Minimum Cost Flow by Successive Shortest Paths

- **Initialize to the 0 flow**
- **Repeat**
  - **Send flow along a shortest path in G_f**

Comments:

- **Correctly computes a minimum-cost flow**
- **Not polynomial time.**
- **Simple bound of $O((nm)(mU))$ time.**

Pseudoflow

Pseudoflow: A pseudoflow is a function on the edges of a graph satisfying

$$0 \leq f(v, w) \leq u(v, w) \;\; \forall (v, w) \in E$$

- Given a pseduflow f, we define the "excess" at v as

$$e(v) = b(v) + \sum_{w \in V} f(w, v) - \sum_{w \in V} f(v, w).$$

- If $e(v) = 0 \;\; \forall v \in V$, then a pseudoflow is a flow.
- We define reduced cost optimality of a pseudoflow f as

$$\exists \pi \textbf{ s.t. } c^{\pi}(v, w) \geq 0 \;\; \forall (v, w) \in G_f$$

Strategy: Maintain an f and π such that f is a pseudoflow satisfying reduced cost optimality. Work to make f a flow. When f is a flow, you know it is optimal.

How do you initialize?

- **You can assume that** $c(v, w) \geq 0 \quad \forall (v, w) \in E$ **. Then the 0-flow satisfies reduced cost optimality.**
- **But what if the assumption doesn't hold?**

How do you initialize?

- **You can assume that** $c(v,w) \geq 0 \;\; \forall (v,w) \in E$ **. Then the 0-flow satisfies reduced cost optimality.**
- **But what if the assumption doesn't hold?**

- **Set** $f(v,w) = u(v,w)$ **for all edges with** $c(v,w) < 0$ **.**
- **Now, all edges in** G_f **, satisfy** $c^{\pi}(v,w) \geq 0$ **.**
- **Update** $e(v)$ **accordingly.**

Successive Shortest Paths for Minimum Cost Flow

Successive Shortest Path

$f = 0; \quad \Pi = 0$
$e(v) = b(v) \; \forall v \in V$
Initialize $E = \{v : e(v) > 0\}$ **and** $D = \{v : e(v) < 0\}$
while $E \neq 0$
    Pick a node $k \in E$ **and** $\ell \in D$**, s.t.** ℓ **is reachable from** k **in** G_f**.**
    Compute $d(v)$**, shortest path distances from** k **in** G_f
        w.r.t. edge distances c^π**.**
    Let P **be a shortest path from** k **to** ℓ**.**
    Set $\pi = \pi - d$
    Let $\delta = \min\{e(k), -e(\ell), \min\{u_f(v,w) : (v,w) \in P\}\}$
    Send δ **units of flow on the path** P
    Update f**,** G_f**,** E**,** D **and** c^π**.**

Correctness of successive shortest path algorithm

Lemma: Let f be a pseudoflow satisfying reduced cost optimality with respect to π . Let $d(v)$ be the shortest path distance from some node s to v in G_f with respect to c^π . Then

- f satisfies reduced cost optimality with respect to $\pi' = \pi - d$.
- $c^{\pi'}(v, w) = 0$ if (v, w) is on a shortest path from s to some other node.

Lemma: Let f' be the pseduoflow at the end of the while loop. Then f' satisfies reduced cost optimality with respect to π'

Correctness of successive shortest path algorithm

Corollary: **After each iteration of the successive shortest paths algorithm, f satisfies reduced cost optimality.**

But still not necessarily polynomial.

Use Capacity Scaling on top of shortest path algorithm

Def:

$$G_f(\Delta) = \{(v,w) \in G_f : u_f(v,w) \geq \Delta\}$$

Capacity Scaling Algorithm for Minimum Cost Flow

Successive Shortest Path

$f = 0; \quad \pi = 0$
$e(v) = b(v)\ \forall v \in V$
$\Delta = 2^{\lfloor \log \rfloor U}$
while $\Delta \geq 1$
  (Δ scaling phase)
  for every edge $(v, w) \in G_f(\Delta)$
    if $u_f(v, w) \geq \Delta$ **and** $c^\pi(v, w) < 0$
      Send $u_f(v, w)$ **units of flow on** (v, w)**; update** f**,** e
  $S(\Delta) = \{v \in V : e(v) \geq \Delta\}$
  $T(\Delta) = \{v \in V : e(v) \leq -\Delta\}$
  while $S(\Delta) \neq 0$ **and** $T(\Delta) \neq 0$
    Pick a node $k \in S(\Delta)$ **and** $\ell \in T(\Delta)$
    Compute $d(v)$**, shortest path distances from** k **in** $G_f(\Delta)$
      w.r.t. edge distances c^π**.**
    Let P **be a shortest path from** k **to** ℓ**.**
    Set $\pi = \pi - d$
    Let $\delta = \min\{e(k), -e(\ell), \min\{u_f(v, w) : (v, w) \in P\}\}$
    Send δ **units of flow on the path** P
    Update f**,** $G_f(\Delta)$**,** $S(\Delta)$**,** $T(\Delta)$ **and** c^π**.**
  $\Delta = \Delta/2$

Analysis of Running Time