Matchings

Definition: Give an undirected graph G , a matching M is a subset of the edges $E \subseteq M$ such that each vertex $v \in V$ is incident to at most one edge from M .

Variants of Matching

- Graph can be bipartite or general
- Graph can be weighted or unweighted

Terms

- A matching M such that, for all edges $e \notin M$, $M \cup \{e\}$ is not a matching, is called **maximal**.
- A maximum cardinality matching is called **maximum**.
- A matching of size $|V|/2$ is called **perfect**.
- The weight of a matching M is $w(M) = \Sigma_{e \in M} w(e)$.
- A maximum weight matching is the matching of maximum weight.
- All variants polynomial time, bipartite matching seems “easier”.

Matchings in Bipartite Graphs

- Can solve via max flow. Ford Fulkerson is $O(m|f|) = O(nm)$.
- Will develop from first principles to understand terminology and to see improvements.

Definitions

- An alternating path with respect to a matching M is a path in which edges alternate between those in M and those not in M .
- A matched vertex is one incident to an edge in M
- An free vertex is a vertex that is not matched
- An augmenting path is an alternating path that starts and ends with a free vertex.
- A shortest augmenting path is an augmenting path of shortest length.
- Symmetric Difference of 2 sets: $A \oplus B = (A \cup B) - (A \cap B)$

Augmenting Paths

Facts

- Let M be a matching and P be an augmenting path relative to M. Then $M \oplus P$ is a matching and $|M \oplus P| = |M| + 1$.
- Let M be a matching and $P_1, P_2, \ldots P_k$ be k vertex-disjoing augmenting path relative to M. Then $M \oplus (P_1 \cup P_2 \cup \ldots \cup P_k)$ is a matching and $|M \oplus (P_1 \cup P_2 \cup \ldots \cup P_k)| = |M| + k$.

Hopkroft-Karp Algorithm

- $M = \emptyset$
- Repeat Until $P = \emptyset$
  - Let $P = (P_1 \cup P_2 \cup \ldots \cup P_k)$ be a maximal set of vertex disjoint shortest augmenting paths with respect to M.
  - $M = M \oplus (P_1 \cup P_2 \cup \ldots \cup P_k)$

Facts

- **You can find a maximal set of vertex-disjoint augmenting paths in $O(m)$ time via breadth-first search.**
- **Given 2 matchings M and M', let $G' = (v, M \oplus M')$, then**
  - **For all vertices, the degree of v in G' is at most 2.**
  - **G' is a set of alternating paths and alternating cycles.**
  - **If $|M| < |M'|$, then G' has at least $|M'| - |M|$ vertex disjoint augmenting paths with respect to M**

Main Lemma

Lemma **Let** ℓ **be the length of a shortest augmenting path with respect to** M **. Let** $P_1, \ldots, P_k$ **be a maximal set of vertex disjiont shortest augmenting paths. Let** $M' = M \oplus (P_1 \cup \ldots P_k)$ **. Let** P **be a shortest augmenting path with respect to** M' **. Then** $|P| > \ell$ **.**

Lemma

Lemma If the shortest augmeting path with respect to M has ℓ edges, and M' is a maximum matching, then

$$|M'| \leq |M| + n/\ell$$

.

Theorem Hopkroft-Karp is an $O(\sqrt{n}m)$ time algorithm for bipartite matching.

A bidding based approach to bipartite matching

- We will think of our bipartite graph as having bidders B and goods G.
- Let $N(i)$ nodes incident to node i .
- We will maintain prices p_j on the goods.
- If (i, j) is an edge in our matching, we will call i the owner of j .
- We will have a parameter $\delta < 1$, that we will set.

Auction Algorithms

1. For each $j \in G$, set $p_j \leftarrow 0$, $\textbf{owner}_j \leftarrow null$.
2. Initiliaze a queue Q to have all bidders B .
3. While $Q \neq \emptyset$
   (a) $i \leftarrow Q.Dequeue()$
   (b) $j = \textbf{argmin}_{j' \in N(i)} p_{j'}$
   (c) if $p_j \leq 1$
      i. Q.enqueue(owner_j)
      ii. $\text{owner}_j \leftarrow i$
      iii. $p_j + = \delta$
4. Return the matching as pairs (owner_j, j)

Analysis

Definition **Bidder i is δ-happy if either**

- i **is unmatched and** $\forall j' \in N(i)\ p_{j'} \geq 1$
- i **is matched to** j **and** $\forall j' \in N(i)\ p_j \leq p_{j'} + \delta$

Facts:

- **If** $j \in G$ **is unmatched, then** $p_j = 0$
- **Once** $j \in G$ **is matched, it stays matched.**

Lemma 1: **All bidders are δ-happy**

Lemma 2: **If all bidders are δ-happy in a matching M, then for any other matching M',** $|M| \geq |M'| - n\delta$

More Analysis

Choose: $\delta = 1/(n+1)$

- **Applying Lemma 2, we have that $|M| \geq |M'| - n/(n+1)$ which implies that M is optimal.**
- **Running time.**
  - **Each iteration either increases a p_j by δ or deletes a node from the Q permanently.**
  - **Thus $O(n/(\delta) + n) = O(n^2)$ iterations.**
  - **Each iteration can be implemeted in amortized $O((m/n) + \log n)$ time, for a total of $O(nm + \log n)$ time.**

Choose: $\delta = 1/\sqrt{n}$

- **Applying Lemma 2, we have that $|M| \geq |M'| - n/\sqrt{n}$ which implies that M is within $\sqrt{n}$ of optimal. Therefore we can finish up with $\sqrt{n}$ augmenting paths.**
- **Running time is $O(\sqrt{n}m)$.**

Assignment Problem

Minimum weighted perfect bipartite matching in a complete graph.

$$\min \sum_{(v,w)\in E} c(v,w)x(v,w)$$

s.t.

$$\begin{aligned} \sum_{v\in V} x(v,w) &= 1 \qquad \forall w \in V \\ \sum_{w\in V} x(v,w) &= 1 \qquad \forall v \in V \\ x(v,w) &\geq 0 \quad \forall (v,w) \in E \end{aligned}$$

Dual:

$$\max \sum_{v\in V} \pi(v)$$

s.t.

$$\pi(v) + \pi(w) \leq c(v,w) \;\; \forall (v,w) \in E$$

Dual is a shortest path problem.

Stable Matching

- Given a set of men X and women W.
- Each man ranks the women, and each woman ranks the men.
- Let $r(m, w)$ be the rank assigned from man m to woman w. Define $r(w, m)$ similarly.
- Given a matching M, a pair (m, w) is unstable if m and w each prefer each other to their current matched partner.
- A stable matching is a matching with no unstable pairs.

Results on Stable Matching

Theorem For any set of preferences, a stable marriage exists and can be found in $O(n^2)$ time.

Algorithm

- Proceed in Rounds
- In each round, any unmatched man proposes to his highest ranked woman who has not yet rejected him.
- Woman accept any proposal if it is preferred to their current v matching (and prefer anyone to being unmatched).

Properties of Algorithm

- **Once a woman is matched, she stays matched.**
- **The partner of a woman only improves over time**
- **Once a woman rejects a man, she would always reject him in the future.**
- **A woman always accepts her first proposal.**

Conclusion: **Every woman is eventually matched.**

Claims:

- **The matching is stable.**
- **Each man is matched to the highest ranked woman he could match in any stable marriage.**
- **Each woman is matched to the lowest ranked man she could match in any stable marriage.**