IEOR 8100. Advanced Topics in IEOR: Matching
Lecture 2: Fractional Perfect Matching

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1 Definitions and Constraints

The variable x_e is 0 if the edge is not in a considered matching, and 1 if it is.

We examine two contraints, and investigate if they're enough for a linear program to give us a matching. We want to avoid adding a constraint that specifies that x_e is integral, as this can turn our LP into which much harder to solve.

1. $\sum\limits_{e \in \delta(v)} x_e \leq 1 \ \forall v \in V$
2. $x_e \geq 0$

We let $\chi^{M'} = (e_1, e_2, ..., e_m)$, where $e_i = 1$ if it is in a M' and 0 otherwise.

We will consider optimization over the matching polytope, for normal matchings and perfect matchings. The matching polytope is defined as:

- $M = conv\{\chi^{M'} | M' \text{ is a matching }\}$
- $PM = conv\{\chi^{M'} | M' \text{ is a perfect matching }\}$

2 Facts About the Matching Polytope

Let P be the feasible region defined by 1 and 2.

We now compare M and P.

Claim. $M \subseteq P$

Proof. Any point in M can be written as $\sum_i \lambda_i \chi_i$, with $\sum_i \lambda_i = 1$. Each χ^i can have at each index at most one edge, so the sum cannot be greater than 1. This satisifies condition 1. Condition 2 is also clearly satisfied, as χ contains only 0s and 1s. ☐

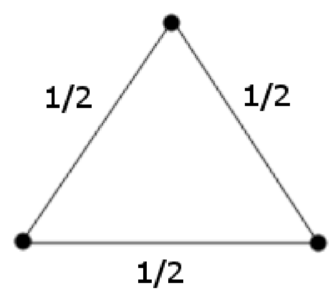

Figure 1: The point corresponding to this graph cannot be in M.

Claim. *$P \not\subseteq M$*

Proof. A simple counter-example is the usual triangle graph (the complete graph on three edges). The vector $(1/2, 1/2, 1/2) \in P$ but is not in M. ☐

3 Bipartite Case

Claim. *$P \subseteq M$ when G is bipartite*

Proof. Assume by way of contradiction that the vertices of P are not integral. Let x be a vertex with at least one non-integral component. Note that "vertex" here refers to vertex of the polytope P, not vertex of the graph.

Let $G' = (V, E_x)$ be the graph defined on the non-integral edges from x. G' is bipartite, so all cycles in G' are of even length.

Case 1. *Assume that G' has a cycle.*

All edges in this cycle are fractional, as all edges of G' are fracitonal. $\exists \epsilon$ s.t. $\epsilon = \min_e\{x_e, 1 - x_e\}$ where x_e are the weights on the edges in the cycle.

Arbitrarily pick an edge on the cycle and place it in a set A. Continue around the cycle, putting the next edge in a set B. We continue in this manner, alternating between putting edges in A and B. Since the cycle is even, this will result in a partition of the edges, where any A edge is surrounded on either side by two B edges, and vice versa.

Let $\alpha_e = \epsilon_e$ if e is in the cycle and in A, $-\epsilon_e$ if e is in the cycle and in B, and 0 otherwise. Basically, we're alternating between adding and subtracting ϵ from the weights on the edges in the cycle.

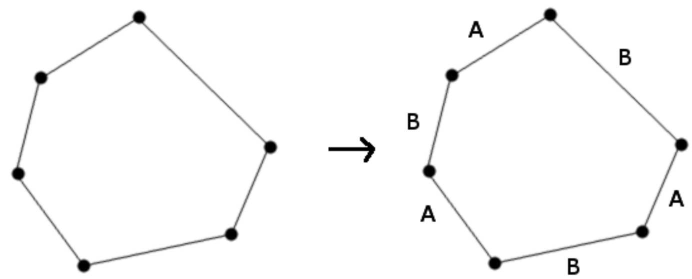

Figure 2: We partition the cycle into two sets.

Consider the two points of P, $z_1 = x + \alpha$ and $z_2 = x - \alpha$. Then $x = 0.5z_1 + 0.5z_2$, meaning x is a linear combination of two verticies in P, meaning x cannot be a vertex of P. Thus, if G' has a cycle, P cannot have any non-integral vertices.

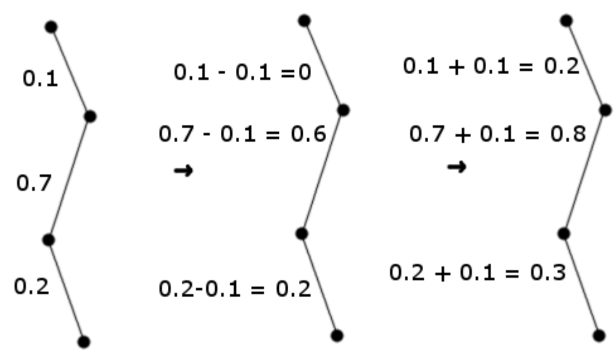

Figure 3: Adding and subtracting epsilon gives two feasible points in P. The linear combination of these points cannot be a vertex.

Case 2. *Assume that G' does not have a cycle.*

Choose a maximal path in G', a path with endpoints of degree 1. We can use a similar argument to the previous case to show that x is not a vertex of P. We divide the edges of the path into two different sets, where no edge is adjacent to an edge in its own set. We can construct two vectors by adding or subtracting some amount. A linear combination of these two vectors will be x, showing that x cannot be a vertex.

Therefore, the vertices of P must be integral. $\square$

4 Fractional Perfect Matching Theorem

Let the region $FPM(G)$ be defined by the two constraints:

1. $\sum\limits_{e \in \delta(v)} x_e = 1 \ \forall v \in V$
2. $x_e \geq 0$

The only difference between these are the previous constraints is that we use equal instead of less than or equal in the first constraint.

Theorem 1. *$x \in FPM(G)$ is a vertex of $FPM(G)$ $\iff$ $x_e \in \{0, 1/2, 1\}$ $\forall e \in E$ and the edges for which $x_e = 1/2$ form node-disjoint odd-cycles*

Proof. First, we will prove in the $\Leftarrow$ direction.

We will show that we are intersecting the region with a half plane, and that the size of the intersection will be just one point. This can only occur at a vertex.

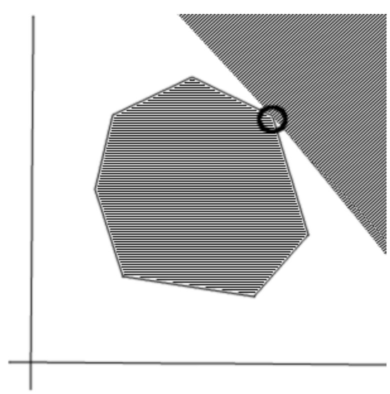

Figure 4: The half plane intersects the polytope at one point, meaning the intersection has to be a vertex.

Suppose we are given an x' that is half-integral. Define $w = -1$ if $x'_e = 0$, and $w = 0$ if $x'_e > 0$. Let $S = FPM(G) \cap \{y : w^T y = 0\}$

Claim. *x' is the only point in S.*

Assume by way of contradiction that $x'' \neq x$ is in S. Let E_1 be the edges for which $x'_e > 0$. Likewise, let $\bar{E}_1$ be the edges for which $x'_e = 0$.

If $x''_e \in S$, then $x''_e = 0 \ \forall e \in \bar{E}$, as $x''_e \in \{y : w^T y = 0\}$ For any e in a cycle, x'' must also have $x''_e = x'_e = 1/2$. Any other edges cannot be in cycles, and thus must have 1, same as x'. Thus, $x''_e = x'_e$.

Now, we will prove in the $\Rightarrow$ direction

Suppose x is a vertex of $FPM(G)$. We will consider a transform from G to G'. If u and v are nodes in G that share an edge, then in G', let us have nodes u', u'', v', and v''. In G', nodes u' and v'' share an edge, and nodes u'' and v' share an edge. Thus, if an edge e is in G, it will have two corresponding edges e' and e'' in G'. Note that $x_e = 0.5(x_{e'} + x_{e''})$.

Note also that G' is bipartite. A node with one prime only has neighbors that have double primes, and vice versa. Since G' is bipartite, that means its matching polytope has integral verticies, as we prove previously.

If it's an odd cycle, exactly one of $x_{e'}$ and $x_{e''}$ will be 1. Because $x_e = 0.5(x_{e'} + x_{e''})$, $x_e = 1/2$. It's easy to show that if x_e is 1/2, it must be in an odd cycle.

This completes the proof. $\square$