Two Applications of Disjunctive Programming

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Can we account for all valid inequalities with small coefficients?

Theorem (B. and Mark Zuckerberg, 2004)

For any fixed integer $\boldsymbol{k} \geq \mathbf{1}$ there exists a *compact, extended* formulation whose solutions satisfy all valid inequalities with coefficients in $\{\mathbf{0}, \mathbf{1}, \ldots, \boldsymbol{k}\}$.

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Corollary: For any fixed positive integer $\boldsymbol{r} \geq \mathbf{1}$ and $\mathbf{0} < \epsilon < \mathbf{1}$,

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$\forall c \in \mathbb{R}^n$:

$$\min c^T x \quad \text{s.t. } x \in \text{projected formulation} \geq$$
$$(1-\epsilon)\left(\min c^T x \quad \text{s.t. } x \in \text{rank-r Gomory closure}\right)$$

Two recent, related papers:

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- Today, a shorter proof +

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Consider a (known) valid inequality

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Gives rise to an alternate scheme for branch-and-bound

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Then, the solution to any **node** of the branch-and-bound (sub)tree thus created satisfies every valid inequality

$$\boldsymbol{\alpha}^{\boldsymbol{T}} \boldsymbol{x} \ \geq \ \mathbf{2}$$

where

- $\alpha_j \ \in \ \{0, 1, 2\}$ for $j = 1, \ldots, n$
- H contained in the **support** of α

Example

Consider a valid inequality

$$\sum_{j \in S} x_j \;\geq\; 2 \tag{1}$$

and suppose we vector-branch on a set covering constraint

$$\sum_{j \in H} x_j \;\geq\; 1, \qquad \text{with } H \subseteq S$$

And now consider a node where $\boldsymbol{x_{j_k} = 1}$ with $j_k \in H$. **But**:

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So the solution to the node must satisfy (1). **Related: Letchford 2001**

Pitch k

Consider a valid inequality of pitch k:

$$\sum_{j \in S} \alpha_j x_j \;\geq\; \alpha_0 \tag{3}$$

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Since (3) is valid, **so is**:

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But, (4) has **pitch** $\leq k - 1$
So all we need is a **recursive** construction

Construction

Construction – a few corners are cut

- Set-covering system $Ax \geq \boldsymbol{e}$.
- Pitch $\boldsymbol{p} \geq \mathbf{2}$
- $\mathcal{Z}^{\boldsymbol{p}-\mathbf{1}}$: recursively constructed formulation whose solutions satisfy all valid inequalities of pitch $\leq \boldsymbol{p} - \mathbf{1}$.
- For $\boldsymbol{p} = \mathbf{2}$,

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- $\mathcal{Z}^{\boldsymbol{p}-\mathbf{1}}$: recursively constructed formulation whose solutions satisfy all valid inequalities of pitch $\leq \boldsymbol{p} - \mathbf{1}$.
- For $\boldsymbol{p} = \mathbf{2}$, $\mathcal{Z}^{\boldsymbol{p}-\mathbf{1}}$ is the original formulation $\boldsymbol{Ax} \geq \boldsymbol{e}$
- Now we will consider a row $\boldsymbol{i}$ of $\boldsymbol{Ax} \geq \boldsymbol{e}$ and, effectively, vector-branch on it
- Actually we will write the corresponding **disjunction**

Let the row be

$$\sum_{j \in S^i} x_j \geq 1$$

where $S^i = \{j_1, j_2, \ldots, j_{|S^i|}\}$.

Row $\boldsymbol{i}$ of $\boldsymbol{Ax} \geq \boldsymbol{e}$: $\sum_{j \in S^i} x_j \geq 1$, where $S^i = \{j_1, \ldots, j_{|S^i|}\}$.

(a) For $\mathbf{1} \leq \boldsymbol{t} \leq |\boldsymbol{S^i}|$, polyhedron $D_i^p(t) \subseteq \mathbb{R}^n$ given by

$$
\begin{aligned}
x_{j_t} &= 1 && (5)\\
x_{j_h} &= 0 \quad \forall\, 1 \leq h < t, \quad \text{and} && (6)\\
x &\in \mathcal{Z}^{p-1} && (7)
\end{aligned}
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(b) Polyhedron $\boldsymbol{D_i^p} \doteq \operatorname{conv}\{\boldsymbol{D_i^p(t)} : \mathbf{1} \leq \boldsymbol{t} \leq |\boldsymbol{S^i}|\}$

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Finally: $\boldsymbol{Z^p} \doteq \bigcap_i \boldsymbol{D_i^{p-1}}$

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Finally: $\boldsymbol{Z^p} \doteq \bigcap_i \boldsymbol{D_i^{p-1}}$

Lemma:

$\boldsymbol{Z^p}$ can be described by a polynomial-size formulation for fixed **p**, and its feasible solutions satisfy all valid inequalities of pitch $\leq \boldsymbol{p}$.

Subapplication 1a: minimum knapsack

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$$\text{s.t.} \quad \sum_j w_j x_j \geq \; \boldsymbol{b}, \qquad x \text{ binary}$$

$\boldsymbol{w} \geq \mathbf{0}, \;\; \boldsymbol{b} > \mathbf{0}$

- "FPTAS" exists

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Open question:

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Open question: Given $\boldsymbol{w}, \boldsymbol{b}$ is there a compact extended formulation that yields a constant factor approximation, $\forall \, \boldsymbol{c}$?

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Open question: Given $\boldsymbol{w}, \boldsymbol{b}$ is there a compact extended formulation that yields a constant factor approximation, $\forall\, \boldsymbol{c}$?

ANY constant whatsoever?

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$\boldsymbol{w} \geq \mathbf{0}$, $\boldsymbol{b} > \mathbf{0}$, **integral**

Well-known result: equivalent to set-covering problem,

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Well-known result: equivalent to set-covering problem, with constraints

$$\sum_{j \in S} x_j \;\geq\; 1, \quad \forall\, S \;\; \textbf{with} \; \sum_{j \in S} w_j \;\geq\; w^* \;\doteq\; \sum_j w_j - b + 1$$

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But exponentially many constraints

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- Compact, extended formulation that yields valid inequalities of **pitch** $\leq k$, for fixed k? X
- Compact, extended formulation that yields valid inequalities with **coefficients** in $0, 1, \ldots, k$, for fixed k?

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- Polynomial-time **near** separation over valid inequalities with **coefficients** in $\mathbf{0, 1, \ldots, k}$, for fixed $\boldsymbol{k}$.

Given $\boldsymbol{y}$, either

- Find a valid inequality with coefficients in $\mathbf{0, 1, \ldots, k}$, violated by $\boldsymbol{y}$, or
- Certify that $\boldsymbol{\alpha^T y \geq \alpha_0} - \mathrm{o}(\mathbf{1})$ for all valid $\alpha^T x \geq \alpha_0$ with $\alpha_j \in \{0, 1, \ldots, k\}$ for all j.

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- Polynomial-time **near** separation over valid inequalities with **coefficients** in $\mathbf{0, 1, \ldots,} \boldsymbol{k}$, for fixed $\boldsymbol{k}$.

Given $\boldsymbol{y}$, either

- Find a valid inequality with coefficients in $\mathbf{0, 1, \ldots,} \boldsymbol{k}$, violated by $\boldsymbol{y}$, or
- Certify that $\boldsymbol{\alpha^T y \geq \alpha_0} - \mathrm{o}\mathbf{(1)}$ for all valid $\alpha^T x \geq \alpha_0$ with $\alpha_j \in \{0, 1, \ldots, k\}$ for all j. e.g. $\mathrm{o}(1) = \boldsymbol{O(1/n)}$

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- For $\boldsymbol{k} = \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}$, solve minimum-knapsack problem

$$\min \sum_j y_j z_j \tag{8}$$

$$\text{s.t.} \sum_{j \neq k} w_j z_j \geq w^*, \qquad z \text{ binary} \tag{9}$$

$$\boldsymbol{z_k} = \mathbf{1},\ \boldsymbol{z_j} = \mathbf{0}\ \forall\, j \text{ with } w_j > w_k \tag{10}$$

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So, get approximate separation, with violation if objective < 2

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Given $\boldsymbol{y}$, does it satisfy every valid inequality $\mathbf{2}\sum_{j \in T} \boldsymbol{x_j} + \sum_{j \in S} \boldsymbol{x_j} \geq \mathbf{2}$?

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Stronger: $2(x_1 + x_2) + x_3 + x_4 + x_5 \geq 2$

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- To separate $\boldsymbol{y}$, for $\boldsymbol{k = 1, 2, \ldots, n}$, solve minimum-knapsack problem

$$\min \ 2 \sum_{j \in B} \tilde{y}_j z_j \quad + \quad \sum_{j \in L} \tilde{y}_j z_j \qquad (\tilde{y} \ = \ y \ \text{“rounded up”})$$

$$\text{s.t.} \ \sum_{j \neq k} w_j z_j \ \geq \ w^*, \qquad z \ \text{binary}$$

$$\boldsymbol{z_k = 1}, \qquad \boldsymbol{L \doteq \{ j : w_j \leq w_k \}} \quad \boldsymbol{B \doteq \{ j : w_j > w_k \}}$$

General case? First, coefficients in 0, 1, 2, 3

Example: $8x_1 + 5x_2 + 4x_3 + 4x_4 + 4x_5 \ \geq \ 13$ (the knapsack)

Valid: $x_1 + 2x_2 + x_3 + x_4 + x_5 \ \geq \ 3$ (non-monotone)

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Separation by **enumeration** of errors; each case is a knapsack; $\boldsymbol{O(n^2)}$ cases

General case? (coefficients in 0, 1,2, ..., k)

Basic principle: an inequality

$$k\,x(S_k) + (k-1)\,x(S_{k-1}) + \ldots + x(S_1) \;\geq\; k \tag{11}$$

is **equivalent** to its set of covers **–**

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Application 2: polynomial optimization problems and NN training

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\begin{aligned}
& \min \;\; c^T x \\
\text{s.t.} \quad & f_i(x) \le 0, \qquad i = 1, \ldots, m \qquad \text{(polynomial ineq.)} \\
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- **Intersection graph**

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  A vertex for each variable and an edge anytime two variables appear in the same $\boldsymbol{f_i}$
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- **Tree-width** of a graph **G**
  Minimum clique number (minus one) over all chordal supergraphs of G

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Theorem (B. and Muñoz 2015, SIOPT 2018).

Suppose:

the intersection graph has tree-width $\boldsymbol{\omega}$ and the $\boldsymbol{f_i}$ of degree $\le \boldsymbol{\rho}$.

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Then, for every $\mathbf{0} < \boldsymbol{\epsilon} < \mathbf{1}$ there is a **disjunctive LP** relaxation with

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Theorem (B. and Muñoz 2015, SIOPT 2018).

Suppose:

the intersection graph has tree-width ω and the f_i of degree $\leq \rho$.

Then, for every $0 < \epsilon < 1$ there is a **disjunctive LP** relaxation with

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Polynomial optimization:

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Optimality and feasibility errors $O(\epsilon)$ (additive)

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$$O\left(2^w D^{nw} \text{poly}(D, n, w)\right)$$

Polynomial in the size of the data set, for fixed $\boldsymbol{n}, \boldsymbol{w}$

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Theorem. For any k, n, w, ϵ **approximate LP** of size

$$O\left(\left(\frac{4}{\epsilon}\right)^{O((k-1)w^2 + nw)} \mathrm{poly}(D, n, w, k) \right)$$