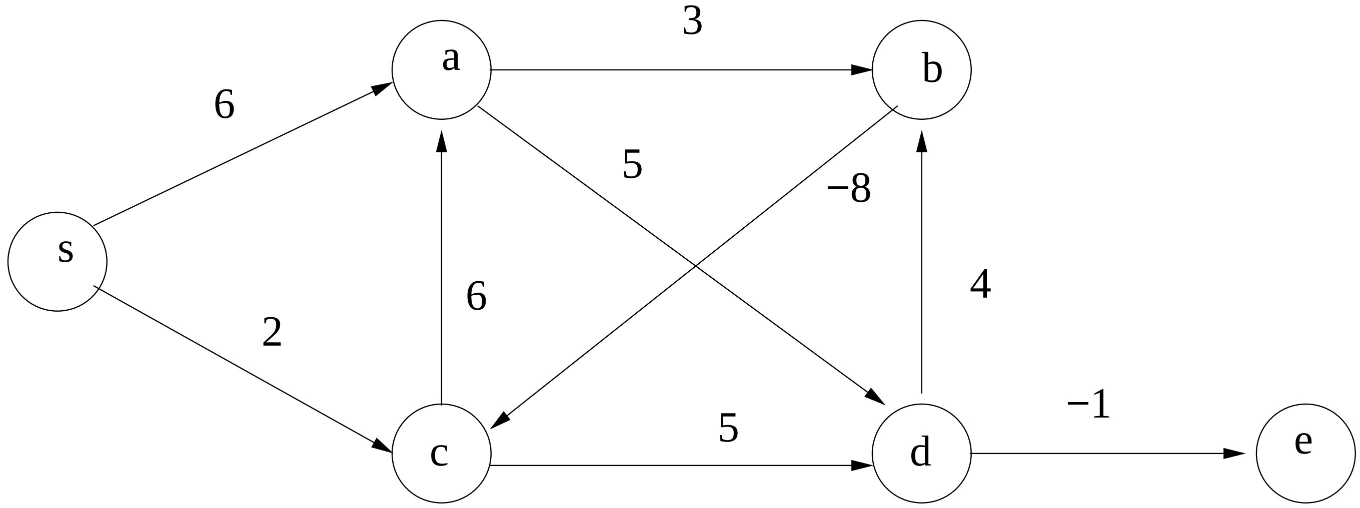
**Bellman-Ford**$(G, w, s)$

```
INITIALIZE-SINGLE-SOURCE(G, s)
for i ← 1 to |V[G]| − 1
        do for each edge (u, v) ∈ E[G]
                do RELAX(u, v, w)
for each edge (u, v) ∈ E[G]
        do if d[v] > d[u] + w(u, v)
                then return FALSE
return TRUE
```

$Initialize - Single - Source(G, s)$

```
for each vertex v ∈ V[G]
        do d[v] ← ∞
           π[v] ← NIL
d[s] ← 0
```

# Example



# Correctness of Bellman Ford

- **Every shortest path must be relaxed in order**
- **If there are negative weight cycles, the algorithm will return false**

**Running Time** $O(VE)$

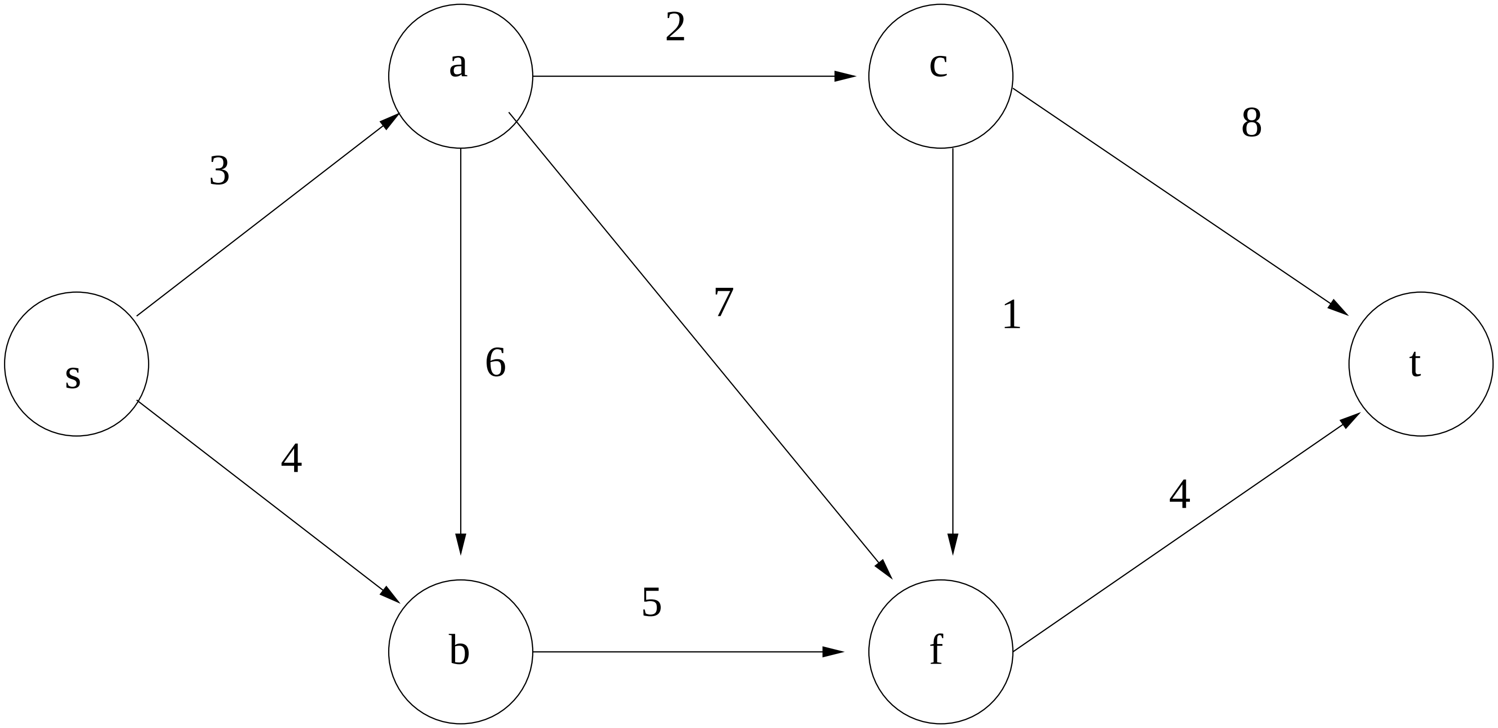
# All edges non-negative

- **Dijkstra's algorithm, a greedy algorithm**
- **Similar in spirit to Prim's algorithm**
- **Idea: Run a discrete event simulation of breadth-first-search. Figure out how to implement it efficiently**
- **Can relax edges out of each vertex exactly once.**

*Dijkstra*$(G, w, s)$

```
Initialize-Single-Source(G, s)
S ← ∅
Q ← V[G]
while Q ≠ ∅
        do u ← Extract-Min(Q)
           S ← S ∪ {u}
           for each vertex v ∈ Adj[u]
               do Relax(u, v, w)
```

# Example



# Running Time and Correctness

**Correctness of Dijkstra's algorithm** Dijkstra's algorithm, run on a weighted, directed graph $G = (V, E)$ with nonnegative weight function $w$ and source $s$, terminates with $d[u] = \delta(s, u)$ for all vertices $u \in V$.

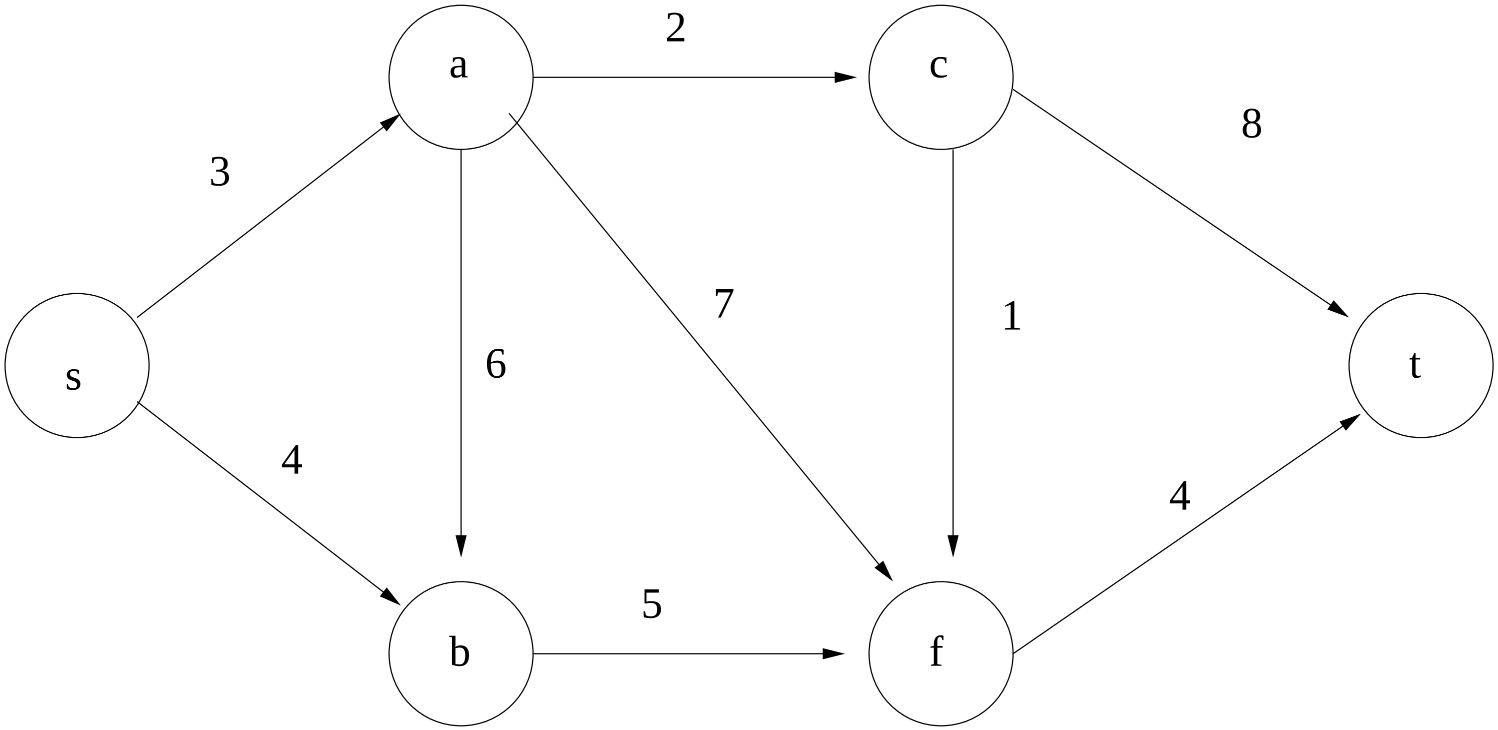
- $E$ decrease keys and $V$ delete-min's
- $O(E \log V)$ using a heap
- $O(E + V \log V)$ using a Fibonacci heap
- We will see algorithm using a Relaxed Heap

# Shortest Path in a DAG

**Dag-Shortest-Paths**$(G, w, s)$

**topologically sort the vertices of** $G$
Initialize-Single-Source′$(G, s)$
**for each** $u$ **taken in topological order**
    **do for each** $v \in Adj[u]$
        **do** Relax$(u, v, w)$

# Example



# Correctness and Running Time

**Correctness** **If a weighted, directed graph $G = (V, E)$ has source vertex $s$ and no cycles, then at the termination of the** DAG-SHORTEST-PATHS **procedure, $d[v] = \delta(s, v)$ for all vertices $v \in V$, and the predecessor subgraph $G_\pi$ is a shortest-paths tree.**

**Running Time**

- **Topological sort is linear time**
- **Each edge is relaxed once**
- **No additional data structure overhead**

$O(V + E)$ **time.**