Carpool Fairness

	Days				
Person	**1**	**2**	**3**	**4**	**5**
1	**X**	**X**	**X**		
2	**X**		**X**		
3	**X**	**X**	**X**	**X**	**X**
4		**X**	**X**	**X**	**X**

What is a fair division of driving?

Carpool Fairness

	Days				
Person	**1**	**2**	**3**	**4**	**5**
1	**X**	**X**	**X**		
2	**X**		**X**		
3	**X**	**X**	**X**	**X**	**X**
4		**X**	**X**	**X**	**X**

What is a fair division of driving?

	Days					
Person	**1**	**2**	**3**	**4**	**5**	**Total responsibility** r_i
1	**1/3**	**1/3**	**1/4**			**11/12**
2	**1/3**		**1/4**			**7/12**
3	**1/3**	**1/3**	**1/4**	**1/2**	**1/2**	**23/12**
4		**1/3**	**1/4**	**1/2**	**1/2**	**19/12**

Proposal: **Person** i **should drive no more than** $\lceil r_i \rceil$ **times.**

Formaulation as a flow problem

Person	Days 1	2	3	4	5	Total responsibility r_i
1	1/3	1/3	1/4			11/12
2	1/3		1/4			7/12
3	1/3	1/3	1/4	1/2	1/2	23/12
4		1/3	1/4	1/2	1/2	19/12

- Bipartite graph, nodes for each person and day.
- Think of r_i as supply for each person
- Think of 1 as demand for each day
- Edges between person and day if they can drive on that day..

Does a flow of value 5 exist?

Fractional/Integral flow

- **A fractional flow of value 5 exists in graph with source-incident capacities $\lceil r_i \rceil$ and flow of r_i**
- **Theorem If capacities are integral a fractional flow of value x exists, then an integral flow for value $\lceil x \rceil$ exists.**
- **Use the integral flow to solve the carpool problem.**

Baseball Elimination

(SportsWriters end of Season Problem)

Team	**Wins** w_i	**Games left** g_i	**Games against** g_{ij}			
			NY	**Bos**	**Tor**	**Bal**
NY Yankees	**93**	**8**	**-**	**1**	**6**	**1**
Boston Red Sox	**89**	**4**	**1**	**-**	**0**	**3**
Toronto Blue Jays	**88**	**7**	**6**	**0**	**-**	**1**
Baltimore Orioles	**86**	**5**	**1**	**3**	**1**	**-**

Question: **Which teams are eliminated and which are not?**

Formalism

- w_i - wins for team i
- g_i - games left for team i
- g_{ij} - games left between i and j

For any subset R of teams T :

- wins in R, $w(R) = \Sigma_{i \in R} w_i$
- games left in R, $g(R) = \Sigma_{i,j \in R, i<j} g_{ij}$

A lower bound on a number of games that some team must win

$$a(R) = \frac{w(R) + g(R)}{|R|}$$

Claim: For $i \in T$, $R \subseteq T - \{i\}$, and $a(R) > w_i + g_i$, then i is eliminated.

Justification: Some team must win the average.

A stronger condition?

Let x_{ij} **be the number of times that** i **beats** j **.**

Team k **is not eliminated if there exist** x_{ij} **s.t. it is possible for team** k **to come in first. That is,**

$$
\begin{aligned}
x_{ij} + x_j \quad &= g_{ij} && \forall i, j \in T && (1)\\
w_k + g_k \quad &\geq w_i + \Sigma_{j \in T - \{k\}} x_{ij} && \forall i \in T && (2)\\
x_{ij} \geq 0 \quad & x_{ij} \in \{0, 1\} && && (3)
\end{aligned}
$$

Can also use flow problem.

Proof that flow problem solves the problem

Claims:

- **If a flow** f **of value** $\sum_{i<j} g_{ij}$ **exists, then team** k **is not eliminated.**
- **If no flow of value** $\sum_{i<j} g_{ij}$ **exists then them** k **is eliminated.**