Johnson's Algorithm for All-Pairs Shortest Paths

- **Input is Graph $G = (V, E)$ with arbitrary edge weights c.**
- **Assume strongly connected.**
- **Assume no negative cycle.**
- **Can run Bellman Ford n times for $O(n^2 m)$.**
- **Can run Floyd-Warshall in $O(n^3)$ time.**
- **If all edge weights are non-negative, can run Dijkstra n times for a running time of $O(nm + n^2 \log n)$. Can we match that for general weights?**
- **Johnson Algorithm**
  - **Run single source shortest paths from one arbitrary node s. (Bellman Ford)**
  - **Use results of previous step to "reweight edges" so that all edges have non-negative weights**
  - **Run single source shortest paths from the other $n - 1$ vertcies. (Dijkstra)**
- **Running Time is $O(nm + n(m + n \log n)) = O(nm + n^2 \log n)$, better than $O(n^3)$ for non-dense graphs.**

How to Reweight

- **Let $p(v)$ be some prices that we put on vertices.**
- **Consider reduced cost of edge vw , $c_p(vw) = c(vw) - p(v) + p(w)$.**
- **For a P , what is the relationship between $c(P)$ and $c_p(P)$?**
- **For a cycle X , what is the relationship between $c(X)$ and $c_p(X)$?**

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Path $P = v_1 v_2 \dots v_k$

$$\begin{aligned} c_p(P) &= c_p(v_1v_2) + c_p(v_2v_3) + \cdots + c_p(v_{k-1}v_k) \\ &= c(v_1v_2) - p(v_1) + p(v_2) + c(v_2v_3) - p(v_2) + p(v_3) + \cdots + c(v_{k-1}v_k) - p(v_{k-1}) + p(v_k) \\ &= c(P) - p(v_1) + p(v_k) \end{aligned}$$

- **The length of each path from v_1 to v_k is increased by the same amount, $p(v_k) - p(v_1)$.**
- **Therefore, the shortest path is still the shortest path**
- **For a cycle $p(v_1) = p(v_k)$, so the distance does not change at all.**

Reweighting for Shortest Paths

- **We will set $p(v)$ to the negative of the shortest path length $d(v)$ from s to v.**
- **We now have that $c_p(vw) = c(vw) - p(v) + p(w) = c(vw) + d(v) - d(w)$.**
- **But we know that, by the optimality condition for shortest paths:**

$$d(w) \leq d(v) + c(vw) \Rightarrow c(wv) + d(v) - d(w) \geq 0 \Rightarrow c_p(vw) \geq 0$$

- **So was have non-negative edge weights, still no negative cycles, and can use Dijkstra's algorithm.**