Unimodularity

Definition A **basis matrix** is a square submatrix with linearly independent columns.

Definition A matrix A is **unimodular** if every basis matrx B of A has $\det(B)$ equal to 1 or -1 .

Theorem Let A be an integer matrix with linearly independent rows. Then the following 3 conditions are equivalent:

1. A is unimodular
2. For any integer vector b , every basic feasible solution to $Ax = b, x \geq 0$ is integral.
3. Every basis matrix B has an integer inverse B^{-1} .

Definition A matrix is **totally unimodular** if each square submatrix has determinant equal to $-1, 0$ or 1 .

Totally unimodualar is a subclass of unimodular.

Totally unimodular

Theorem **The node-arc incidence matrix of a directed network is totally unimodular.**

Non-bipartite matching

$$\max \sum_{(i,j)\in E} x_{ij} \tag{1}$$

$$\textbf{s.t.} \tag{2}$$

$$\sum_{(i,j)\in E} x_{ij} \le 1 \quad \forall i \in V \tag{3}$$

$$x_{ij} \in \{0,1\} \tag{4}$$

$$\tag{5}$$

This program is not totally unimodular.

We can give a graph for which the optimal fraction matching and the optimal integral matching have different values.

Ideas for non-bipartite matching algorithm

- Emulate the bipartite algorithm, and fix it when it breaks.
- unique label property: In the search algorithm for augmenting paths, label nodes as even or odd, given their distance from the first free vertex. If the label of a node is independent of the choices of the search algorithm, then the unique label property holds.
- (redefinition) An augmenting path is an alternating path starting at a free vertex, ending at a free vertex, and the end is labelled odd.

Lemma For two matchings M and M', let $A = M \oplus M'$. Then the connected components of A are of six types:

- empty
- alternating cycle
- alternating path (with four choices for endpoints)

Augmenting Path Lemma: If p is unmatched in a matching M, and there is no augmenting path starting at p, then there is a maximum matching in which p is unmatched.

Augmenting Path

- **Augmenting path lemma implies that if a matching is not optimal, an augmenting path exists**
- **Finding it may be difficult.**

Ideas

- **A stem is an even length alternating path starting at a root p and ending at a vertex w ($p = w$ is possible).**
- **A blossom is an odd length alternating cycle starting and ending at w .**
- **Claim: Every node in a blossom is reachable by both an odd length and an even length alternating path.**
- **Idea: Label the whole blossom as "even."**
- **Implementation of idea: Contract the blossom.**
- **contract(v_1, v_2) - replace v_1 and v_2 by a new vertex smathv' where v' has an edge to any neighbor of v_1 or v_2**

Correctness of Algorithm

- **Let G^C be G with a contracted blossom.**
- **If there is an augmenting path in G^C then there is an augmenting path in G**
- **If there is an augmenting path in G then there is an augmenting path in G^C**

Running time

- **At most n augmenting paths**
- **Each search takes $O(m)$ time to either find a path, or contract a blossom, for a total of $O(nm)$ time per path.**
- **Total time of $O(n^2m)$.**
- **Running time of $O(\sqrt{n}m)$ is possible.**