

Matchings

Definition: Give an undirected graph G , a matching M is a subset of the edges $E \subseteq M$ such that each vertex $v \in V$ is incident to at most one edge from M .

Variants of Matching

- Graph can be bipartite or general
- Graph can be weighted or unweighted

Terms

- A matching M such that, for all edges $e \notin M$, $M \cup \{e\}$ is not a matching, is called **maximal**.
- A maximum cardinality matching is called **maximum**.
- A matching of size $|V|/2$ is called **perfect**.
- The weight of a matching M is $w(M) = \sum_{e \in M} w(e)$.
- A maximum weight matching is the matching of maximum weight.
- All variants polynomial time, bipartite matching seems “easier”.

Matchings in Bipartite Graphs

- Can solve via max flow. Ford Fulkerson is $O(m|f|) = O(nm)$.
- Will develop from first principles to understand terminology and to see improvements.

Definitions

- An **alternating path** with respect to a matching M is a path in which edges alternate between those in M and those not in M .
- A **matched vertex** is one incident to an edge in M
- An **free vertex** is a vertex that is not matched
- An **augmenting path** is an alternating path that starts and ends with a free vertex.
- A **shortest augmenting path** is an augmenting path of shortest length.
- **Symmetric Difference of 2 sets:** $A \oplus B = (A \cup B) - (A \cap B)$

Augmenting Paths

Facts

- Let M be a matching and P be an augmenting path relative to M . Then $M \oplus P$ is a matching and $|M \oplus P| = |M| + 1$.
- Let M be a matching and P_1, P_2, \dots, P_k be k vertex-disjoint augmenting paths relative to M . Then $M \oplus (P_1 \cup P_2 \cup \dots \cup P_k)$ is a matching and $|M \oplus (P_1 \cup P_2 \cup \dots \cup P_k)| = |M| + k$.

Hopcroft-Karp Algorithm

- $M = \emptyset$
- Repeat Until $P = \emptyset$
 - Let $P = (P_1 \cup P_2 \cup \dots \cup P_k)$ be a maximal set of vertex disjoint shortest augmenting paths with respect to M .
 - $M = M \oplus (P_1 \cup P_2 \cup \dots \cup P_k)$

Facts

- You can find a maximal set of vertex-disjoint augmenting paths in $O(m)$ time via breadth-first search.
- Given 2 matchings M and M' , let $G' = (v, M \oplus M')$, then
 - For all vertices, the degree of v in G' is at most 2.
 - G' is a set of alternating paths and alternating cycles.
 - If $|M| < |M'|$, then G' has at least $|M'| - |M|$ vertex disjoint augmenting paths with respect to M

Main Lemma

Lemma Let ℓ be the length of a shortest augmenting path with respect to M . Let P_1, \dots, P_k be a maximal set of vertex disjoint shortest augmenting paths. Let $M' = M \oplus (P_1 \cup \dots \cup P_k)$. Let P be a shortest augmenting path with respect to M' . Then $|P| > \ell$.

Lemma

Lemma If the shortest augmenting path with respect to M has ℓ edges, and M' is a maximum matching, then

$$|M'| \leq |M| + n/\ell$$

.

Theorem Hopcroft-Karp is an $O(\sqrt{nm})$ time algorithm for bipartite matching.

A bidding based approach to bipartite matching

- We will think of our bipartite graph as having bidders B and goods G .
- Let $N(i)$ nodes incident to node i .
- We will maintain prices p_j on the goods.
- If (i, j) is an edge in our matching, we will call i the owner of j .
- We will have a parameter $\delta < 1$, that we will set.

Auction Algorithms

1. For each $j \in G$, set $p_j \leftarrow 0$, $\text{owner}_j \leftarrow \text{null}$.
2. Initialize a queue Q to have all bidders B .
3. While $Q \neq \emptyset$
 - (a) $i \leftarrow Q.\text{Dequeue}()$
 - (b) $j = \text{argmin}_{j' \in N(i)} p_{j'}$
 - (c) if $p_j \leq 1$
 - i. $Q.\text{enqueue}(\text{owner}_j)$
 - ii. $\text{owner}_j \leftarrow i$
 - iii. $p_{j+} = \delta$
4. Return the matching as pairs (owner_j, j)

Analysis

Definition Bidder i is δ -happy if either

- i is unmatched and $\forall j' \in N(i) p_{j'} \geq 1$
- i is matched to j and $\forall j' \in N(i) p_j \leq p_{j'} + \delta$

Facts:

- If $j \in G$ is unmatched, then $p_j = 0$
- Once $j \in G$ is matched, it stays matched.

Lemma 1: All bidders are δ -happy

Lemma 2: If all bidders are δ -happy in a matching M , then for any other matching M' , $|M| \geq |M'| - n\delta$

More Analysis

Choose: $\delta = 1/(n + 1)$

- Applying Lemma 2, we have that $|M| \geq |M'| - n/(n + 1)$ which implies that M is optimal.
- Running time.
 - Each iteration either increases a p_j by δ or deletes a node from the Q permanently.
 - Thus $O(n/(\delta) + n) = O(n^2)$ iterations.
 - Each iteration can be implemented in amortized $O((m/n) + \log n)$ time, for a total of $O(nm + \log n)$ time.

Choose: $\delta = 1/\sqrt{n}$

- Applying Lemma 2, we have that $|M| \geq |M'| - n/\sqrt{n}$ which implies that M is within \sqrt{n} of optima. Therefore we can finish up with \sqrt{n} augmenting paths.
- Running time is $O(\sqrt{nm})$.

Assignment Problem

Minimum weighted perfect bipartite matching in a complete graph.

$$\begin{aligned} \min \quad & \sum_{(v,w) \in E} c(v,w)x(v,w) \\ \text{s.t.} \quad & \\ & \sum_{v \in V} x(v,w) = 1 \quad \forall w \in V \\ & \sum_{w \in V} x(v,w) = 1 \quad \forall v \in V \\ & x(v,w) \geq 0 \quad \forall (v,w) \in E \end{aligned}$$

Dual:

$$\begin{aligned} \max \quad & \sum_{v \in V} \pi(v) \\ \text{s.t.} \quad & \\ & \pi(v) + \pi(w) \leq c(v,w) \quad \forall (v,w) \in E \end{aligned}$$

Dual is a shortest path problem.

Stable Matching

- Given a set of men X and women W .
- Each man ranks the women, and each woman ranks the men.
- Let $r(m, w)$ be the rank assigned from man m to woman w . Define $r(w, m)$ similarly.
- Given a matching M , a pair (m, w) is **unstable** if m and w each prefer each other to their current matched partner.
- A **stable matching** is a matching with no unstable pairs.

Results on Stable Matching

Theorem For any set of preferences, a stable marriage exists and can be found in $O(n^2)$ time.

Algorithm

- Proceed in Rounds
- In each round, any unmatched man proposes to his highest ranked woman who has not yet rejected him.
- Woman accept any proposal if it is preferred to their current v matching (and prefer anyone to being unmatched).

Properties of Algorithm

- Once a woman is matched, she stays matched.
- The partner of a woman only improves over time
- Once a woman rejects a man, she would always reject him in the future.
- A woman always accepts her first proposal.

Conclusion: Every woman is eventually matched.

Claims:

- The matching is stable.
- Each man is matched to the highest ranked woman he could match in any stable marriage.
- Each woman is matched to the lowest ranked man she could match in any stable marriage.