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Approximate fixed-rank closures of covering problems[*†]

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Abstract

Consider a 0/1 integer program $\min\{c^T x : Ax \geq b,\, x \in \{0,1\}^n\}$ where A is nonnegative. We show that if the number of minimal covers of $Ax \geq b$ is polynomially bounded, then there is a polynomially large lift-and-project relaxation whose value is arbitrarily close to being at least as good as that given by the rank $\leq q$ cuts, for any fixed q. A special case of this result is that given by set-covering problems, or, generally, problems where the coefficients in A and b are bounded.

1 Introduction

Consider a 0/1 integer programming problem with nonnegative constraints,

$$\min\{c^T x : Ax \geq b,\, x \in \{0,1\}^n\}. \tag{1}$$

Here A is an $m \times n$ nonnegative matrix, and $b \in R^m_+$. Let

$$\mathcal{Z}_{A,b} \doteq \{\, x \in \{0,1\}^n \;:\; Ax \geq b \,\}$$

denote the feasible region of (1). For an integer $q \geq 0$, denote by $\mathrm{K}^q_{A,b}$ the polytope defined by the Chvátal-Gomory inequalities of rank $\leq q$, and write

$$\tau^{(q)}_{A,b}(c) \doteq \min\{c^T x : x \in \mathrm{K}^q_{A,b}\}.$$

For $P \subseteq R^N$ $(N \geq n)$ we denote by $\pi(P)$ the projection of P to R^n. A special case of our results is the following:

Theorem 1.1 *For each integer $q \geq 0$ and $0 < \epsilon < 1$ there are integers $d = d(q, \epsilon)$ and $u = u(q, \epsilon)$ with the following property. Consider the feasible region $\mathcal{Z}_{A,e}$ for a set-covering problem, where A is an $m \times n$, 0/1 matrix and e is the vector of m 1s. There is a polyhedron $\hat{\mathcal{R}}$ such that:*

(i) *$\hat{\mathcal{R}}$ is defined by a linear system with at most $O\left((n+m)^d\right)$ variables and inequalities, which is computable in time $O\left((n+m)^d\right)$,*

(ii) *the coefficients in this linear system are integral and have absolute value at most u,*

(iii) *$\mathcal{Z}_{A,e} \subseteq \pi(\hat{\mathcal{R}})$, and*

(iv) *For all $c \in R^n_+$, $\min\{c^T x : x \in \pi(\hat{\mathcal{R}})\} \geq (1-\epsilon)\tau^{(q)}_{A,e}(c)$.*

■

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[†]formerly: Set covering problems and Chvátal-Gomory cuts

In contrast to this result, we show that there exist examples of set-covering problems where the Sherali-Adams [SA90] and Lovász-Schrijver [LS91] "lift-and-project" methods provably require exponential time to achieve accuracy as in Theorem 1.1 (iv), even for $q = 1$ and for any fixed $\epsilon < 1/2$ (see Theorem 5.7).

Chvátal-Gomory cuts (see [NW88]) have long received attention; recently there has been renewed interest as a result of computational success. In particular, the separation problem for Chvátal-Gomory closures of arbitrary integer programs was shown to be NP-hard in [E99], also see [CL01]. Caprara and Letchford [CL03] have shown that it is NP-hard to separate over those rank-1 cuts (for set-covering) obtained by using multipliers $0, 1/2$. Recently, Letchford [Le04] has extended this result to set-packing and set-covering problems. The issue of exact optimization over the rank-1 Chvátal-Gomory closure of a set-covering problem remains open and appears quite interesting, though in view of the result in [Le04] the answer may be negative.

Theorem 1.1 is a special case of a more general result, which uses the standard concept of *covers* [NW88].

Definition 1.2 *Given an inequality $\alpha^T x \geq \beta$ with $\alpha \geq 0$, a* cover *is a set $C \subseteq suppt(\alpha)$ such that $\sum_{j \notin C} \alpha_j < \beta$. The resulting* cover inequality *is $\sum_{j \in C} x_j \geq 1$. The cover is called* minimal *if it is inclusion-minimal.*

Let $\mathcal{R}_{A,b} \doteq \{\, x \in [0,1]^n \ : \ Ax \geq b \,\}$ denote the continuous relaxation of (1).

Theorem 1.3 *For each integer $q > 0$ and $0 < \epsilon < 1$ there are integers $d = d(q, \epsilon)$ and $u = u(q, \epsilon)$ with the following property. Let A be nonnegative and $m \times n$ and $b \in R^m_+$. Assume we are given the set of minimal covers arising from each row of $Ax \geq b$, and let $\nu_{A,b}$ denote the number of such covers. Then, in time $O((n + m + \nu_{A,b})^d)$ we can compute a formulation described by at most $O\left((n + m + \nu_{A,b})^d\right)$ rows and columns, with coefficients that are integral and with absolute value at most u, and whose feasible region $\hat{\mathcal{R}}$ satisfies*

(i) $\mathcal{Z}_{A,b} \subseteq \pi(\hat{\mathcal{R}})$,

(ii) for any $c \in R^n_+$, $\min\{c^T x \,:\, x \in \pi(\hat{\mathcal{R}}) \cap \mathcal{R}_{A,b}\} \ \geq \ (1 - \epsilon)\tau^{(q)}_{A,b}(c)$.

■

Note that under the conditions of the Theorem, $\mathcal{Z}_{A,b}$ equals the set of 0/1 solutions to $Mx \geq e$, where M is the matrix whose rows are the minimal covers of $Ax \geq b$. However, even for $q = 1$, it does not follow that $\mathrm{K}^q_{A,b} = \mathrm{K}^q_{M,e}$. Thus, even though Theorem 1.1 directly follows from Theorem 1.3, the converse does not necessarily hold.

There are several cases where Theorem 1.3 implies polynomial-time approximation of the polyhedron defined by bounded rank Chvátal-Gomory cuts.

(i) Suppose that all the coefficients in A and b take values $0, 1, \cdots, p$ for some integer p. Then $\nu_{A,b} = O(n^p)$ and there is a polynomial-size relaxation with value at least $(1 - \epsilon)\tau^{(q)}_{A,b}(c)$, for fixed q, ϵ and p.

(ii) More generally, the same result as in (i) holds if every constraint in $Ax \geq b$ has *pitch* at most p. The concept of *pitch* was introduced in [BZ02a] (or [BZ02b] for a more comprehensive version):

Definition 1.4 *The* **pitch** *of an inequality $\alpha^T x \geq \beta$ with nonnegative coefficients is the smallest integer k, such that the sum of the k smallest nonzero α_j is at least β.*

In particular, an inequality $\alpha^T x \geq \alpha_0$ with nonnegative (integral) coefficients $\alpha_j \in \{0, 1, \cdots, p\}$ $(0 \leq j \leq n)$ has pitch $\leq p$.

Also see Section 5 for a more complex case where a similar approximability result holds.

1.1 Outline

We will first provide a brief outline of our approach, in particular in terms of Theorem 1.1. A critical ingredient is provided by a result proved in [BZ02a] which we describe next.

Definition 1.5 *Consider a problem of type (1). For integral $k \geq 2$, let $P^k_{A,b}$ denote the set of n-vectors $\hat{x}$, such that:*

1. $0 \leq \hat{x}_j \leq 1$ *for* $1 \leq j \leq n$,
2. $\hat{x}$ *satisfies all inequalities* $a^T x \geq a_0$ *of pitch* $\leq k$ *that are valid for* $\mathcal{Z}_{A,b}$.

In the case of set-covering, the polyhedra $P^k_{A,b}$ were considered in [BZ02a], where the following result was proved.

Theorem 1.6 *[BZ02a] For each integer $k \geq 2$ there is an integer g_k with the following property. Given the feasible region $\mathcal{Z}_{A,e}$ for a set-covering problem with m rows and n columns, there is a polyhedron $\mathcal{L}_k$ such that:*

(i) $\mathcal{L}_k$ *is described by a system of* $O((n+m)^{g_k})$ *inequalities in* $O((n+m)^{g_k})$ *variables, that is computable in time* $O((n+m)^{g_k})$ *, and whose coefficients are integral and with absolute value at most* k,

(ii) $\mathcal{Z}_{A,e} \subseteq \pi(\mathcal{L}_k) \subseteq P^k_{A,e}$.

■

In other words, $\mathcal{L}_k$ is a valid "lifting" of the continuous relaxation of $\mathcal{Z}$, with polynomially many variables and constraints.

In order to see how Theorem 1.6 is relevant towards the proof of Theorem 1.1, consider the special case $q = 1$. Suppose we choose k large, and let $\hat{x}$ be the solution to $\min\{c^T x \,:\, x \in \pi(\mathcal{L}_k)\}$.

Theorem 1.6 implies that $\hat{x}$ "nearly" satisfies all rank-1 inequalities – this will be shown in a more general context in Section 2. In fact, for any rank-1 inequality $\alpha^T x \geq \alpha_0$, we will show that $a^T \hat{x} \geq \frac{k}{k+1}\alpha_0$, and this is at least $(1-\epsilon)\alpha_0$ for $k+1 > \epsilon^{-1}$.

This fact *almost* suffices to obtain Theorem 1.1 in the rank-1 case. In particular, if we were to define

$$\hat{y} = (1-\epsilon)^{-1}\hat{x},$$

then we would have that $\hat{y}$ satisfies every *nonnegative* rank-1 inequality, while at the same time $c^T\hat{y} = (1-\epsilon)^{-1}c^T\hat{x}$. However, the problem with this approach is that $\hat{y}$, as defined, may not be feasible: it may be the case that $\hat{y}_j > 1$ for some coordinate(s) j. In other words, simply scaling up $\hat{x}$ does not work. Instead we will show that using $\bar{y}_j = \min\{1, (1-\epsilon)^{-1}\hat{x}_j\}$ (for all j) **does** work, that is to say, $\bar{y}$ satisfies all rank-1 inequalities. Proving this fact, in turn, requires a structural result pertaining rank-1 cuts.

1.2 Chvátal-Gomory cuts

The Chvátal-Gomory cutting-plane procedure for integer programs with nonnegative variables (see [NW88]) can be defined as follows. Consider a set $S = \left\{\bar{A}x \geq \bar{b},\, x \in Z^n_+\right\}$, where $\bar{A}$ has m rows. Let $\pi \in R^m_+$. Then the inequality $\sum_j \lceil (\sum_i \pi_i \bar{a}_{ij}) \rceil x_j \geq \lceil \pi^T b \rceil$ is valid for S and is called a Chvátal-Gomory inequality. Note that in the 0/1 case, the rows of $\bar{A}x \geq \bar{b}$ include $-x_j \geq -1$ for all j.

The *Chvátal-Gomory rank* (for short, simply rank) of an inequality is defined as follows. First, any nonnegative linear combination of the rows of $\bar{A}x \geq \bar{b}$ is said to have rank-0, and any inequality that is dominated by a rank-0 inequality is also said to have rank-0. Proceeding inductively, suppose that for integer $q \geq 0$ we have defined the rank $\leq q$ inequalities. Then, any inequality $\alpha^T x \geq \alpha_0$, which does not have rank $\leq q$, and is dominated by an inequality obtained by applying the Chvátal-Gomory procedure to the set of all inequalites of rank $\leq q$, is said to have rank-$(q+1)$.

Definition 1.7 *Given two inequalities $\alpha^T x \geq \alpha_0$ and $\beta^T x \geq \beta_0$ that are valid for an integer program with nonnegative variables, we say that the first dominates the second if $\alpha_j \leq \beta_j$ for $1 \leq j \leq n$ and $\alpha_0 \geq \beta_0$.*

Consider a system $Ax \geq b$ of inequalities where A and b are nonnegative. We are interested in the nondominated valid inequalities for the polyhedron defined by the rank $\leq q$ inequalities for any given integer $q \geq 0$. In what follows, we will refer to inequalities of the form

$$x_j \geq 0 \quad \text{or} \; - x_j \geq -1$$

as, respectively, type (A) and type (B) inequalities.

Definition 1.8 *Let $Ax \geq b$ be a system of inequalities with A and b nonnegative. Let $q \geq 0$ be an integer. An inequality $\alpha^T x \geq \alpha_0$ of is said to be of type $(C^q_{A,b})$ if $\alpha_j \geq 0$ for $0 \leq j \leq n$, and*

(i) When $q = 0$, then $\alpha^T x \geq \alpha_0$ is one of the rows of $Ax \geq b$,

(ii) When $q > 0$, then for some integer $p > 0$, there is a family of p inequalities

$$\sum_{j=1}^{n} s_{i,j} x_j \;\; \geq s_{i,0}, \quad \textit{of type } (C^{q-1}_{A,b}) \;\; (1 \leq i \leq p), \tag{2}$$

nonnegative multipliers π_i $(1 \leq i \leq p)$, and multipliers $0 \leq \gamma_j < 1$ $(1 \leq j \leq n)$, such that

$$\alpha_j \;\; = \;\; \left\lceil \sum_{i=1}^{p} \pi_i s_{i,j} \; - \; \gamma_j \right\rceil \;\; \textit{for} \;\; 1 \leq j \leq n, \;\; \textit{and} \tag{3}$$

$$\alpha_0 \;\; = \;\; \left\lceil \sum_{i=1}^{p} \pi_i s_{i,0} \; - \; \sum_{j=1}^{n} \gamma_j \right\rceil. \tag{4}$$

Informally, the type $(C^q_{A,b})$ have nonnegative coefficients, and are obtained using type $(C^{q-1}_{A,b})$ inequalities, and also inequalities $-x_j \geq -1$ with multipliers strictly smaller than 1. Note: an inequality of type $(C^q_{A,b})$ is also of type $(C^{q+1}_{A,b})$. For completeness and future reference, we state the following result:

Lemma 1.9 *Consider a system of inequalities $Ax \geq b$ with A and b nonnegative, and let $q \geq 0$.*

(a) Any type $(C^q_{A,b})$ inequality has rank $\leq q$.

(b) Any inequality valid for $\mathrm{K}^q_{A,b}$ is dominated by a nonnegative linear combination of inequalities of type (A), (B), and $(C^q_{A,b})$. ■

The proof of this Lemma is routine, with part (b) by induction on q.

1.3 Terminology and simple facts

For convenience, we list here various objects and simple results used throughout the paper. All definitions concern a system $Ax \geq b$ with A and b nonnegative.

- $\mathcal{Z}_{A,b} \; = \; \{ \, x \in \{0,1\}^n \; : \; Ax \geq b \, \}$ is the set of feasible 0/1 points.
- $\mathcal{R}_{A,b} \; = \; \{ \, x \in [0,1]^n \; : \; Ax \geq b \, \}$ is the continuous relaxation of $\mathcal{Z}_{A,b}$.
- $\mathrm{K}^q_{A,b}$ is the polytope defined by the Chvátal-Gomory inequalities of rank $\leq q$, where $q \geq 0$ is an integer.
- $\tau^{(q)}_{A,b}(c) \; = \; \min\{c^T x \; : \; x \in \mathrm{K}^q_{A,b}\}$; here $c \in R^n$.
- $P^k_{A,b}$ is the relaxation defined by the set of inequalities of pitch $\leq k$ that are valid for $\mathcal{Z}_{A,b}$, together with $0 \leq \hat{x}_j \leq 1 \; \forall j$. Here $k \geq 1$ is an integer.

Remark 1.10 *For any reals f and g, $\lceil f \rceil + \lceil g \rceil \geq \lceil f + g \rceil$.*

Remark 1.11 *Consider a system $Ax \geq b$ where A and b are nonnegative. Then a 0/1 vector y satisfies $Ay \geq b$ if and only if y satisfies every cover inequality arising from each row of $Ax \geq b$.*

2 Main result

In this section we present a proof of Theorem 1.3. This proof will be contingent upon a structural result stated below (Lemma 2.1) whose proof we will give later. First we outline our approach.

Suppose we have a system $Ax \geq b$ with A and b nonnegative. Let k be a large integer, and suppose that $\hat{x} \in P^k_{A,b} \cap \mathcal{R}_{A,b}$. We would like to argue that $\hat{x}$ 'nearly' satisfies all rank ≤ 1 inequalities valid for $\mathcal{Z}_{A,b}$. Since $\hat{x} \in \mathcal{R}_{A,b}$ it satisfies all rank-0 inequalities, so consider any rank-1, type $(C^1_{A,b})$ inequality

$$\sum \alpha_j x_j \quad \geq \quad \alpha_0. \tag{5}$$

Suppose first that $\alpha_0 \leq k$. Then, since $\sum_j \min\{\alpha_0, \alpha_j\} x_j \geq \alpha_0$ is valid for $\mathcal{Z}_{A,b}$, and of pitch $\leq k$, it follows that $\hat{x}$ satisfies this latter inequality and therefore $\hat{x}$ satisfies (5).

On the other hand, suppose now that $\alpha_0 > k$. By definition of the rank-1, type $(C^1_{A,b})$ inequalities, it follows that for some real $\bar{\alpha}_0 > k$ with $\alpha_0 = \lceil \bar{\alpha}_0 \rceil$, $\sum \alpha_j x_j \geq \bar{\alpha}_0$ is a rank-0 inequality, and as such is satisfied by $\hat{x}$. Hence, in any case, $\sum \alpha_j \hat{x}_j \geq \frac{k}{k+1}\alpha_0$ – this is what we mean by $\hat{x}$ nearly satisfying (5).

There are two issues that arise from this analysis. First, we must extend the analysis to inequalities of rank higher than 1. The second issue was indirectly mentioned before – suppose we have a vector $y \in [0,1]^n$ such that y "nearly" satisfies some set of inequalities $\bar{A}x \geq \bar{b}$. Despite this, it may still be the case that, for some $c \in R^n_+$, $c^T y$ is much smaller than $\min\{c^T x \,:\, \bar{A}x \geq \bar{b},\ x \in [0,1]^n\}$.

In order to overcome these hurdles, our analysis makes use of two ingredients. The first one is that we will not merely rely on "nearly" satisfying the rank $\leq q$ inequalities. Instead, our vector $\hat{x}$ satisfies all pitch $\leq k$ inequalities exactly, regardless of their rank, and we are using a large k. The second ingredient is that, as we will show, for each fixed q the type $(C^q_{A,b})$ inequalities have a very special structure. The combination of the two ingredients suffices to bridge the gap between 'nearly' and *exactly* satisfying the rank $\leq q$ inequalities.

In order to describe our key structural result we need a simple construction. In what follows, given an n-vector y and nonnegative integers k and q, define the vector $y^{(k,q)}$ by

$$y^{(k,q)}_j \quad = \quad \min\left\{1\ ,\ \left(\frac{k+1}{k}\right)^q y_j\right\} \quad \text{for} \quad 1 \leq j \leq n. \tag{6}$$

Lemma 2.1 *Let $Ax \geq b$ be a system of inequalities with nonnegative coefficients. Let $k > 0$ be an integer, and suppose $\hat{x} \in P^k_{A,b} \bigcap \mathcal{R}_{A,b}$. Then for any integer $q \geq 0$, $\hat{x}^{(k,q)} \in \mathrm{K}^q_{A,b}$.*

A proof of Lemma 2.1 is given in Section 4. The key observation here is that for each fixed q, we will have that $\hat{x}^{(k,q)} \approx \hat{x}$ if we choose k large enough. We can now prove Theorem 1.3.

Proof of Theorem 1.3. Let $q > 0$ integer and $0 < \epsilon < 1$ be as in the statement of Theorem 1.3. Choose $k > 0$ integral large enough that

$$\left(\frac{k+1}{k}\right)^q \;\leq\; (1-\epsilon)^{-1}.$$

Let $Mx \geq e$ be the system of all minimal cover inequalities arising from all rows of $Ax \geq b$. By definition, M has $\nu_{A,b}$ rows. By Theorem 1.6, for some integer $g_k > 0$ there is a polyhedron $\mathcal{L}_k$ defined by a linear system on $O((n+m+\nu_{A,b})^{g_k})$ constraints and variables, and with coefficients with absolute value at most k, such that $\mathcal{Z}_{M,e} \subseteq \pi(\mathcal{L}_k) \subseteq P^k_{M,e}$. By Remark 1.11 $\mathcal{Z}_{A,b} = \mathcal{Z}_{M,e}$. Thus, writing

$$\sigma \;=\; \min\left\{c^T x \,:\, x \in \pi(\mathcal{L}_K) \bigcap \mathcal{R}_{A,b}\right\},$$

in order to complete the proof we just need to show that $\tau^{(q)}_{A,b}(c) \;\leq\; (1-\epsilon)^{-1}\sigma$.

Choose $\hat{x} \in \pi(\mathcal{L}_k) \bigcap \mathcal{R}_{A,b}$ so that $c^T \hat{x} \;=\; \sigma$.

$\mathcal{Z}_{A,b} = \mathcal{Z}_{M,e}$ implies $P^k_{M,e} = P^k_{A,b}$. Hence $\hat{x} \in P^k_{A,b} \bigcap \mathcal{R}_{A,b}$, and by Lemma 2.1, $\hat{x}^{(k,q)} \in \mathrm{K}^q_{A,b}$, and therefore

$$\tau^{(q)}_{A,b}(c) \;\leq\; c^T \hat{x}^{(k,q)} \;\leq (1-\epsilon)^{-1}\sigma,$$

where the second inequality follows from our choice of k. The proof is complete. ■

Thus, Lemma 2.1 is the key to our main result. Our proof of Lemma 2.1 will be by induction on q. This requires an appropriate inductive hypothesis, which brings about our second ingredient. In order to motivate our approach in this case, we will sketch a proof of Lemma 2.1 in the set-covering case, when $q = 1$.

In the set-covering case, A is a 0/1 matrix and $b = e$, the m-vector of 1s. To handle the rank-1 case, assume that $\hat{x} \in P^k_{A,e} \bigcap \mathcal{R}_{A,e}$. Let $\hat{y} = \hat{x}^{(k,1)}$.

Let $\alpha^T x \geq \alpha_0$ be any rank-1, type $(C^1_{A,b})$ inequality. For simplicity, we will assume that $\alpha^T x \geq \alpha_0$ is obtained by rounding a nonnegative linear combination of the original set-covering constraints (i.e., none of the constraints $-x_j \geq -1$ are used).

We wish to argue that $\alpha^T \hat{y} \geq \alpha_0$. Certainly (since $\alpha^T \hat{\geq} \frac{k}{k+1}\alpha_0$) this is going to be the case if $\frac{k+1}{k}\hat{x}_j \leq 1$ for all j with $\alpha_j > 0$. Thus, let $U(\alpha)$ be the set of indices j with $\alpha_j > 0$ and $\frac{k+1}{k}\hat{x}_j > 1$. We will show that $\alpha^T \hat{y} \geq \alpha_0$ by induction on $|U(\alpha)|$; the case $|U(\alpha)| = 0$ is clear as we just outlined.

So assume, without loss of generality, that $1 \in U(\alpha)$.

Since $\alpha^T x \geq \alpha_0$ has rank-1, for some integer $p > 0$ there are p set-covering inequalities of the original formulation

$$\sum_{j \in S_i} x_j \quad \geq \quad 1, \quad 1 \leq i \leq p, \tag{7}$$

(e.g. S_i is the support of the i^{th} inequality) and multipliers $\pi_i > 0$ $(1 \leq i \leq p)$ such that:

$$\alpha_j \quad = \quad \left\lceil \sum_{i \,:\, j \in S_i} \pi_i \right\rceil \;\; (1 \leq j \leq n) \;\text{ and }\; \alpha_0 \quad = \quad \left\lceil \sum_{i=1}^{p} \pi_i \right\rceil. \tag{8}$$

Since we are assuming $\alpha_1 > 0$, we have that $1 \in S_i$ for at least one index $1 \leq i \leq p$. Without loss of generality, we may assume that there is an index h such that $1 \in S_i$ for $1 \leq i \leq h$, and $1 \notin S_i$ for $i > h$.

Hence we can rewrite:

$$\alpha_1 \quad = \quad \left\lceil \sum_{i=1}^{h} \pi_i \right\rceil. \tag{9}$$

Note that if $h = p$ then $\alpha_1 = \alpha_0$, and so $\alpha^T \hat{y} \geq \alpha_1 \hat{y}_1 = \alpha_0$, and we are done.

So assume $h < p$, and consider the rank ≤ 1 inequality $\breve{\alpha}^T x \geq \breve{\alpha}_0$ which is obtained by rounding the linear combination yielded by using multiplier π_i on the i^{th} constraint (7) for $h < i \leq p$. Thus

$$\breve{\alpha}_0 \quad = \quad \left\lceil \sum_{i=h+1}^{p} \pi_i \right\rceil, \tag{10}$$

$$\breve{\alpha}_j \quad = \quad \left\lceil \sum_{h < i \leq p \,:\, j \in S_i} \pi_i \right\rceil, \quad \text{for } 1 \leq j \leq n, \tag{11}$$

Note that $\breve{\alpha}_1 \; = 0$ definition of h. So $|U(\breve{\alpha})| < |U(\alpha)|$, and by induction,

$$\breve{\alpha}^T \hat{y} \geq \breve{\alpha}_0.$$

But clearly, $\alpha \geq \breve{\alpha}$. Hence,

$$\alpha^T \hat{y} \quad = \quad \alpha_1 + \sum_{j=2}^{n} \alpha_j \hat{y}_j \tag{12}$$

$$\geq \quad \alpha_1 + \sum_{j=2}^{n} \breve{\alpha}_j \hat{y}_j \tag{13}$$

$$\geq \quad \alpha_1 + \breve{\alpha}_0 = \left\lceil \sum_{i=1}^{h} \pi_i \right\rceil + \left\lceil \sum_{i=h+1}^{p} \pi_i \right\rceil \tag{14}$$

$$\geq \quad \alpha_0, \tag{15}$$

where the last inequality follows from Remark 1.10. This completes our sketch of the proof of Lemma 2.1 in this special case. ■

This proof sketch suggests that rank-1 inequalities for set-covering can be "decomposed" in a way that the right-hand sides behave in an additive manner. It turns out that this view extends to higher rank inequalities, as well. This fact, properly applied, is our second ingredient, and will be used in our inductive proof of Lemma 2.1.

3 The additive property

Definition 3.1 *A system Γ of inequalities with nonnegative coefficients is called* **additive** *if for each inequality $\alpha^T x \geq \alpha_0$ in Γ and each integer h, $1 \leq h \leq n$, there is an inequality $\beta^T x \geq \beta_0$, such that:*

(h.i) $\beta^T x \geq \beta_0$ *is in* Γ,

(h.ii) $\beta_h = 0$,

(h.iii) $\beta_j \leq \alpha_j$, $1 \leq j \leq n$, *and*

(h.iv) $\alpha_h + \beta_0 \geq \alpha_0$.

Theorem 3.2 *Consider an additive system $Ax \geq b$. Then the type $(C^1_{A,b})$ inequalities form an additive system.*

Proof. Let $\alpha^T x \geq \alpha_0$ be a type $(C^1_{A,b})$ inequality. Thus, for some integer $p > 0$ there are p inequalities chosen from $Ax \geq b$,

$$\sum_{j=1}^{n} s_{i,j} x_j \quad \geq \quad s_{i,0}, \quad 1 \leq i \leq p, \tag{16}$$

multipliers $0 \leq \pi_i$ $(1 \leq i \leq p)$ and $0 \leq \gamma_j < 1$ $(1 \leq j \leq n)$, such that:

$$\alpha_j \quad = \quad \left\lceil \sum_{i=1}^{p} \pi_i s_{i,j} \; - \; \gamma_j \right\rceil, \quad 1 \leq j \leq n, \;\; \text{and} \tag{17}$$

$$\alpha_0 \quad = \quad \left\lceil \sum_{i=1}^{p} \pi_i s_{i,0} \; - \; \sum_{j=1}^{n} \gamma_j \right\rceil. \tag{18}$$

We want to show that for any $1 \leq h \leq n$ there is an inequality $\beta^T x \geq \beta_0$ satisfying (h.i)-(h.iv). For simplicity of notation we set $h = 1$.

Since the system $Ax \geq b$ is made up of additive inequalities, for each $1 \leq i \leq p$ there is an inequality

$$\sum_{j=1}^{n} t_{i,j} x_j \quad \geq \quad t_{i,0}, \tag{19}$$

in $Ax \geq b$ such that

$$s_{i,j} \quad \geq \quad t_{i,j} \quad \forall j > 1, \tag{20}$$

$$t_{i,1} \quad = \quad 0, \quad \text{and} \tag{21}$$

$$s_{i,1} + t_{i,0} \quad \geq \quad s_{i,0}. \tag{22}$$

Consider the type $(C^1_{A,b})$ inequality

$$\sum_{j=1}^{n} \beta_j x_j \quad \geq \quad \beta_0 \tag{23}$$

where

$$\beta_j = \left\lceil \sum_{i=1}^{p} \pi_i t_{i,j} - \gamma_j \right\rceil \quad \text{for } 1 \le j \le n, \tag{24}$$

$$\beta_0 = \left\lceil \sum_{i=1}^{p} \pi_i t_{i,0} - \sum_{j>1} \gamma_j \right\rceil \tag{25}$$

We claim that (23) satisfies (1.i) - (1.iv). By construction (1.i) follows. Also, $\beta_1 = 0$ is implied by (21) and $\gamma_1 \ge 0$, and thus (1.ii) holds. By (17) and (20), $\alpha_j \ge \beta_j$ for $j > 1$, hence (1.iii) holds. Finally

$$\alpha_1 + \beta_0 = \left\lceil \sum_{i=1}^{p} \pi_i s_{i,1} - \gamma_1 \right\rceil + \left\lceil \sum_{i=1}^{p} \pi_i t_{i,0} - \sum_{j>1} \gamma_j \right\rceil \tag{26}$$

$$\ge \left\lceil \sum_{i=1}^{p} \pi_i (s_{i,1} + t_{i,0}) - \sum_{j=1}^{n} \gamma_j \right\rceil, \tag{27}$$

where the inequality follows from Remark 1.10. Continuing with (27), and using (22), we obtain

$$\alpha_1 + \beta_0 \ge \left\lceil \sum_{i=1}^{p} \pi_i s_{i,0} - \sum_{j=1}^{n} \gamma_j \right\rceil = \alpha_0, \tag{28}$$

by (18). So (1.iv) is satisfied. ■

Corollary 3.3 *Consider an additive system $Ax \ge b$. Then for any integer $q \ge 0$ the type $(C^q_{A,b})$ inequalities form an additive system.*

Lemma 3.4 *Let $Ax \ge b$ be a system of inequalities with nonnegative coefficients. Let $\breve{A}x \ge \breve{b}$ be the system containing each inequality*

$$\sum_{j \notin J} a_{ij} x_j \ge \max \left\{ 0 \, , \, b_i - \sum_{j \in J} a_{ij} \right\}$$

for every choice of row $\sum_j a_{ij} x_j \ge b_i$ of $Ax \ge b$ and subset J of $\{1, 2, \cdots, n\}$. Then

(i) $\breve{A}x \ge \breve{b}$ is additive.

(ii) For any integer $q \ge 0$, any type $(C^q_{A,b})$ inequality is a type $(C^q_{\breve{A},\breve{b}})$ inequality.

(iii) For any integer $q \ge 0$, $\mathrm{K}^r_{\breve{A},\breve{b}} = \mathrm{K}^q_{A,b}$.

Proof. (i) follows by definition of additiveness. Also note that $Ax \ge b$ is a subsystem of $\breve{A}x \ge \breve{b}$. Thus (ii) holds. Finally, each inequality of $\breve{A}x \ge \breve{b}$ has rank 0 with respect to the system $Ax \ge b$, $0 \le x_j \le 1$ $\forall j$, and hence (iii) holds as well. ■

In essence, Lemma 3.4 shows that given a system $Ax \ge b$ there is an "equivalent" additive system.

4 Proof of Lemma 1.3

Our proof of Lemma 1.3 will proceed in three steps.

Lemma 4.1 *Consider a system of inequalities $Ax \ge b$ with both A and b nonnegative. Let $0 < q$ be an integer, and let $\alpha^T x \ge \alpha_0$ be a type $(C^q_{A,b})$ inequality of rank q. Suppose z is a vector such that $z \in P^k_{A,b}$, for some integer $k > 0$, and such that in addition $z \in \mathrm{K}^{q-1}_{A,b}$. Then*

$$\alpha^T z \ge \left(\frac{k}{k+1} \right) \alpha_0.$$

Proof. The inequality $\sum_j \min\{\alpha_0, \alpha_j\} x_j \geq \alpha_0$ is valid for $\mathcal{Z}_{A,b}$; if it has pitch $\leq k$ the Lemma is clear by hypothesis. Hence we assume that $\alpha_0 \geq k+1$.

Now since $\alpha^T x \geq \alpha_0$ is of type $(C^q_{A,b})$, by definition we have that for some integer $p > 0$, there exist multipliers $0 \leq \pi_i$ $(1 \leq i \leq p)$ and $0 \leq \gamma_j < 1$ $(1 \leq j \leq n)$, such that:

$$\alpha_j = \left\lceil \sum_{i=1}^{p} \pi_i s_{i,j} - \gamma_j \right\rceil \quad \text{for } 1 \leq j \leq n, \text{ and} \tag{29}$$

$$\alpha_0 = \left\lceil \sum_{i=1}^{p} \pi_i s_{i,0} - \sum_{j=1}^{n} \gamma_j \right\rceil, \tag{30}$$

where for $1 \leq i \leq p$, the inequality

$$\sum_{j=1}^{n} s_{i,j} x_j \geq s_{i,0} \tag{31}$$

is of type $(C^{q-1}_{A,b})$, and thus of rank $< q$. Since $z \geq 0$,

$$\sum_{j=1}^{n} \alpha_j z_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{p} \pi_i s_{i,j} - \gamma_j \right) z_j \tag{32}$$

and since z satisfies all inequalities (31) (because they have rank $< q$), the right-hand side of (32) is at least

$$\sum_{i=1}^{p} \pi_i s_{i,0} - \sum_{j=1}^{n} \gamma_j z_j \geq \sum_{i=1}^{p} \pi_i s_{i,0} - \sum_{j=1}^{n} \gamma_j, \tag{33}$$

where (33) follows since $z_j \leq 1 \ \forall j$. By (30), the difference between the right-hand side in (33) and α_0 is smaller than 1; since we are assuming that $\alpha_0 \geq k+1$ the proof is now complete. ■

Lemma 4.2 *Consider an additive system $Ax \geq b$. Let $\hat{x} \in [0,1]^n$ be such that $A\hat{x} \geq b$. Suppose $0 \leq \epsilon < 1$ is such that*

$$\alpha^T \hat{x} \geq (1-\epsilon)\alpha_0$$

for each type $(C^1_{A,b})$ inequality $\alpha^T x \geq \alpha_0$. Define

$$\hat{y}_j = \min\left\{1, \frac{\hat{x}_j}{1-\epsilon}\right\}, \quad 1 \leq j \leq n.$$

Then $\hat{y} \in \mathrm{K}^1_{A,b}$.

Proof. All we need to show is that any type $(C^1_{A,b})$ inequality $\alpha^T x \geq \alpha_0$ satisfies $\alpha^T \hat{y} \geq \alpha_0$. Let

$$U(\alpha) = \{1 \leq j \leq n : \hat{y}_j = 1 \text{ and } \alpha_j > 0\}.$$

The proof will be by induction on $|U(\alpha)|$. Suppose first that $|U(\alpha)| = 0$. Then the result clearly follows. Assume now that for some $1 \leq h \leq n$ we have $h \in U(\alpha)$. Let

$$\beta^T x \geq \beta_0 \tag{34}$$

satisfy (h.i) - (h.iv). Then since $\hat{y}_h = 1$, we have that (h.iii), (h.iv), and the inductive hypothesis (since $|U(\beta)| < |U(\alpha)|$ by (h.ii)) imply:

$$\alpha^T \hat{y} \geq \alpha_h + \beta^T \hat{y} \geq \alpha_h + \beta_0 \geq \alpha_0 \tag{35}$$

as desired. ■

Remark 4.3 *Note that in the proof above we do not really need $z \in P^k_{A,b}$ – we only need that z satisfy all valid inequalities with coefficients in $\{0, 1, \cdots, k\}$.*

Proof of Lemma 2.1. By induction on q, with the case $q = 0$ clear. Thus, assume $q > 0$, and, inductively, that $\hat{x}^{(k,q-1)}$ satisfies all inequalities valid for $\mathcal{Z}_{A,b}$ of rank $\leq q-1$. Since $\hat{x}^{(k,q)} \in [0,1]^n$, all we need to show is that $\hat{x}^{(k,q)}$ satisfies all type $(C^q_{A,b})$ inequalities, and since $\hat{x}^{(k,q)} \geq \hat{x}^{(k,q-1)}$, we just need to look at the rank-q, type $(C^q_{A,b})$ inequalities.

Consider a type $(C^q_{A,b})$ inequality $\alpha^T x \geq \alpha_0$ of rank q. Since $\hat{x}^{(k,q-1)} \geq \hat{x} \in P^k_{A,b}$, by Lemma 4.1 we have that

$$\alpha^T \hat{x}^{(k,q-1)} \quad \geq \quad \left(\frac{k}{k+1}\right) \alpha_0. \tag{36}$$

Let $\breve{A}x \geq \breve{b}$ be the system described in Lemma 3.4. By Lemma 3.4 (ii), $\alpha^T x \geq \alpha_0$ is a type $(C^q_{\breve{A},\breve{b}})$ inequality.

By the inductive hypothesis, and Lemma 3.4 (iii), we know that $\hat{x}^{(k,q-1)}$ satisfies all type $(C^{q-1}_{\breve{A},\breve{b}})$ inequalities since these have rank $< q$. Further, by Lemma 3.4 (i) and Corollary 3.3 the type $(C^{q-1}_{\breve{A},\breve{b}})$ inequalities form an additive system. Since

$$\hat{x}^{(k,q)}_j \; = \; \min\left\{1, \frac{k+1}{k}\, \hat{x}^{(k,q-1)}_j\right\}, \quad 1 \leq j \leq n,$$

by Lemma 4.2 and (36) we have that $\alpha^T \hat{x}^{(k,q)} \geq \alpha_0$, as desired. ■

5 Related results

The proof of Lemma 2.1 shows that the system $Mx \geq e$ of minimal covers derived from $Ax \geq b$ plays a critical role – if we can "quickly" optimize over the system $P^k_{M,e}$ of pitch $\leq k$ valid inequalities for $Mx \geq e$ then we can "explain" all low rank valid inequalities for $Ax \geq b$. In the proof of Theorem 1.3 we explicitly list out all inequalities in $Mx \geq e$; as we saw above in the case where each inequality in $Ax \geq b$ has pitch bounded by a fixed constant p, $Mx \geq e$ will have a polynomial number of rows.

It is possible to generalize this situation to cases where $Mx \geq e$ does not need to be explicitly stated, and still one obtains polynomial-time approximation algorithms.

Theorem 5.1 *Consider an optimization problem* $\min\{c^T x : Ax \geq b,\ x \in \{0,1\}^n\}$. *Let* $u \geq 0$ *and* $t \geq 0$ *be given integers, such that each entry in* A *is nonnegative, integral, and with value at most* u*; and that the support of any row of* A *intersects the support of at most* t *other rows. Then for each rank* q *and* $0 \leq \epsilon < 1$ *there is a relaxation with value at least* $(1-\epsilon)\tau^q_{A,b}(c)$ *that can be computed in polynomial time (for fixed* u*,* t*,* q *and* ϵ*).*

Proof. The key point in the proof is that, because no bound is assumed on the entries of b, the number of minimal cover inequalities can be exponentially large. However, because of the bounds u and t, it turns out that there is a succinct representation of the set of minimal covers.

In order to capture this representation, our proof will use an extended formulation using n variables y in addition to the x variables, and three sets of constraints in addition to $Ax \geq b$. One set of constraints involve the y variables only, and it consists of a certain set-covering system $My \geq e$. The second set of constraints involves both x and y variables. The third set of constraints exercises Theorem 1.6 on the set-covering system $My \geq e$. The value k, as before, is such that $(1+1/k)^q \leq (1-\epsilon)^{-1}$.

We will prove:

(a) that this formulation is valid,

(b) that any point in the projection of this formulation to the space of the x variables is contained in $P^k_{A,b}$, and

(c) that a linear program over this formulation can be solved in polynomial time. The number of constraints and variables in the first and third sets of constraints is polynomial. The number of constraints in the second set is exponential, but we show that they can be separated in polynomial time.

Together, (a), (b) and (c) complete the proof.

In what follows the support of row i of A, $1 \leq i \leq m$, will be denoted by $suppt(i)$.

A *cell* will be an inclusion-maximal set of columns C, with the property that for each i, $1 \leq i \leq m$ all entries a_{ij} for $j \in C$ have the same value (which of course depends on i, and which could equal 0.) If $C \cap suppt(i) \neq \emptyset$, we say that C is a cell *for row i*.

Remark 5.2 *For any row i there are at most* $O(u^t)$ *cells* C *for row i.* □

Suppose that S is a minimal cover for some row of $Ax \geq b$, let C be a cell, and let $J \subseteq C$ be such that $|J| = |S \cap C$. By definition of cell, it follows that $S' = (S \setminus (S \cap C)) \bigcup J$ is also a minimum cover. It follows that given a row i of $Ax \geq b$, we can specify all minimal covers arising from row i, up to permutation of the elements of the cover within in each cell.

Moreover, we can use this observation to implicitly enumerate all minimal covers arising from any row i in polynomial time: we simply specify, for each cell C for row i, an nonnegative integer $\nu_C \leq |C|$. This integer serves a candidate for $|S \cap C|$ for some (hypothetical) minimimal cover S. Clearly, once we specify a value ν_C for each cell C then it is a simple matter to check whether indeed there is a minimal cover whose intersection with each C has cardinality ν_C.

In what follows, we will refer to any vector of values ν_C that does give rise to a minimal cover as a *specification.* By Remark 5.2 the total number of specifications corresponding to a row i, and the work needed to enumerate them,is $O(n^{O(u^t)})$.

Now we proceed to present our extended formulation.

For each cell C, we define new 0/1 variables $y^{C,1}, y^{C,2}, \ldots y^{C,|C|}$. Although this will altogether define n new variables, the labeling we use will be helpful.

Let $\nu = \{\nu_C : C \text{ a cell for row } i\}$ be a specification. Suppose we choose, for each cell C for row i with $\nu_C > 0$,

(i) An integer $0 \leq r(C) \leq |C|$,

(ii) a subset $J(C) \subseteq \{r(C)+1, r(C)+2, \ldots, \min\{r(C)+\nu_C+k-1, |C|\}\}$ with $|J(C)| = \nu_C$.

Then we impose the constraint

$$K(\nu, r, J): \quad \sum_{C\,:\,\nu_C>0} \sum_{j \in J(C)} y^{C,j} \quad \geq \quad 1. \tag{37}$$

Note that the total number of terms in the left-hand side of (37) is exactly $\sum_C \nu_C$ – so the inequality can be viewed as a generic representation of all the minimal covers with specification ν. We note that we form constraint (37) for each specification ν and for each combination of choices $J(C, \nu)$. Constraints (37) constitute the set-covering system mentioned before.

Next, we impose the following constraints. Let C be a cell. Let $L_0, L_1, L_2, \ldots, L_k$ be k disjoint subsets of C, some of which may be empty, which form a partition of C. For $0 \leq h \leq k$, write $c_h \doteq |L_0| + |L_1| + \ldots |L_h|$, and $c_{-1} \doteq 0$. Then we impose

$$\sum_{h=0}^{k} \sum_{j \in L_h} h x_j \; - \; \sum_{h=0}^{k} \sum_{j=c_{h-1}+1}^{c_h} h y^{L,j} \quad \geq \quad 0, \tag{38}$$

where we interpret a sum as being zero if its range is empty.

The final set of constraints is that obtained by applying Theorem 1.6 to the set-covering system given by constraints (37) – these new constraints use additional variables $z \in R^N$, where N is polynomially large for each fixed k, and the projection of its continuous relaxation to the space of the y variables satisfies all inequalities valid for $\mathcal{Z}_{M,e}$ with pitch $\leq k$. The system used by Theorem 1.6 is of the form

$$Py + Qz \quad \geq \quad d, \tag{39}$$

for appropriate P, Q and d.

Now we prove the desired results. First, as a consequence of the above observations regarding the number of specifications, we have:

Remark 5.3 *The number of constraints (37) is polynomial for each fixed k, t and u.*

Proof. This follows because for given ν, C, and $r(C)$, the number of choices for $J(C,\nu)$ is $O(\nu_C^k) = O(|C|^k)$. $\square$

Next,

Remark 5.4 *Suppose $\bar{x} \in \mathcal{Z}_{A,b}$. Then there exists a 0/1 vector $\bar{y}$ such that $(\bar{x}, \bar{y})$ satisfy (37) and (38).*

Proof of Remark 5.4. Let C be a cell. Then we set $\bar{y}^{C,j} = 1$ for every $j \leq \min\{r(C), |suppt(\bar{x}) \cap C|\}$, and zero otherwise. To put it differently, $\bar{y}^{C,j}$ equals the j^{th} largest $\bar{x}_i$ with $i \in C$. It is easily seen that this satisfies (37) and (38), as desired. $\square$

Remark 5.5 *Inequalities (38) can be separated in polynomial time.*

Proof of Remark 5.5. Suppose $(\hat{x}, \hat{y}) \in [0,1]^{2n}$. Let C be a cell. Note that there are polynomially number of ways to choose nonnegative integers c_h, $0 \leq h \leq k$, such that $c_0 \leq c_1 \leq \ldots \leq c_k = k$. For each choice of such numbers, the second term of inequality (38) is determined. The inequality becomes most binding when we choose L_0 to consist of the c_0 largest $\hat{x}_j$ with $j \in C$, L_1 to consist of the next $c_1 - c_0$ largest $\hat{x}_j$ with $j \in C$, and so on. Thus we can indeed check whether (38) is satisfied by $(\hat{x}, \hat{y})$, as desired.$\square$

Finally, we have:

Remark 5.6 *Suppose $(\hat{x}, \hat{y}, \hat{z}) \in [0,1]^{2n+N}$ satisfies constraints (37), (38) and (39). Then $\hat{x} \in P^k_{A,b}$.*

Proof of Remark 5.6. Let

$$\alpha^T x \quad \geq \quad \alpha_0 \tag{40}$$

be valid for $\mathcal{Z}_{A,b}$ and such that for $0 \leq j \leq n$, α_j is integral and $0 \leq \alpha_j \leq k$. We wish to show that $\hat{x}$ satisfies (40). Consider the inequality

$$\sum_C \sum_{1 \leq j \leq |C|} \beta^{C,j} y^{C,j} \quad \geq \quad \alpha_0, \tag{41}$$

constructed as follows. Consider any cell C. Then, for $1 \leq j \leq |C|$, let $h = h(j) \in C$ be such that α_h is the j^{th} **smallest** α_q with $q \in C$. We then set $\beta^{C,j} = \alpha_h$.

If we can show that (41) is valid for the set of $y \in \{0,1\}^n$ satisfying (37) then we have that $\hat{y}$ satisfies (41), and therefore, since $(\hat{x}, \hat{y})$ satisfy (38), $\hat{x}$ satisfies (40), as desired.

In order to show that (41) is valid, consider, for each cell C, a (possibly empty) subset W_C such that $\beta^{C,j} > 0$ for all $j \in W_C$ and

$$\sum_C \sum_{j \in W_C} \beta^{C,j} < \alpha_0.$$

What we need to show is that the inequality

$$\sum_C \sum_{j \notin W_C} y^{C,j} \quad \geq \quad 1 \tag{42}$$

is valid for the set of $y \in \{0,1\}^n$ satisfying (37) (in which case it must be dominated by one of the constraints in (37)). We will show that an inequality of the form (37) dominates (42).

To see that this is the case, for any cell C let $\bar{W}_C = \{h(j)\,:\, j \in W_C\}$. It follows that

$$\sum_C \sum_{h \in \bar{W}_C} \alpha_h < \alpha_0,$$

and consequently there is a minimal cover $S \subseteq suppt(\alpha)$ with $(S \cap C) \cap \hat{W}_C = \emptyset$ for each cell C. Let ν be the specification of S (i.e. $\nu_C = |S \cap C|$ for each cell C).

Consider any cell C. Since all the α_j are nonnegative integers, and $\alpha_0 \leq k$, it follows that $|W_C| < k$. Furthermore, $S \cap C \subseteq suppt(\alpha) \cap (C \setminus \bar{W}_C)$. Since $|\bar{W}_C| = |W_C|$, we conclude that

$$\nu_c + |W_C| + \left|\left\{1 \leq j \leq |C| \,:\, \beta^{C,j} = 0\right\}\right| \,\leq\, |C|.$$

We use this now to construct the desired inequality (37). For each cell C, let:

(i) $r(C) = \max\{j \leq |C| \,:\, \beta^{C,j} = 0\}$,

(ii) $J(C) = \{r(C)+1, r(C)+2, \ldots, r(C) + \nu_C + |W_C|\} \setminus W_C$.

It is clear that the resulting inequality (37) dominates (41). This concludes the proof of Remark 5.6. □

The theorem is now proved. ■

Another issue concerns the effectiveness of standard lift-and-project methods, such as the t-iterate Lovász-Schrijver procedure N_+^t [LS91], the level-t Sherali-Adams procedure $S^{(t)}$ [SA90], and others (see [BCC96], [Las01], [Lau01]) in approximating Chvátal-Gomory closures of, say, set-covering problems. Here we have a negative result, extending one from [BZ02a], [BZ02b]. This concerns a set-covering problem with a *full-circulant* constraint matrix,

$$\sum_{j \neq k} x_j \geq 1, \qquad \text{for each } \; k, \; 1 \leq k \leq n \tag{43}$$
$$x \in \{0,1\}^n$$

which gives rise to the rank-1 inequality

$$\sum_{j=1}^{n} x_j \,\geq\, 2. \tag{44}$$

Theorem 5.7 *Let $0 < t < n-2$ integral. Consider the set-covering problem given by the full-circulant matrix on n rows. The vector*

$$x_j^* \;=\; \frac{n-1}{n(n-1)/2 + (n-t-1)(n-t-2)/2} \qquad (1 \leq j \leq n) \tag{45}$$

satisfies the constraints of the t-iterate Lovász-Schrijver procedure N_+^t and of the level-t Sherali-Adams procedure $S^{(t)}$. ■

This result can be proved using a straightforward extension of the proof of a closely related result given in [BZ02b]. For completeness, a proof of this fact is given in the Appendix, where we also provide a definition of the Sherali-Adams operators. In fact the vector x^* is consistent with the constraints of an operator that is exponentially stronger than the combination of N_+^t and $S^{(t)}$. In any case, for fixed $t > 0$ and n large,

$$\sum_j x_j^* \,\sim\, 1 + \frac{t+1}{n},$$

i.e. x^* violates (44) by nearly a factor of 2, for t fixed and n large.

Related results concerning the N_+ operator are given in [CD01] [GT01], [Lau01]. In particular, [CD01] shows that starting from the system $\{\sum_j x_j \geq 1/2\}$, the N_+-rank of the inequality $\sum_j x_j \geq 1$ is n. [GT01] gives an example of a set-covering problem (with an exponential number of constraints) where a certain valid inequality has high N_+-rank. On the other hand, Letchford [Le01] has produced a disjunctive procedure (on the original space of variables) that, when applied to system (43), guarantees that (44) is satisfied. We note that many results are known regarding the effectiveness of lift-and-project methods on *packing* problems; some of the earliest are in [LS91]. For some recent results, and a more thorough bibliography, see [BO04].

An open issue concerns the rank of pitch $\leq k$ inequalities for (say) a set-covering problem: is it bounded as a function of k? We do not know the answer to this; but there is a (perhaps not unexpectedly) positive answer to a simpler question:

Theorem 5.8 *Consider an optimization problem* $\min\{c^T x \,:\, Ax \geq b,\, x \in \{0,1\}^n\}$ *where A is nonnegative. For integer $k > 0$, the Chvátal-Gomory rank of any valid inequality $\alpha^T x \geq \beta$ with all coefficients in $\{0, 1, \cdots, k\}$ is at most k.*

Proof. First we claim that any cover inequality derived from any row of $Ax \geq b$ has rank (at most) 1. To see this, consider a cover inequality

$$\sum_{j \in J} x_j \geq 1 \tag{46}$$

derived from some row i of A. Write $K = b_i + \sum_j a_{ij}$, and consider the linear combination obtained by using multiplier $\frac{1}{K}$ for row i of $Ax \geq b$, and, for each $j \notin J$, multiplier $\frac{a_{ij}}{K}$ for $-x_j \geq -1$. This yields

$$\sum_{j \in J} \frac{a_{ij}}{K} x_j \quad \geq \quad \left(b_i - \sum_{j \notin J} a_{ij} \right) /K. \tag{47}$$

Rounding (47) yields (46), as desired, and the claim is proved.

Further, if an inequality of the form

$$\sum_{j \in J} x_j \geq 1 \tag{48}$$

is valid then J must be a cover for at least one of the rows of $Ax \geq b$, and hence (48) has rank at most 1.

Consider now an inequality of the form

$$\sum_{j \in J} \alpha_j x_j \geq \beta \tag{49}$$

where every α_j and β are in $\{1, \cdots, k\}$. We will show that (49) has rank at most β, by induction on β (with the case $\beta = 1$ clear).

Since (49) is valid, then so is $\sum_{j \in J} \min\{\alpha_j, \beta\} x_j \geq \beta$, so without loss of generality, $\alpha_j \leq \beta$ for all $1 \leq j \leq n$. Let $T = \{j \in J \,:\, \alpha_j < \beta\}$, and $t = |T|$. If $t = 0$ we are done, since in this case (49) is a multiple of a cover inequality.

Assuming otherwise, since (49) is valid, then for each $h \in T$ the following inequality is also valid:

$$(\alpha_h - 1) x_h + \sum_{j \neq h} \alpha_j x_j \quad \geq \quad \beta - 1 \tag{50}$$

Further, by induction, each inequality (50) has rank at most $(\beta - 1)$. Noting that $\alpha_j = \beta$ for all $j \in J \setminus T$, the arithmetic average of the t inequalities (50) is:

$$\sum_{j \in T} \left(\alpha_h - \frac{1}{s} \right) x_j + \beta \sum_{j \in J \setminus T} x_j \quad \geq \quad \beta - 1 \tag{51}$$

Now since (49) is valid, then so is:

$$\beta \sum_{j \in J} x_j \quad \geq \quad \beta. \tag{52}$$

Since (52) is a multiple of a cover inequality its rank is at most 1. Let $0 < \gamma < 1$ be such that

$$\gamma \beta + (1 - \gamma) \left(\alpha_h - \frac{1}{s} \right) < \alpha_h,$$

for all $h \in T$. Then if we multiply (52) by γ, and (51) by $1 - \gamma$, add, and round up, we obtain an inequality that dominates (49), as desired. ■

In summary, then, on the one hand given a set covering problem the valid inequalities with small coefficients have small rank. On the other hand we can separate in polynomial time over the valid inequalities with small coefficients, which results in "nearly" optimizing in polynomial time over all inequalities with sufficiently small rank.

Appendix – A rank-1 inequality that is hard to approximate by Sherali-Adams and N_+

In this section we consider a set-covering problem defined by a full-circulant matrix, i.e., a feasible region of the form (43) which we restate here for convenience

$$\sum_{j \neq k} x_j \geq 1, \qquad \text{for each } \; k, \; 1 \leq k \leq n \tag{53}$$

$$x \in \{0,1\}^n \tag{54}$$

for $n > 1$, and show that for any fixed integer $t > 0$, there is a point satisfying the constraints of the N_+^t procedure and the level-t Sherali-Adams procedure, which violates the valid inequality

$$\sum_j x_j \quad \geq \quad 2 \tag{55}$$

by a factor of 2 (asymptotically) as t remains fixed and n grows arbitrarily large. The analysis here is a strengthening of a similar proof given in [BZ02b], which only shows that (55) has high rank.

First, we describe the level-t ($0 \leq t \leq n$) Sherali-Adams procedure. As before, we indicate this procedure by $S^{(t)}$. The procedure creates a variable $w(Q,P)$ that approximates the polynomial

$$f(Q,P) \doteq \prod_{j \in Q} x_j \cdot \prod_{j \in P} (1 - x_j)$$

for each pair of disjoint subsets Q, P of $\{1, 2, \cdots, n\}$ with $|Q \cup P| \leq \min\{t+1, n\}$. The procedure operates as follows. Suppose we have a set of the form

$$\begin{array}{rcl} Ax & \geq & b \\ x & \in & \{0,1\}^n. \end{array} \tag{56}$$

For each polynomial $f(Q,P)$ where $|Q \cup P| = t$, we multiply each constraint r of (56) by $f(Q,P)$, obtaining a (valid) polynomial inequality of the form

$$(a_r x - b_r) f(Q,P) \quad \geq \quad 0. \tag{57}$$

(where a_r denotes the r^{th} row of A). In addition, we write the (valid) inequalities

$$f(Q \cup j, P) + f(Q, P \cup j) \quad = \quad f(Q,P), \;\; \forall \; j \notin Q \cup P, \tag{58}$$

$$f(Q,P) \quad \geq \quad 0, \;\; \forall \; Q, \; P, \;\; \text{with} \;\; |Q \cup P| \leq \min\{t+1, n\}, \tag{59}$$

where we abbreviate $\{j\}$ as j. Next, we linearize the constraints (57), (58) and (59): for each j, any positive power of x_j (resp., $1 - x_j$) is replaced with x_j (resp., $1 - x_j$), and any expression containing $x_j(1 - x_j)$ is replaced with 0. Finally, we require $f(\emptyset, \emptyset) = 1$. We are left with a system of linear inequalities on the polynomials $f(Q,P)$ where $|Q \cup P| \leq \min\{t+1, n\}$. Each of these polynomials is then replaced by a new variable $w(Q,P)$, thus obtaining a linear system on the variables w. We will refer to this linear system as the $S^{(t)}$ *constraints*; see [SA90] for further background.

Now we return to the full-circulant set-covering example (53). Suppose we consider an assignment to the variables $w(Q,P)$ of the form $w(Q,P) = z(|Q|, |P|)$, for an appropriate function z. It is not difficult to see that this assignment satisfies the $S^{(t)}$ constraints if $z \geq 0$ and:

$$z(q+1,p) + z(q,p+1) - z(q,p) \quad = \quad 0 \quad \text{if } \; q + p \leq t \tag{60}$$

$$(q-2)z(q,p) + (n-q-p)z(q+1,p) \quad \geq \quad 0 \quad \text{if } \; q+p \leq t \; \text{ and } \; q > 0 \tag{61}$$

$$(n-p-1)z(1,p) - z(0,p) \quad \geq \quad 0 \quad \text{if } \; p \leq t \tag{62}$$

$$z(0,0) \quad = \quad 1. \tag{63}$$

We will next show how to choose such values $z(q,p)$ so as to violate (55) as desired. In fact, we will do something stronger: we will define $z(q,p)$ for any nonnegative q,p with $q + p \leq n$, and we will require

$$z(q+1,p) + z(q,p+1) - z(q,p) \quad = \quad 0 \quad \text{if } \; q+p \leq n-1, \tag{64}$$

instead of (60). These additional conditions will be of use later.

To this effect, define

$$\alpha = \frac{(n-t-1)(n-t-2)}{n(n-1) + (n-t-1)(n-t-2)}, \tag{65}$$

$$\beta = \frac{2}{n(n-1) + (n-t-1)(n-t-2)}. \tag{66}$$

and set, for $0 \leq p \leq n-2$,

$$z(2,p) = \beta, \tag{67}$$

$$z(1,p) = (n-p-1)\beta, \quad \text{and} \tag{68}$$

$$z(0,p) = \alpha + \frac{(n-p)(n-p-1)}{2}\beta \tag{69}$$

For all other cases of q, p, define $z(q,p) = 0$. Then a calculation shows that (64), (61)-(63) are satisfied. Thus, indeed the values $w(Q,P) = z(|Q|,|P|)$ satisfy the $S^{(t)}$ constraints, and these values are a lifting of the vector x^* defined by

$$x_j^* = z(1,0) = \frac{n-1}{n(n-1)/2 + (n-t-1)(n-t-2)/2} \quad (1 \leq j \leq n). \tag{70}$$

What is more,

$$\sum_j x_j^* = \frac{n(n-1)}{n(n-1) - (t+1)n + (t+1)(t+2)/2} \tag{71}$$

$$\sim 1 + \frac{t+1}{n}, \tag{72}$$

for fixed t as n grows large. This proves half of Theorem 5.7. Clearly x^* satisfies the constraints imposed by N^t (because S^t is a stronger operator, see e.g. [Lau01]); we need a little further analysis to show that it satisfies the constraints imposed by N_+^t. This is done as in [BZ02b].

Let $V_n = \{1, 2, \cdots, n\}$. For each $h \subseteq V_n$ let ζ^h be the $zeta-vector$ of the subset lattice of V_n corresponding to h (this is the 0-1 vector with an entry for each subset of V_n, where the entry for subset s equals 1 iff $s \subseteq h$). Let

$$y = \sum_{h \subseteq V_n} \lambda_h \zeta^h,$$

where

$$\lambda_h = \begin{cases} \alpha & \text{if } h = \emptyset, \\ \beta & \text{if } |h| = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{73}$$

By construction, it is not difficult to show that for any $h \subseteq V_n$, $y_h = z(|h|, 0) = w(h, \emptyset)$. Further, consider the $2^n \times 2^n$-matrix W^y defined by:

$$W^y = \sum_{h \subseteq V_n} \lambda_h \zeta^h (\zeta^h)^T. \tag{74}$$

By construction, W^y can be equivalently defined as follows:

$$W^y_{p,q} = y_{p \cup q}, \quad \forall\, p,\, q \subseteq V_n. \tag{75}$$

Clearly W^y is symmetric positive-semidefinite, and its $\emptyset$-row and -column, and its main diagonal, all equal y – and as a consequence, $W^y_{\emptyset,\emptyset} = 1$.

We have already shown that x^* satisfies the $S^{(t)}$ constraints; the fact that W^y is of the form (74) with λ defined as in (73) implies that x^* satisfies the N_+^t constraints as well. Formally, this is proved as follows.

Assume that the $\emptyset$-column of W^y is its 0^{th} column, and that for $1 \leq j \leq n$, the j^{th} column of W^y corresponds to the singleton $\{j\}$. Since w satisfies the constraints imposed by $S^{(t)}$ (in particular, since z satisfies (64)), it is clear that for any $h \subseteq V_n$, the h^{th}-entry of $(e_0 - e_j)^T W^y$ equals $w(h, j)$. In other words, the appropriate subvector of $(e_0 - e_j)^T W^y$ satisfies the $S^{(t-1)}$ constraints. Further, define $P(j)$ to be the set of pairs $\{i,k\}$ of distinct elements of V_n with $i \neq j$ and $k \neq j$. Then:

$$(e_0 - e_j)^T W^y \;=\; \alpha \zeta^{\emptyset} \;+\; \beta \sum_{\{i,k\} \in P(j)} \zeta^{\{i,k\}}.$$

which shows that $(e_0 - e_j)^T W^y$ can be lifted to a symmetric, positive-semidefinite $(2^n \times 2^n)$ matrix. Inductively, these facts shows that (the singletons subvector of) $(e_0 - e_j)^T W^y$ satisfies the N_+^{t-1} constraints.

Also,

$$e_j^T W^y \;=\; \beta \sum_{\{j,k\},\, j \neq k} \zeta^{\{j,k\}},$$

But for any pair $\{j,k\}$ any vector x with $x_j = x_k = 1$ is a feasible solution to the system (53)-(54). Since $e_j^T W^y$ is a sum of nonnegative multiples of vectors $\zeta^{\{j,k\}}$ it must satisfy all constraints consistent with the N_+^n operator. This concludes the proof.

Note that we have proved that x^* satisfies the constraints of a far stronger operator than either $S^{(t)}$ or N_+^t. This is because the matrix W^y constructed above is $2^n \times 2^n$. This stronger procedure lifts a vector x to a $2^n \times 2^n$ matrix W, with rows and columns indexed by the subsets of V_n, such that

(a) The vector occupying entries $1, 2, \cdots, n$ of the $\emptyset$-column of W equals x,

(b) $W \succeq 0$ and $W_{\emptyset,\emptyset} = 1$. For any subsets p, q, p', q' of V_n, if $p \cup q \;=\; p' \cup q'$ then $W_{p,q} = W_{p',q'}$.

(c) For any $1 \leq j \leq n$, the restrictions of both $e_j^T W$ and $(e_\emptyset - e_j)^T W^y$ to the $(t+1)$-tuples satisfy the $S^{(t-1)}$ constraints, and the restriction of these vectors to the singletons satisfy the N_+^{t-1} constraints.

Conditions (b) and (c) embody the difference between this procedure and N_+^t or $S^{(t)}$ – they constrain the entire $2^n \times 2^n$ matrix.

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