Minimum Cost Flow by Successive Shortest Paths

- Initialize to the 0 flow
- Repeat
  - Send flow along a shortest path in $G_f$

Comments:
- Correctly computes a minimum-cost flow
- Not polynomial time.
- Simple bound of $O((nm)(mU))$ time.
**Pseudoflow**

**Pseudoflow:** A pseudoflow is a function on the edges of a graph satisfying

\[ 0 \leq f(v, w) \leq u(v, w) \quad \forall (v, w) \in E \]

- Given a pseudoflow \( f \), we define the “excess” at \( v \) as

\[ e(v) = b(v) + \sum_{w \in V} f(w, v) - \sum_{w \in V} f(v, w). \]

- If \( e(v) = 0 \ \forall v \in V \), then a pseudoflow is a flow.
- We define reduced cost optimality of a pseudoflow \( f \) as

\[ \exists \pi \text{ s.t. } c^\pi(v, w) \geq 0 \ \forall (v, w) \in G_f \]

**Strategy:** Maintain an \( f \) and \( \pi \) such that \( f \) is a pseudoflow satisfying reduced cost optimality. Work to make \( f \) a flow. When \( f \) is a flow, you know it is optimal.
How do you initialize?

- You can assume that $c(v, w) \geq 0 \quad \forall (v, w) \in E$. Then the 0-flow satisfies reduced cost optimality.
- But what if the assumption doesn’t hold?
How do you initialize?

- You can assume that $c(v, w) \geq 0 \ \forall (v, w) \in E$. Then the 0-flow satisfies reduced cost optimality.
- But what if the assumption doesn’t hold?

- Set $f(v, w) = u(v, w)$ for all edges with $c(v, w) < 0$.
- Now, all edges in $G_f$, satisfy $c^\pi(v, w) \geq 0$.
- Update $e(v)$ accordingly.
Successive Shortest Paths for Minimum Cost Flow

Successive Shortest Path

1. \( f = 0; \quad \Pi = 0 \)
2. \( e(v) = b(v) \quad \forall v \in V \)
3. Initialize \( E = \{v : e(v) > 0\} \) and \( D = \{v : e(v) < 0\} \)
4. while \( E \neq 0 \)
5. \hspace{1em} Pick a node \( k \in E \) and \( \ell \in D \), s.t. \( \ell \) is reachable from \( k \) in \( G_f \).
6. \hspace{1em} Compute \( d(v) \), shortest path distances from \( k \) in \( G_f \)
   \hspace{2em} w.r.t. edge distances \( c^\pi \).
7. \hspace{1em} Let \( P \) be a shortest path from \( k \) to \( \ell \).
8. \hspace{1em} Set \( \pi = \pi - d \)
9. \hspace{1em} Let \( \delta = \min\{e(k), -e(\ell), \min\{u_f(v, w) : (v, w) \in P\}\} \)
10. \hspace{1em} Send \( \delta \) units of flow on the path \( P \)
11. \hspace{1em} Update \( f, G_f, E, D \) and \( c^\pi \).
Correctness of successive shortest path algorithm

Lemma: Let $f$ be a pseudoflow satisfying reduced cost optimality with respect to $\pi$. Let $d(v)$ be the shortest path distance from some node $s$ to $v$ in $G_f$ with respect to $c^\pi$. Then

- $f$ satisfies reduced cost optimality with respect to $\pi' = \pi - d$.
- $c^\pi'(v, w) = 0$ if $(v, w)$ is on a shortest path from $s$ to some other node.

Lemma: Let $f'$ be the pseudoflow at the end of the while loop. Then $f'$ satisfies reduced cost optimality with respect to $\pi'$. 
Correctness of successive shortest path algorithm

**Corollary:** After each iteration of the successive shortest paths algorithm, \( f \) satisfies reduced cost optimality.

But still not necessarily polynomial.
Use Capacity Scaling on top of shortest path algorithm

Def:

\[ G_f(\Delta) = \{(v, w) \in G_f : u_f(v, w) \geq \Delta\} \]
Capacity Scaling Algorithm for Minimum Cost Flow

Successive Shortest Path

1. \( f = 0; \quad \pi = 0 \)
2. \( e(v) = b(v) \quad \forall v \in V \)
3. \( \Delta = 2^{\lfloor \log U \rfloor} \)
4. while \( \Delta \geq 1 \)
   (\( \Delta \) scaling phase )
5. for every edge \((v, w) \in G_f(\Delta)\)
6.   if \( u_f(v, w) \geq \Delta \) and \( c^\pi(v, w) < 0 \)
7.   Send \( u_f(v, w) \) units of flow on \((v, w)\); update \( f, e \)
8. \( S(\Delta) = \{v \in V : e(v) \geq \Delta\} \)
9. \( T(\Delta) = \{v \in V : e(v) \leq -\Delta\} \)
10. while \( S(\Delta) \neq 0 \) and \( T(\Delta) \neq 0 \)
11.   Pick a node \( k \in S(\Delta) \) and \( \ell \in T(\Delta) \)
12.   Compute \( d(v) \), shortest path distances from \( k \) in \( G_f(\Delta) \)
    w.r.t. edge distances \( c^\pi \).
13.   Let \( P \) be a shortest path from \( k \) to \( \ell \).
14.   Set \( \pi = \pi - d \)
15. Let \( \delta = \min \{e(k), -e(\ell), \min \{u_f(v, w) : (v, w) \in P\}\} \)
16. Send \( \delta \) units of flow on the path \( P \)
17. Update \( f, G_f(\Delta), S(\Delta), T(\Delta) \) and \( c^\pi \).
18. \( \Delta = \Delta/2 \)
Analysis of Running Time