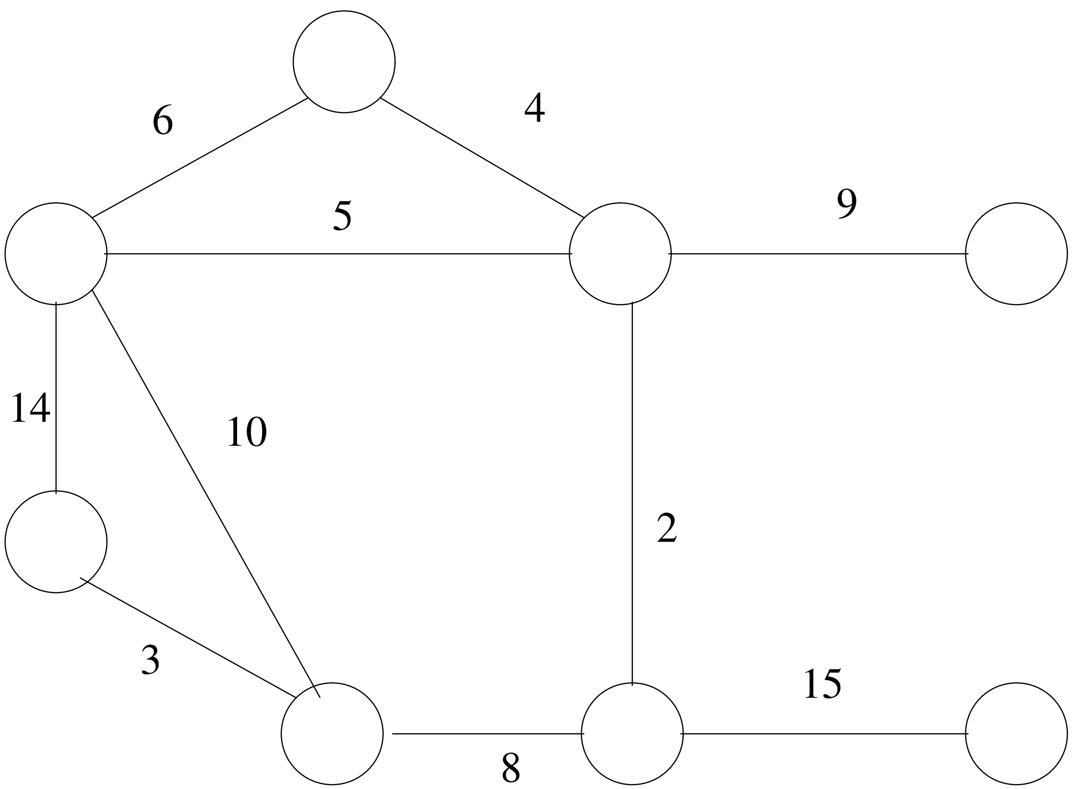
Minimum Spanning Trees

- $G = (V, E)$ **is an undirected graph with non-negative edge weights** $w : E \to Z^+$
- **We assume wlog that edge weights are distinct**
- **A spanning tree is a tree with** $V - 1$ **edges, i.e. a tree that connects all the vertices.**
- **The total cost (weight) of a spanning tree T is defined as** $\Sigma_{e \in T} w(e)$
- **A minimum spanning tree is a tree of minimum total weight.**



Minimum Spanning Trees

- $G = (V, E)$ **is an undirected graph with non-negative edge weights** $w : E \rightarrow Z^+$

- **We assume wlog that edge weights are distinct**
- **A spanning tree is a tree with** $V - 1$ **edges, i.e. a tree that connects all the vertices.**
- **The total cost (weight) of a spanning tree T is defined as** $\Sigma_{e \in T} w(e)$
- **A minimum spanning tree is a tree of minimum total weight.**

Greedy Property

Greedy Property: **The minimum weight edge crossing a cut is in the minimum spanning tree.**

Proof Idea: **Assume not, then remove an edge crossing the cut and replace it with the minimum weight edge.**

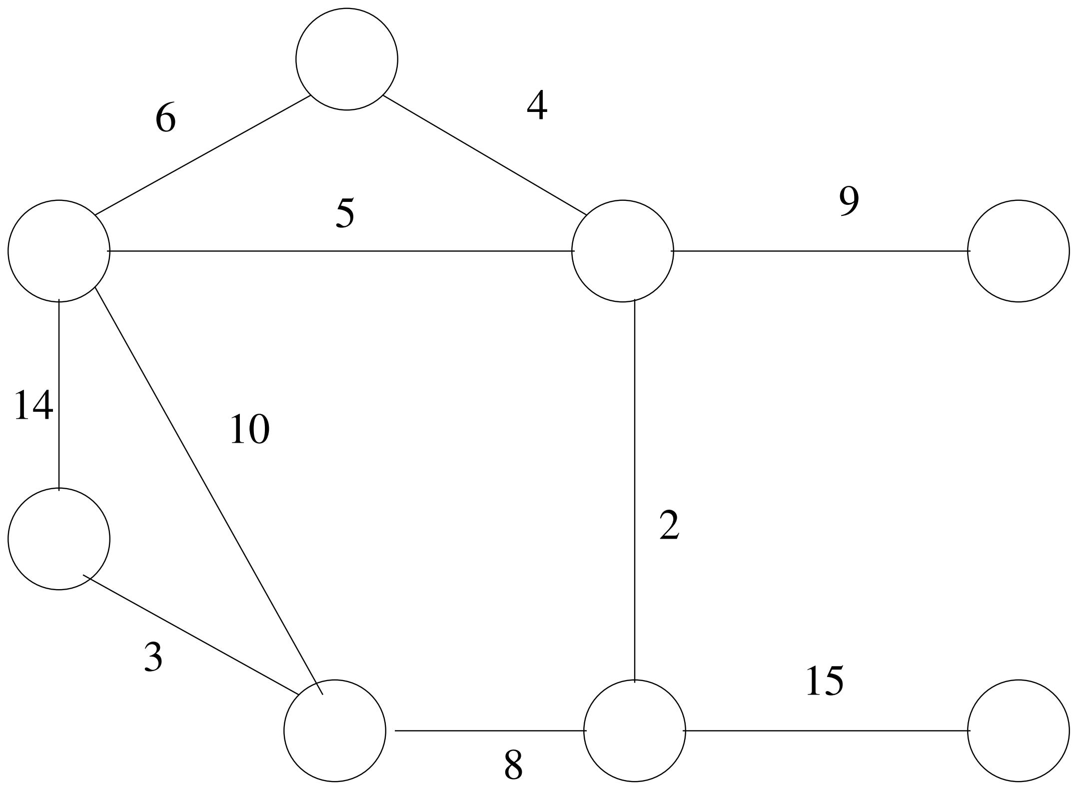
Restatement Lemma: **Let $G = (V, E)$ be an undirected graph with edge weights w. Let $A \subseteq E$ be a set of edges that are part of a minimum spanning tree. Let (S, T) be a cut with no edges from A crossing it. Then the minimum weight edge crossing (S, T) can be added to A.**

Recall Kruskal's algorithm

MST-Kruskal(G, w)

$A \leftarrow \emptyset$
for each vertex $v \in V[G]$
    do MAKE-SET(v)
sort the edges of E **into nondecreasing order by weight** w
for each edge $(u, v) \in E$**, taken in nondecreasing order by weight**
    do if FIND-SET$(u) \neq$ FIND-SET(v)
        then $A \leftarrow A \cup \{(u, v)\}$
            UNION(u, v)
return A

Example



Baruvka's Algorithm

- **Repeat**
  - **Every node picks its minimum incoming edge and adds to the spanning tree T**
  - **Contract all edges in T**

- **How much progress is made?**
- **How implement an iteration?**
- **Total running time?**

Baruvka's Algorithm

- **Repeat**
  - **Every node picks its minimum incoming edge and adds to the spanning tree T**
  - **Contract all edges in T**

- **How much progress is made?**
- **How implement an iteration?**
- **Total running time?**

$$T(n, m) = T(n/2, m - n) + O(n + m)$$

Problem: **Edges don't decrease fast enough**

Eliminating Edges

- **The heaviest edge on any cycle is not in the MST.**
- **Let** F **be a forest**
- **Let** $w_F(u,v)$ **be the maximum weight of an edge on the path from** u **to** v **in** F **(or** ∞ **if the path does not exist.**
- **Edge** (u,v) **is F-heavy if** $w(u,v) > w_F(u,v)$ **and F-light otherwise.**

Claim: **Let** F **be any forest, let** (u,v) **be any edge. If** (u,v) **is F-heavy, then** (u,v) **is not in the MST.**

Ideas

- **It is good to eliminate F-heavy edges**
- **The MST T would let us eliminate all non-MST edges.**
- **What can an F that is not an MST eliminate**

Fact to accept without proof: Given G and a forrest F, we can eliminate all F-heavy edges in $O(n + m)$ time (spanning tree verification).

Algorithm $MST(G)$

1. Run 3 Bruvka phases to get G' . Let C be the contracted edges.
2. Let G'' be G' with each edge included with prob. $1/2$.
3. Recursively compute $F'' = MST(G'')$.
4. Identify the F"-heavy edges in G' . Delete them to obtain G''' .
5. Recursively compute $F''' = MST(G''')$
6. Return $F''' \cup C$

Note: The recursion bottons out on a graph with $O(1)$ nodes.

Key Lemma: Let H be a subgraph of G where each edge is included with probability p . Let F be a Minimum Spanning Forest of H . Then the expected number of F-light edges in G is at most n/p .

Recurrence $T(n,m) \leq O(n+m) + T(n/8, m/2) + T(n/8, n/4) = O(n+m)$