Minimum Spanning Tree Formulation

- **Let x_{ij} be 1 if edge ij is in the tree T .**
- **Need constraints to ensure that:**
  - $n-1$ **edges in** T
  - **no cycles in** T **.**

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First constraint:

$$\sum_{ij \in E} x_{ij} = n - 1$$

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Second constraint. **Subtour elimination constraint. Any subset of k vertices must have at most $k-1$ edges contained in that subset.**

$$\sum_{ij \in E: i \in S, j \in S} x_{ij} \leq |S| - 1 \quad \forall S \subseteq V$$

IP formulation

$$
\begin{aligned}
\textbf{minimize} \sum_{ij \in E} c_{ij} x_{ij} & \\
\textbf{subject to} \sum_{ij \in E} x_{ij} &= n - 1 \\
\sum_{ij \in E: i \in S, j \in S} x_{ij} &\leq |S| - 1 \quad \forall S \subseteq V \\
x_{ij} &\in \{0, 1\} \quad \forall ij \in E
\end{aligned}
$$

- **This is an exponential-sized IP. One can imagine that it is not a great idea to write down and solve this directly.**
- **One can formulate the LP relaxation:**

$$
\begin{aligned}
\textbf{minimize} \sum_{ij \in E} c_{ij} x_{ij} & \\
\textbf{subject to} \sum_{ij \in E} x_{ij} &= n - 1 \\
\sum_{ij \in E: i \in S, j \in S} x_{ij} &\leq |S| - 1 \quad \forall S \subseteq V \\
x_{ij} &\geq 0 \quad \forall ij \in E
\end{aligned}
$$

LP relaxation

$$
\begin{aligned}
\textbf{minimize} \sum_{ij \in E} c_{ij} x_{ij} & \\
\textbf{subject to} \sum_{ij \in E} x_{ij} & = n - 1 \\
\sum_{ij \in E : i \in S, j \in S} x_{ij} & \leq |S| - 1 \quad \forall S \subseteq V \\
x_{ij} & \geq 0 \quad \forall ij \in E
\end{aligned}
$$

- $x_{ij} \leq 1$ **constraints implied by two vertex sets** S **.**
- **Still exponential and not an efficient directly solution method.**
- **One can show that the MST is an optimal solution to the relaxation (proof omitted). That is, the LP has integer extreme points.**
- **Similar formulation is used in many harder problems, e.g. TSP, Steiner Tree.**

Other Formulations

Cut Formulation For every cut, at least one edge must cross the cut. For a subset $S \subset V$, let $\delta(S)$ be the edges crossing the cut (one endpoint in S, one in $V - S$.

$$
\begin{aligned}
\textbf{minimize} \sum_{ij \in E} c_{ij} x_{ij} & \\
\textbf{subject to} \sum_{ij \in E} x_{ij} &= n - 1 \\
\sum_{ij \in E : ij \in \delta(S)} x_{ij} &\geq 1 \qquad \forall S \subseteq V, \emptyset \neq S \neq V \\
x_{ij} &\in \{0, 1\} \ \ \forall ij \in E
\end{aligned}
$$

- Also exponential, also useful in other problems.
- For the LP relaxation, there may be fractional extreme points.