Shortest Paths

- **Input: weighted, directed graph $G = (V, E)$, with weight function $w : E \to \mathbf{R}$.**
- **The weight of path $p =< v_0, v_1, \ldots, v_k >$ is the sum of the weights of its constituent edges:**

$$w(p) = \sum_{i=1}^{k} w(v_{i-1}, v_i) \ .$$

- **The shortest-path weight from u to v is**

$$\delta(u,v) = \begin{cases} \min\{w(p)\} & \textbf{if there is a path } p \textbf{ from } u \textbf{ to } v \ , \\ \infty & \textbf{otherwise} \ . \end{cases}$$

- **A shortest path from vertex u to vertex v is then defined as any path p with weight $w(p) = \delta(u, v)$.**

To do:

- **Quickly Review Basics**
- **Talk About Data Structures for Dijkstra**
- **Talk About Some Details in Implementation**

Shortest Paths

Key Property: Subpaths of shortest paths are shortest paths Given a weighted, directed graph $G = (V, E)$ with weight function $w : E \to \mathbf{R}$, let $p =< v_1, v_2, \ldots, v_k >$ be a shortest path from vertex v_1 to vertex v_k and, for any i and j such that $1 \leq i \leq j \leq k$, let $p_{ij} =< v_i, v_{i+1}, \ldots, v_j >$ be the subpath of p from vertex v_i to vertex v_j. Then, p_{ij} is a shortest path from v_i to v_j.

Note: this is optimal substructure

Corollary 1 For all edges $(u, v) \in E$,

$$\delta(v) \leq \delta(u) + w(u, v)$$

Corollary 2 Shortest paths follow a tree of edges for which

$$\delta(v) = \delta(u) + w(u, v)$$

More precisely, any edge in a shortest path must satisfy

$$\delta(v) = \delta(u) + w(u, v)$$

Relaxation

Relax(u, v, w)

if $d[v] > d[u] + w(u, v)$
    then $d[v] \leftarrow d[u] + w(u, v)$
        $\pi[v] \leftarrow u$ **(keep track of actual path)**

Lemma: Assume that we initialize all $d(v)$ to ∞, $d(s) = 0$ and execute a series of Relax operations. Then for all v, $d(v) \geq \delta(v)$.

Lemma: Let $P = e_1, \ldots, e_k$ be a shortest path from s to v. After initialization, suppose that we relax the edges of P in order (but not necessarily consecutively). Then $d(v) = \delta(v)$.

Algorithms

Goal of an algorithm: **Relax the edges in a shortest path in order (but not necessarily consecutively).**

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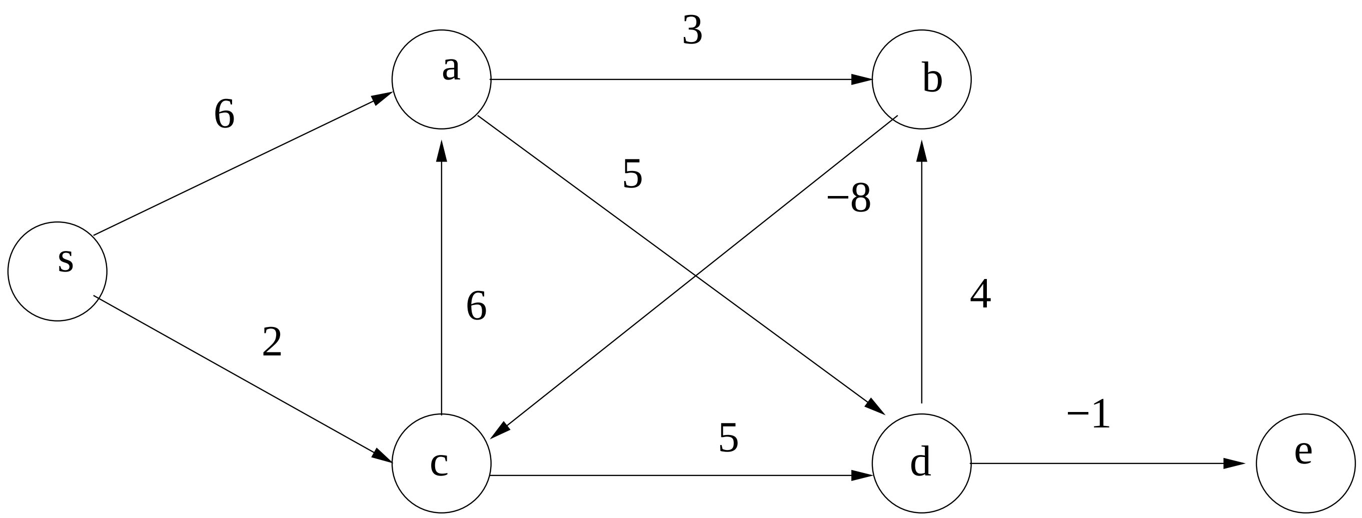
Bellman-Ford(G, w, s)

Initialize-Single-Source(G, s)
for $i \leftarrow 1$ **to** $|V[G]| - 1$
    do for each edge $(u, v) \in E[G]$
        do Relax(u, v, w)
for each edge $(u, v) \in E[G]$
    do if $d[v] > d[u] + w(u, v)$
        then return FALSE
return TRUE

$Initialize - Single - Source(G, s)$

for each vertex $v \in V[G]$
    do $d[v] \leftarrow \infty$
        $\pi[v] \leftarrow$ NIL
$d[s] \leftarrow 0$

Example



Correctness of Bellman Ford

- **Every shortest path must be relaxed in order**
- **If there are negative weight cycles, the algorithm will return false**

Running Time $O(VE)$

All edges non-negative

- **Dijkstra's algorithm, a greedy algorithm**
- **Can relax edges out of each vertex exactly once.**

Dijkstra(G, w, s)

```
Initialize-Single-Source(G, s)
S ← ∅
for each vertex v ∈ V
        do Insert(Q,v)
while Q ≠ ∅
        do u ← Extract-Min(Q)
             S ← S ∪ {u}
             for each vertex v ∈ Adj[u]
                    do Relax(u, v, w)
                         if relax changed d(v)
                             then Decrease-Key(v, d(v))
```

Running Time and Correctness

Correctness of Dijkstra's algorithm **Dijkstra's algorithm, run on a weighted, directed graph $G = (V, E)$ with nonnegative weight function w and source s, terminates with $d[u] = \delta(s, u)$ for all vertices $u \in V$.**

- E **decrease keys and** V **delete-min's**
- $O(E \log V)$ **using a heap**
- $O(E + V \log V)$ **using a Fibonacci heap**

Question: **What can we do when the weights come from a restricted range?**