Solving Nonlinear Problems via Disjunctions

Daniel Bienstock

Columbia University

October 2019

Some of my papers that were influenced by Egon's work and which relate to today's talk

- *Subset Algebra Lift Operators for 0-1 Integer Programming*, with M. Zuckerberg (2004).
- *Tree-width and the Sherali-Adams operator*, with N. Özbay (2004)
- *Strong formulations for convex functions over nonconvex sets*, with A. Michalka (2014).
- *LP formulations for polynomial optimization problems*, with G. Muñoz (2015).
- *Outer-Product-Free Sets for Polynomial Optimization and Oracle-Based Cuts*, with C. Chen and G. Muñoz (2018).
- *Principled Deep Neural Network Training through Linear Programming*, with G. Muñoz and S. Pokutta (2019).

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We wanteed to talk about ongoing computational work

But it is not ready

So we will settle for some theory

Approximate optimization of well-behaved functions

Prototype problem:

$$\begin{aligned} \min \ & c^T x \\ \text{s.t. } & f_i(x) \leq 0, \qquad i = 1, \ldots, m \\ & x \in [0,1]^n, \ x_j \in \{0,1\}, \ j \in J \end{aligned}$$

Each $\boldsymbol{f_i}$ is "well-behaved": Lipschitz constant $\boldsymbol{\mathcal{L}_i}$
Note: it appears redundant to say that some variables are binary

Toolset:

- **Intersection graph**
  A vertex for each variable and an edge whenever two variables appear in the same $\boldsymbol{f_i}$
- **Tree-width** Min clique number (minus one) over all chordal supergraphs of G

Prototype problem:

$$\begin{aligned} \min \ & c^T x \\ \text{s.t. } & f_i(x) \le 0, \qquad i = 1, \ldots, m \\ & x \in [0,1]^n, \ x_j \in \{0,1\}, \, j \in J \end{aligned}$$

An extension of work in B. and Muñoz 2015, SIOPT 2018.

Suppose:

the intersection graph has tree-width ω and $\boldsymbol{f_i}$ has Lipschitz constant $\boldsymbol{\mathcal{L}_i} \le \boldsymbol{\mathcal{L}}$.

If problem is feasible, for every $\mathbf{0} < \epsilon < \mathbf{1}$ there is an **LP** relaxation with

$\boldsymbol{O\left((\mathcal{L}/\epsilon)^{\omega+1}\,(n+m)\log(\mathcal{L}/\epsilon)\right)}$ variables and constraints, and

optimality and feasibility errors $\boldsymbol{O(\epsilon)}$

Main technique: approximation through pure-binary problems

Glover, 1975 (extended)
Let x be a variable, with bounds $0 \le x \le 1$. Let $0 < \gamma < 1$. Then we can approximate

$$x \approx \sum_{h=1}^{K} 2^{-h} y_h$$

where each y_h is a **binary variable**. In fact, choosing $K = \lceil \log_2 \gamma^{-1} \rceil$, we have

$$x \le \sum_{h=1}^{K} 2^{-h} y_h \le x + \gamma.$$

→ Given a mixed-integer well-behaved problem
apply this technique to each continuous variable x_j

The main result

$$\begin{aligned} c^* \doteq \min \ & c^T x \\ \text{s.t. } & f_i(x) \le 0, \qquad i = 1, \ldots, m \\ & x \in [0,1]^n, \quad x_j \in \{0,1\}, j \in J \end{aligned}$$

- Intersection graph with tree-width ω,
- Each f_i has Lipschitz constant $\mathcal{L}_i \le \mathcal{L}$.

For $0 < \epsilon < 1$, an **LP** relaxation of size $O\left((1/\epsilon)^{\omega+1} (n+m) \log(1/\epsilon)\right)$

yields $\hat{x} \in [0,1]^n, \ \hat{x}_J \in \{0,1\}^J$ with

- $c^T \hat{x} \ \le \ c^* + O(\|c\|_1 \epsilon)$
- $f_i(\hat{x}) \le O(\mathcal{L}_i \epsilon), \qquad i = 1, \ldots, m$

How is it done?

- **Lifting hierarchies** in 0-1 linear integer programming
- Balas, Lovász-Schrijver, Sherali-Adams, Balas-Ceria-Cornuéjols
- Specific version: zeta-function idea of Lovász-Schrijver

$$x \in \{0,1\}^n \ \rightarrow \ X \in \{0,1\}^{2^{[n]}}$$

Applications

- Fixed-charge network flow problems, on networks with small treewidth

- ACOPF problem, on networks with small treewidth

- In both cases, LP of size $\boldsymbol{O}\left((\mathbf{1}/\epsilon)^{\omega+1}\, \boldsymbol{n} \log(\mathbf{1}/\epsilon)\right)$

$$\begin{aligned} \omega = &\ \text{treewidth of network} \\ \leq &\ \text{treewidth of intersection graph of formulation} \end{aligned}$$

- A famous scientist: "lifting hierarchies do not work"

Another “application”

Positivestellensätze for semi-algebraic systems

- Given a system of **polynomial** inequalities $\boldsymbol{p_i(x) \leq 0, \quad i \in I}$
- A positivstellensatz is a **proof of infeasibility** of the system
- Under assumptions (e.g. compactness) such statements exist (rich literature)

(2017) Amir-Ali Ahmadi, Georgina Hall $\Rightarrow$ a new positivstellensatz, under compactness (containment in a known ball)

The lifted LP relaxation hierarchy gives a similar result

We are given a system

$$f_i(x) \le 0, \qquad i \in I \tag{1a}$$
$$x \in [0,1]^n, \quad x_j \in \{0,1\}, j \in J. \tag{1b}$$

where each $\boldsymbol{f_i}$ is a function with Lipschitz constant $\boldsymbol{\mathcal{L}}$.

$\rightarrow$ If the system is **infeasible**, is there a short proof thereof? Yes:

1. Given ϵ, our **LP** $\mathbf{= LP(\epsilon)}$, **if feasible**, yields $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\boldsymbol{\epsilon})$ with

$$\boldsymbol{f_i(\hat{x}) \le O(\mathcal{L}\epsilon) \quad \forall i \in I, \;\; \hat{x} \in [0,1]^n, \;\; \hat{x}_j \in \{0,1\}, j \in J}$$

2. Let $\epsilon \rightarrow \mathbf{0}$: $\hat{\boldsymbol{x}}$ has accumulation point $\boldsymbol{x^*}$
3. But system (1a), (1b) is infeasible, so $\boldsymbol{x^*}$ cannot be feasible!
4. **Conclusion:** **LP(ϵ) is not feasible for** some ϵ small enough
5. But **LP(ϵ)** is a relaxation for (1a), (1b)

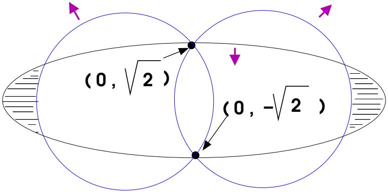
A bad QCQP

```
Maximize x2

s.t.
       x3 - x1 = -1
       x4 - x1 =  1
   o1: [ x3^2 + x2^2 - sneaky^2 ] >= 3
   o2: [ x4^2 + x2^2 ] >= 3
   e1: [ .1 x1^2 + x2^2 ] <= 2

  bad:  distraction + [ sneaky^2 ]         >= 0.1

joke1: - a + [ distraction^2 + sneaky^2 ]  <= 0.0

cruel:  - sneaky + [ a^2 + sneaky^2 ]      <= 0.0

Bounds
x1 free
x2 free
x3 free
x4 free
End
```

→ Gurobi, SCIP, other codes: value ≈ **1.4142**
Wrong, actual value ≈ **1.22**

What's going on?

$$\begin{array}{rrcl} & \max & x_2 & \\ \text{s.t.} & (x_1-1)^2 + x_2^2 & \geq & 3 \\ & (x_1+1)^2 + x_2^2 & \geq & 3 \\ & \frac{x_1^2}{10} + x_2^2 & \leq & 2 \end{array}$$



But this is NOT the problem being solved ..

The bad QCQP

```
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```

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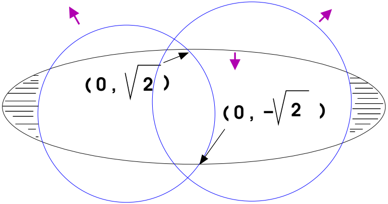
Wrong, actual value ≈ **1.22**

Actually THIS is the problem being solved:

$$
\begin{array}{rrcl}
 & \max \quad x_2 & & \\
\text{s.t.} & (x_1-1)^2 + x_2^2 & \geq & 3 + \phi \qquad\qquad (\phi > 0) \\
 & (x_1+1)^2 + x_2^2 & \geq & 3 \\
 & \dfrac{x_1^2}{10} + x_2^2 & \leq & 2
\end{array}
$$



Fri.Oct.25.162243.2019@littleboy