Approximate formulations for 0-1 knapsack sets[1]

Abstract

We show that for each $0 < \epsilon \leq 1$ there exists an extended formulation for the knapsack problem, of size polynomial in the number of variables, whose value is at most $(1+\epsilon)$ times the value of the integer program.

Keywords: Integer Programming, Approximation Algorithms, Lift-and-Project.

1 Introduction

Consider the feasible set for a $0-1$ knapsack problem,

$$\mathcal{F}^{KNAP} \;=\; \left\{ x \,\in\, \{0\,,\,1\}^n \;:\; \sum_{j=1}^{n} a_j x_j \;\leq\; a_0 \right\}, \tag{1}$$

where $a_j \geq 0$ for $0 \leq j \leq n$. Here we prove the following result:

Theorem 1.1 *Let* $0 < \epsilon \leq 1$. *There exists an extended formulation*

$$Ax \,+\, A'x' \;\leq\; b,$$

with $O\left(\epsilon^{-1} n^{1+\lceil 1/\epsilon \rceil}\right)$ *variables and* $O\left(\epsilon^{-1} n^{2+\lceil 1/\epsilon \rceil}\right)$ *constraints such that*

$$\mathcal{F}^{KNAP} \;\subseteq\; \{x \in R^n \,:\, \exists\,(y,z) \text{ s.t. } Ax \,+\, A'x' \;\leq\; b\},$$

and for any $w \in R^n_+$,

$$\max\left\{w^T x \,:\, x \,\in\, \mathcal{F}^{KNAP}\right\} \;\;\geq\;\; (1-\epsilon)\;\max\left\{w^T x \,:\, \exists\,(y,z) \;\text{ s.t. }\; Ax \,+\, A'x' \,\leq\, b\right\}.$$

1.1 Motivation

The knapsack problem

$$\max\left\{w^T x \,:\, \sum_{j=1}^{n} a_j x_j \;\leq\; a_0,\; x \,\in\, \{0\,,\,1\}^n\right\}$$

is, possibly, the simplest combinatorial optimization problem. A celebrated classical result [8, 9], proves the existence of *fully polynomial-time approximation schemes* for the knapsack problem – that is to say, for each $0 < \epsilon \leq 1$ there is an algorithm whose complexity grows as a polynomial in ϵ^{-1} and n, and which yields a solution guaranteed to have value at least $1-\epsilon$ times the optimum.

The constructions in [8, 9] are quintessentially combinatorial: they rely on scaling and dynamic programming. Nevertheless, intuitively (because of the equivalence of optimization and separation) one would expect that the same result should be achievable through the use of linear programming techniques.

One can take this expectation in a different direction: can we find a "compact" and "strong" system of constraints that approximates the knapsack polytope? In our context, what we mean by this is that for each $0 < \epsilon \leq 1$ there is a system of constraints

$$Bx \;\;\leq\;\; b, \tag{2}$$

such that

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(a) Every vector $x \in \{0\,,\,1\}^n$ with $\sum_{j=1}^n a_j x_j \ \le \ a_0$ also satisfies $Bx \ \le \ b$,

(b) The number of constraints in (2) is polynomial in ϵ^{-1} and n, and

(c) For every $w \in R_+^n$,

$$\max \left\{ w^T x \,:\, \sum_{j=1}^n a_j x_j \ \le \ a_0, \ x \ \in \ \{0\,,\,1\}^n \right\} \ \ \ge \ \ (1-\epsilon) \max \left\{ w^T x \,:\, Bx \le b \right\}.$$

We stress that, in (c) the approximation result must hold for *every* objective vector $w \in R_+^n$ (and, hence, for every $w \in R^n$). Finding a system (2) that satisfies (a)-(c) is interesting at the very least in the sense of theoretical completeness. In addition, there is a practical aspect to this goal: given a mixed-integer program, any knapsack row could be strengthened while paying a polynomial cost in formulation size.

It turns out that, in a sense, it may be too much to expect that a such a system does exist, because we ask that the system be restricted to the original space of variables x. This observation stems from a different paradigm, which in turn suggests an alternate approach: there are examples of polyhedra $P \subseteq R^n$ such that P is the projection (onto R^n) of a higher dimensional polyhedron $Q \subseteq R^N$ (i.e., $N > n$), and such that Q is a simpler object than P, in the sense that fewer inequalities are required to describe Q. In such a case, since P is a projection of Q, we can solve an an optimization problem $\min \left\{ w^T x \,:\, x \in P \right\}$ by solving a similar problem over Q, and this may be preferable since Q is simpler. Or, using the equivalence between separation and optimization, we can always work in R^n but separate vectors from Q, thus obtaining a cutting-plane algorithm. This is the genesis of so-called *lift-and-project* reformulation operators, see [10, 13, 2, 12, 11, 4]. Given a $0-1$ integer program, such operators create “lifted” formulations that are polynomially larger, both in terms of constraints and variables, but which are provably strong. In particular, [10] shows how, in the case of the clique problem, the resulting formulation provably satisfies strong inequalities. Also see [4, 5, 3, 6, 7].

Proceeding along these lines, in [14] Van Vyve and Wolsey ask whether, given a knapsack $\sum_{j=1}^n a_j x_j \ \le \ a_0$, and $0 < \epsilon \le 1$, there is a formulation of the form $Ax + A'x' \le b$, such that

(d) For each vector $x \in \{0\,,\,1\}^n$ with $\sum_{j=1}^n a_j x_j \ \le \ a_0$ there exists x' such that $Ax + A'x' \le b$,

(e) The number of variables x' and rows of A and A' is polynomial in n and/or ϵ^{-1}, and

(f) For every $w \in R_+^n$,

$$\max \left\{ w^T x \,:\, \sum_{j=1}^n a_j x_j \ \le \ a_0, \ x \ \in \ \{0\,,\,1\}^n \right\} \ \ \ge \ \ (1-\epsilon) \max \left\{ w^T x \,:\, Ax + A'x' \le b \right\}.$$

In this paper we answer this question in the affirmative, in the sense of “or” in (e): our formulation is polynomially large in n for each fixed ϵ. The “and” case remains open.

1.2 The disjunctive procedure

The lifting techniques described in the previous section are underlaid by an older idea, that of *disjunctive programming* [1]. The gist of this idea can be described as follows. Let $P \subseteq [0,1]^n$ be an arbitrary set, and suppose we are interested in solving an optimization problem of the form

$\max\left\{w^T x \,:\, x \in P\right\}$ for some $w \in R^n$. In the typical application, P is a non-convex set. Suppose that there are polyhedra $Q^1, Q^2, \ldots, Q^L$ (contained in $[0,1]^n$) such that

$$P \;\subseteq\; Q^1 \cup Q^2 \cup \ldots \cup Q^L. \tag{3}$$

(3) is called a *disjunction*, and the Q^i, its *terms*. If (3) holds, $Q = conv\left(Q^1 \cup \ldots \cup Q^L\right)$ is a relaxation for P, and therefore

$$\max\left\{w^T x \,:\, x \in P\right\} \;\leq\; \max\left\{w^T x \,:\, x \in Q\right\}. \tag{4}$$

The optimization problem over Q can be written as:

$$\min\; w^T x \tag{5}$$

$$x - \sum_{i=1}^{L} \lambda^i x^i \;=\; 0 \tag{6}$$

$$\sum_{i=1}^{L} \lambda^i \;=\; 1 \tag{7}$$

$$\lambda^i \geq 0 \text{ and } x^i \in Q^i, \;\; 1 \leq i \leq L. \tag{8}$$

This is a nonlinear formulation since both the λ^i and x^i are variables. However, suppose that for $1 \leq i \leq L$, $Q^i = \{x \in [0,1]^n \,:\, A^i x \,\leq\, b^i\}$, for appropriate matrices A^i and vectors b^i. Then, as is well-known, problem (5)-(8) can be linearized (through homogenization) as follows:

$$\min\; w^T x \tag{9}$$

$$x_j - \sum_{i=1}^{L} \bar{x}^i_j \;=\; 0, \quad \text{for every } 1 \leq j \leq n \tag{10}$$

$$\sum_{i=1}^{L} \bar{x}^i_0 \;=\; 1 \tag{11}$$

$$0 \leq \bar{x}^i_0, \;\; 1 \leq i \leq L, \tag{12}$$

$$A^i \bar{x}^i \;-\; \bar{x}^i_0\, b^i \;\leq\; 0, \;\; \text{for all } \; 1 \leq i \leq L, \tag{13}$$

$$\bar{x}^i_j \;-\; \bar{x}^i_0 \;\leq\; 0, \;\; \text{for all } \; 1 \leq j \leq n \; \text{ and } \; 1 \leq i \leq L. \tag{14}$$

Problems (5)-(8) and (9)-(14) are equivalent: for example, given a solution to 9)-(14), for $1 \leq i \leq L$ we set $\lambda^i = \bar{x}^i_0$, and whenever, $\bar{x}^i_0 > 0$, we set $x^i_j = \bar{x}^i_j / \bar{x}^i_0$ for $1 \leq j \leq n$ (and if $\bar{x}^i_0 = 0$ we choose x^i arbitrarily). We stress that the above arguments hold even if some of the Q^i are empty.

Moreover, let $W = \max\left\{w^T x \,:\, x \in P\right\}$, and suppose there is a constant $0 \leq \gamma < 1$ such that $W \;\geq\; (1-\gamma)\max\left\{w^T x \,:\, x \in Q^i\right\}$ for each $1 \;\leq\; i \;\leq\; L$. Then, if $\left(x, \bar{x}^1, \ldots, \bar{x}^L\right)$ is an optimal solution to (9)-(14) we have

$$(1-\gamma) \sum_{j=1}^{n} w_j \bar{x}^i_j \;\leq\; \bar{x}^i_0\, W,$$

(by (12)-(14)), and therefore $(1-\gamma)\sum_{j=1}^{n} w_j x_j \;\leq\; W$, by (10) and (11), thereby complementing the bound (4). We will apply this technique in our construction.

The strength of the disjunctive programming approach lies in the fact that we can use a disjunction (3) to generate a convex approximation to P that enforces combinatorial structure. It amounts to, effectively, enumerating a number of cases; but the enumeration is implicit in that we end up with one convex optimization problem.

2 The construction

To produce the construction in Theorem 1.1 we will apply the disjunctive procedure. We will first motivate our approach and later give the actual formulation. Let $H = \left\lceil \frac{1}{\epsilon} \right\rceil$. We assume, without loss of generality, that $n \geq 2H$.

To motivate our approach, given $x \in \{0,1\}^n$ define

$$suppt(x) \,=\, \{1 \leq j \leq n \,:\, \hat{x}_j = 1\}\,.$$

Our disjunction will include terms of two types, corresponding to the cardinality of $suppt(x)$ for $x \in \mathcal{F}^{KNAP}$. For each set $S \subseteq \{1, 2, \ldots, n\}$ with $\sum_{j \in S} a_j \leq a_0$ and $|S| \leq H$, our disjunction contains a term Q^S, as follows:

(i) Suppose $|S| < H$. Then Q^S consists of the single point x with $suppt(x) = S$. The formulation for Q^S is trivial, namely, we simply enforce $x_j = 1$ for all $j \in S$, and $x_j = 0$ for all $j \notin S$.

(ii) Suppose now $|S| = H$. Say that $x \in \{0,1\}^n$ is *S-good* if $S \subseteq suppt(x)$ and for each $j \in suppt(x) - S$ we have $a_j \leq \min_{i \in S}\{a_i\}$. In other words, $\{a_j : j \in S\}$ consists of the H largest a_j in $\{a_j : j \in suppt(x)\}$. Our goal is to we define Q^S so that $Q^S \cap \{0,1\}^n$ consists exactly of the S-good points. While there may exist many polyhedra Q^S that achieve this goal (for example, $conv\{x : x \text{ is } S\text{-good}\}$) they may not necessarily be endowed with simple formulations. However, we can find such a Q^S with a compact formulation: $x_j = 1$ for each $j \in S$, $x_j = 0$ if $a_j > \min_{i \in S}\{a_i\}$, $0 \leq x_j \leq 1$ for all j, and $\sum_{j=1}^n a_j x_j \leq a_0$.

Clearly, if $x \in \mathcal{F}^{KNAP}$ then $x \in Q^S$ for at least one S enumerated in (i) or (ii); in other words, we have a valid disjunction. The purpose of the terms in (ii) is to cut-off fractional extreme points $\hat{x}$ where $0 < \hat{x}_j < 1$ for some index j with "large" a_j – loosely speaking, we explicitly enumerate the H largest a_j where $x_j > 0$; and force $x_j = 1$ for every such index and $x_j = 0$ for indices corresponding to a_j larger than any of the enumerated elements.

2.1 Formal description

Here we will provide a formal description and analysis of our lifted formulation, which is a homogenized version of the constraints given in (i) and (ii) above. However our analysis is self-contained and does not directly appeal to the disjunctive procedure.

(a) For each integer $0 \leq h < H$, and each subset $S \subseteq \{1, 2, \ldots, n\}$ with $|S| = h$ and $\sum_{j \in S} a_j \leq a_0$, we have variables y_j^S, for $0 \leq j \leq n$, as well as the constraints:

$$y_j^S \;\geq\; 0, \;\; 0 \leq j \leq n, \tag{15}$$

$$y_j^S \;-\; y_0^S \;=\; 0, \;\; \forall\, j \in S, \tag{16}$$

$$y_j^S \;=\; 0, \;\; \forall\, j \notin S \cup \{0\}. \tag{17}$$

(b) For each each subset $S \subseteq \{1, 2, \ldots, n\}$ with $|S| = H$ and $\sum_{j \in S} a_j \leq a_0$, we have variables z_j^S, for $0 \leq j \leq n$, as well as the constraints:

$$
\begin{aligned}
& z_j^S \;\geq\; 0, \;\; 0 \leq j \leq n, && (18)\\
& z_j^S \;-\; z_0^S \;\leq\; 0, \;\; 1 \leq j \leq n, && (19)\\
& z_j^S \;-\; z_0^S \;=\; 0, \;\; \forall\, j \in S, && (20)\\
& z_j^S \;=\; 0, \;\; \text{if}\, j \notin S \cup \{0\} \;\text{ and }\; a_j > \min_{i \in S}\{a_i\}, && (21)\\
& \sum_{j=1}^{n} a_j z_j^S \;-\; a_0\, z_0^S \;\leq\; 0. && (22)
\end{aligned}
$$

(c) Let $\mathcal{F}$ be the family of sets enumerated in (a) and (b). We have the additional constraints:

$$
\sum_{S \in \mathcal{F}} y_j^S \;+\; \sum_{S \in \mathcal{F}} z_j^S \;-\; x_j \;=0, \qquad \text{for each index } 1 \leq j \leq n, \tag{23}
$$

$$
\sum_{S \in \mathcal{F}} y_0^S \;+\; \sum_{S \in \mathcal{F}} z_0^S \;=1. \tag{24}
$$

Lemma 2.1 *Formulation (15)-(24) has* $O\left(\epsilon^{-1} n^{1+\lceil 1/\epsilon \rceil}\right)$ *variables and* $O\left(\epsilon^{-1} n^{2+\lceil 1/\epsilon \rceil}\right)$ *constraints.*

Proof. Recall that we assume, without loss of generality, that $n \geq 2H$. The total number of vectors y^S is at most

$$
\sum_{h=0}^{H-1} \binom{n}{h} \;\leq\; (H-1)\binom{n}{H},
$$

whereas the total number of vectors z^S is at most $\binom{n}{H}$. Consequently the total number of variables is, at most,

$$
H(n+1)n^H \;=\; O\left(\epsilon^{-1} n^{1+\lceil 1/\epsilon \rceil}\right),
$$

as desired. Since we impose $O(n)$ constraints on each vector, the total number of constraints is as stated. ■

Lemma 2.2 *Constraints (15)-(24) define a valid relaxation for* $\mathcal{F}^{KNAP}$*, i.e. the projection of the feasible set for (15)-(24) to the space of the* x *variables contains the feasible set for* $\mathcal{F}^{KNAP}$*.*

Proof. This follows from our discussion the disjunctive procedure, but a direct proof is as follows. Consider a 0-1 vector $\hat{x} \in \mathcal{F}^{KNAP}$. Write $J = suppt(\hat{x})$.

Suppose first that $|J| < H$. Then we define $y_j^J = \hat{x}_j$ for $1 \leq j \leq n$, and $y_0^J = 1$; and set $y_j^S = 0$ for all other sets S and all j, and all $z_j^S = 0$. Note that this argument is correct even when $J = \emptyset$.

Suppose now that $|J| \geq H$. Let $S \subseteq J$ consist of the H indices $j \in J$ with largest a_j (ties arbitrarily broken). Then we set $z_j^S = 1$ for all $j \in J$, $z_0^S = 1$, and set $z_j^T = 0$ for all other combinations of T and j; and all y to 0. ■

Write $W^* = \max\left\{w^T x \,:\, \sum_{j=1}^{n} a_j x_j \;\leq\; a_0, \;\; x \,\in\, \{0\,,\,1\}^n\right\}$.

Lemma 2.3 *Suppose* $(\hat{x}, \hat{y}, \hat{z})$ *satisfy (15)-(24). Let* $w \in R_+^n$*. Then* *(i) For any set S included in case (a) of the construction,*

$$W^* \hat{y}_0^S \;\geq\; \sum_{j=1}^{n} w_j \hat{y}_j^S.$$

(ii) For any S included in case (b) of the construction,

$$W^* \hat{z}_0^S \;\geq\; (1-\epsilon) \sum_{j=1}^{n} w_j \hat{z}_j^S.$$

Proof. (i) If $\hat{y}_0^S = 0$ the result is clear, and if $\hat{y}_0^S > 0$ then the $0-1$ vector with entries $\hat{y}_j^S / \hat{y}_0^S$ $(1 \leq j \leq n)$ satisfies (1) from which the result follows.

(ii) As in (i) assume that $\hat{z}_0^S > 0$, and define $\bar{x}_j = \hat{z}_j^S / \hat{z}_0^S$ for $1 \leq j \leq n$. By construction in case (b), we have that $\bar{x}$ is a feasible solution to the linear program:

$$\begin{aligned}
\tilde{W} \;&=\; \max \sum_{j=1}^{n} w_j x_j && (25)\\
&\text{Subject to:} && (26)\\
&0 \;\leq\; x_j \;\leq\; 1, \;\; 1 \leq j \leq n, && (27)\\
&x_j \;=\; 1, \;\; \forall\, j \in S, && (28)\\
&x_j \;=\; 0, \;\; \text{if}\, j \notin S \;\text{ and }\; a_j > \min_{i \in S}\{a_i\}, && (29)\\
&\sum_{j=1}^{n} a_j x_j \;\leq\; a_0. && (30)
\end{aligned}$$

Thus, in order to conclude with case (ii) it suffices to prove that $W^* \geq (1-\epsilon)\tilde{W}$. To this end, let $\tilde{x}$ be an extreme point optimal solution to the LP (25)-(30). We assume $\tilde{x}$ is not integral for otherwise the result is clear.

Clearly, there exists exactly one index p such that $0 < \tilde{x}_p < 1$.

Let $i = \mathrm{argmin}_{j \in S}\{w_j\}$, and suppose that $w_i < w_p$. Then we increase $\tilde{x}_p$ by $1 - \tilde{x}_p$, decrease $\tilde{x}_i$ by $1 - \tilde{x}_p$, and reset $S \leftarrow S - \{i\} \cup \{p\}$. By (29), we have $a_i \geq a_p$. Thus, after the change, the vector $\tilde{x}$ still satisfies (30), as well as (27). Moreover, the objective value of $\tilde{x}$ has increased.

Thus (whether the change was performed or not), we have:

(C.1) $0 < \tilde{x}_q < 1$ for one entry q,

(C.2) There is a set S with $|S| = H$ such that $\tilde{x}_i = 1$ for all $i \in S$, and if an index q as in (C.1) exists, then $w_q \leq \min_{i \in S}\{w_i\}$.

(C.3) $\tilde{x}$ satisfies (30),

(C.4) $\sum_j w_j \tilde{x}_j \geq \tilde{W}$.

Consider the $0-1$ vector $\tilde{x}$ defined by $\tilde{x}_j = \lfloor \tilde{x}_j \rfloor$ for $1 \leq j \leq n$. By (C.3) this vector satisfies the knapsack constraint (1). Furthermore, by (C.1) and (C.2), we have that

$$\frac{\sum_j w_j \tilde{x}_j \;-\; \sum_j w_j \tilde{x}_j}{\sum_j w_j \tilde{x}_j} \;\leq\; \frac{1}{H} \;\leq\; \epsilon,$$

and therefore

$$(1-\epsilon) \sum_j w_j \tilde{x}_j \ \leq \ \sum_j w_j \tilde{x}_j \ \leq W^*,$$

as desired. ■

Lemma (2.3), together with constraints (23) and (24) of our system, complete the proof of Theorem 1.1.

3 Conclusion

An interesting open question is whether the lift-and-project operators in [10, 13, 12] can be used to obtain a result similar to Theorem 1.1. However, these operators do not create disjunctions based on the structure of the constraints of an integer program, in particular, the numerical value of coefficients. In contrast, this is a critical feature in our approach (namely in our construction of the z^S vectors), and we feel that this is an ingredient that is necessary to cut-off highly fractional extreme points. See [6, 4, 5] for results concerning the weakness of standard lift-and-project operators.

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