Multicommodity Flow

- **Given a directed network with edge capacities u and possibly costs c.**
- **Give a set K of k commodities, where a commodity i is defined by a triple (s_i, t_i, d_i) – source, sink and demand.**
- **For each commodity, you want to find a feasible flow, subject to joint capacity constraints.**

Formulation

$f_i(v,w)$ **is the flow of commodity** i **on edge** (v,w) **.**

$$\sum_w f_i(v,w) - \sum_w f_i(w,v) = \begin{cases} 0 & \textbf{if } v \neq s \textbf{ and } v \neq t \\ d_i & \textbf{if } v = s \\ -d_i & \textbf{if } v = t \end{cases} \quad \forall v \in V, i \in K$$

$$\sum_{i \in K} f_i(v,w) \leq u(v,w) \quad \forall (v,w) \in E$$

$$f_i(v,w) \geq 0 \quad \forall (v,w) \in E$$

- **Single commodity flow:** m **variables,** $m+n$ **constraints**
- **Multicommdodity flow:** km **variables,** $kn+m$ **constraints,** km **non-negativity constraints**

Size of A matrix: $km(kn+m) = k^2nm + km^2$

A computationally challenging problem

Facts About Multicommodity Flow

- LP is big
- A matrix is not Totally Unimodular.
- Optimal solution to a multicommodity flow LP might be fractional.
- All feasible solutions might be fractional.

Optimization Variants

- Given costs on edges, $c(v,w)$, find a feasible flow minimizing $\sum_i \sum_{vw} c(v,w) f_i(v,w)$

- No given demands, maximize total flow
- No given demands maximize total flow cost
- Send at least z percent of each demand, maximize z . (concurrent flow)
- Send demands, find minimum α such that the flow is still feasible with capacities $\alpha c(v,w)$. (equivalent to previous problem)

Solutions

- **Optimal fractional solution is solvable by LP in polynomial time**
- **Polynomial is large (degree 6 or so).**
- **No known polynomial time algorithms for multicommodity flow that do not use LP ("easiest" such problem without a combinatorial algorithm).**
- **There are combinatorial algorithms that find a $(1+\epsilon)$ -optimal solution to concurrent flow in polynomial time (many algorithms e.g. $O(\epsilon^{-2}knm)$ time).**
- **Finding a feasible integer solution is NP-complete. Even the disjoint paths version is NP-complete.**

Approximation for Concurrent Flow

- **Consider integer problem, $u = 1, d = 1$.**
- **Objective is to maximize fraction of demand sent.**
- **Equivalent problem: Send one unit of each demand, allow capacity constraints to be violated, but minimize $\lambda = \max_{(v,w)} \Sigma_i f_i(v, w)$**
- **Assume wlog, that a fractional flow exists**

Algorithm

- **Find the optimal fractional flow, via LP**
- **"Round" the fractions (carefully...)**

How to Round

- **Decompose the flow for commodity i into a set of β_i $s_i - -t_i$ paths, $P_1^i, \ldots, P_{\beta_i}^i$ with values $f_1^i, \ldots, f_{\beta_i}^i$.**
- **Interpret the flow values as probabilities and choose a path for commodity i according to the probability distribution defined by the flows.**

Analysis

Use a Chernoff Bound.

Let x_i **be** $0 - 1$ **random variables, where** $x_i = 1$ **with probability** p_i **. Let** $M = E(\sum x_i) = \sum p_i$. **Then, for** $0 < \beta < 1$ **, we have**

$$\Pr(\sum x_i > (1 + \beta)M) \leq e^{-\beta^2 M/2}$$

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