Matchings Class 3 Notes
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1 Proof that $P(G) = PM$

Let G be some graph with a perfect matching. We consider the perfect matching polytpe $P(G)$ in Class 2 well as the convex hull of all perfect matching characteristic vectors be $PM(G) = conv\ \{\ \chi_{M'} | M'$ is a perfect matching $\}$. To the perfect matching polytope $P(G)$, which was defined last class as encompassing the constraints $x_e \geq 0\ \forall e \in E(G)$, $\sum_{\delta(v)} x_e = 1 \forall\ v \in V(G)$, we add exponencially many **cut-conditions**: $\sum_{\delta(S)} x_e \geq 1\ \forall$ sets S which comprise a subset of the vertecies of G and have $|S| > 3$ odd. From now on, we designate this new set as $P(G)$. We show below that $P(G) = PM$ step by step.

$PM(G) \subset P(G)$ One direction is easy. If we have an $x \in PM(G)$ then any of the perfect matchings which are in the expanded expression for x are in the set P by definition and so since the constraints which define $P(G)$ are linear (specifically, convex) we have that x is infact in $P(G)$ also.

$P(G) \subset PM(G)$ Next, we show that the reverse is also true. To this end, assume not and let G be the smallest graph (in terms of number of edges) which contradictions the above assertion so that there is some $x \in P(G)$ and $x \notin PM(G)$. Sequencially, we derive some conditions that x must satisfy and arrive at a contradiction.

(1) We must have $0 < x_e < 1$ for else we could delete the edges which correspond to $x_e = 0$ or $x_e = 1$ and thus get a smaller conterexample, contradicting the minimality of G.

(2) Notice G cannot have any isolated vertecies for if it did then we would not have a perfect matching at all and we assume that we do. Also, notice that G cannot have any vertecies of degree 1 for if it did then we would have $x_e = 1$ on the edge into that vertex v and then get back to case (1) above.

(3) Notice that G must have a vertex of degree more than 2 for if not then we a contradiction. To see this, assume that all vertecies have degree 2 and then notice that we can decompose G into a bunch of cycles (a standard result). Consider some cycle. If it has odd length, then we have we violate the cut condition and hence we are not in $P(G)$. If it is even, then the edges will be of value $a, 1-a, a, 1-a, ...$ for some $0 < a < 1$ and hence the flow on that cycle is a convex combination of two perfect matchings. Hence, since both cases led to a problem, we must have that $\exists\ v$ such that $deg(v) > 2$.

(4) Since the sum of the degrees of a graph is $2|E|$ and all degrees are more than 1 we have $|E| > |V|$. Now look back at the LP. At a vertex of the LP, we will

have that the number of tight constraints is equal to the number of variables and thus the number of tight constraints is $> |V|$ so we must have that there is some set of verticies W with $|W|$ odd so that $\sum_{\delta(W)} x_e = 1$ (we have $|V|$ vertex constrains, and so since we must have $|E|$ active and $|E| > |V|$ we must have a cut constaint active; also, by (1) we have $x_e > 0$ so it really has to be a cut constraint).

(5) Now if $x \in P(G)$ then there is a vertex (basic feasible solution) in $P(G)$ that has the above property and x is just fractional. Notice that x can't be integral since then we'd get a smaller counterexample.

(6) Now look at the contraction of G; specifically contract W and $\overline{W}$ to a pseudonodes w' and w'' in the usual way and consider the two graphs thus produced. Parallel edges now give an additive effect. Label $G' = c(G/W)$ and $G'' = c(G/\overline{W})$ be the two contracted graphs.

(7) By the minimality of G we must have that P and PM are the same for both G' and G'' and hence if x' and x'' is the flow is G' and G'' respectively then we can express each of these as a convex combination of perfect matchings in their own set. Next, observe that x is rational since it is a basic feasible solution of $P(G)$ and, because of this, the same is true of $x' \in P(G')$ and $x'' \in P(G'')$. Due to rationality, we have that there is some integer k and perfect matchings $M_1', M_2', ..., M_k'$ of G' such that $x' = \frac{1}{k}\sum_{i=1}^k \chi_i^{M_i'}$ and similarly there are perfect matchings $M_1'', M_2'', ..., M_k''$ of G'' such that $x'' = \frac{1}{k}\sum_{i=1}^k \chi_i^{M_i''}$.

(8) Now let $e_1, ..., e_h$ be the edges which are in the cut $\delta(W)$. Since we have that $x'(\delta(w')) = 1$ and w' is in every perfect matching, we have that e_j is in exactly $kx'(e_j) = kx(e_j)$ matchings of $M_1', ..., M_k'$ and similarly e_j is in exactly $kx(e_j)$ matchings $M_1'', ..., M_k''$. Note now that we have $\sum_{j=1}^n kx(e_j) = k$ and moreover, exactly one of $e_1, ..., e_h$ can be in M_i' and M_i''. We can thus assume by renumbering if needed that M_i' and M_i'' share exactly one edge from $e_1, ..., e_h$. Then, we have that $M_i = M_i' \cup M_i''$ is a perfect matching of G. Hence, we have that $x = \frac{1}{k}\sum_{i=1}^k \chi^{M_i}$ implies that x is in PM and we have the contradiction, as needed.