Some theorems on nonconvex optimization

Daniel Bienstock and Alexander Michalka

Columbia University

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1.

Cardinality constrained, convex quadratic programming.

$$
\begin{aligned}
\min \ & x^T Q x + c^T x \\
\text{s.t.} \ & Ax \leq b \\
& x \geq 0, \quad \|x\|_0 \leq k
\end{aligned}
$$

$\|x\|_0$ = number of nonzero entries in x.

- $Q \succeq 0$
- $x \in \mathbb{R}^n$ for n possibly large
- k relatively small, e.g. $k = 100$ for $n = 10000$
- VERY hard problem – just getting good bounds is tough

2b.

Sparse vector in column space (Spielwan, Wang, Wright '12)

Given a matrix $Y \in \mathbb{R}^{n \times p}$ (n large)

$$\begin{aligned} \min \quad & \|Y - AX\|_2 \\ \text{s.t.} \quad & A \in \mathbb{R}^{n \times n},\ X \in \mathbb{R}^{n \times p} \\ & X \quad \text{``sparse''} \end{aligned} \tag{1}$$

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- Both A and x are variables
- Usual "convexification" approach may not work
- Again, looks VERY hard

. AC-OPF problem in rectangular coordinates

Given a power grid, determine voltages at every node so as to minimize a convex objective

$$\begin{aligned} \min \ & v^T A v \\ \text{s.t.} \ & L_k \ \leq \ v^T F_k v \ \leq \ U_k, \quad k = 1, \ldots, K \\ & v \in \mathbb{R}^{2n}, \quad (n = \text{number of nodes}) \end{aligned}$$

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- voltages are complex numbers; v is the vector of voltages in rectangular coordinates (real and imaginary parts)
- $A \succeq 0$
- n could be in the tens of thousands, or more
- the F_k are very sparse (neighborhood structure for every node)
- Problem HARD when grid under distress and $L_k \approx U_k$.

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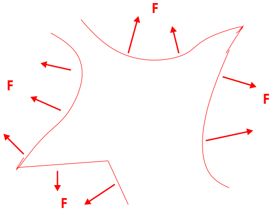

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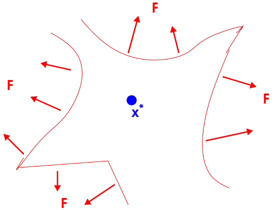
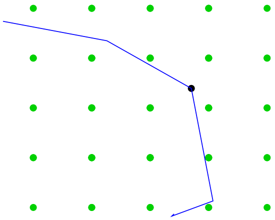


x^* solves $\min \left\{ Q(x), \quad : \quad x \in \hat{F} \right\}$ where $F \subset \hat{F}$ and $\hat{F}$ convex

Lattice-free cuts for **linear** integer programming

Generic problem: $\min \; c^T x, \quad s.t. \quad Ax \leq b, \quad z \in Z^n$



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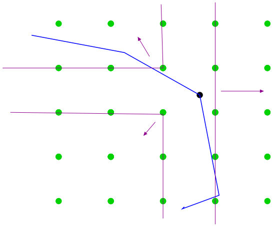

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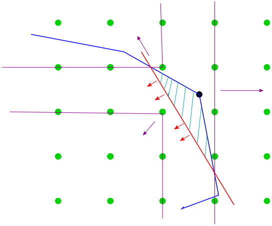

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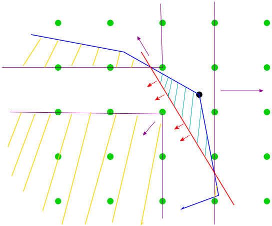

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- Optimal solution at **extreme point** (x^*, z^*) of $\mathrm{conv}\,\{(x,z) \,:\, z \geq Q(x),\ x \in F\}$
- So $x^* \in F$

Exclude-and-cut

$$\min z, \quad s.t. \;\; z \geq Q(x), \;\; x \in F$$

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2. Find an **open** set S s.t. $x^* \in S$ and $S \cap F = \emptyset$.
Examples: lattice-free sets, geometry

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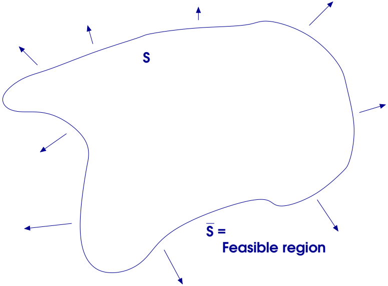
3. Add to the formulation an inequality $\mathbf{az} \;+\; \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{x} \;\geq\; \boldsymbol{\alpha}_{\mathbf{0}}$ valid for

$$\{\,(x,z)\,:\, x \in \overline{S},\;\; z \geq Q(x)\,\}$$

but violated by (x^*, z^*).

Valid **linear** inequalities for $\{\, (x,z) \; : \; x \in \overline{S}, \;\; z \geq Q(x) \,\}$.



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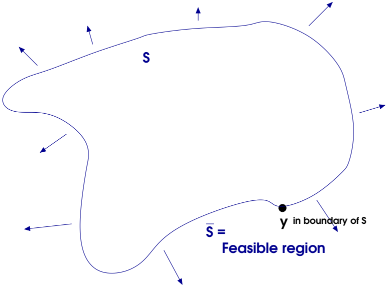

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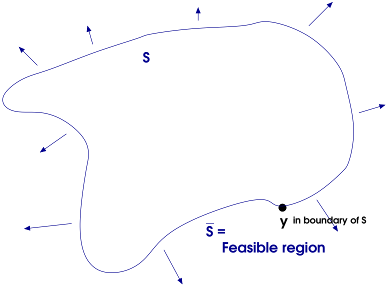

First order inequality:

$$z \ \geq \ [\nabla Q(y)]^T (x - y) + Q(y)$$

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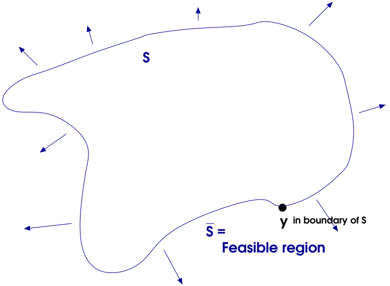

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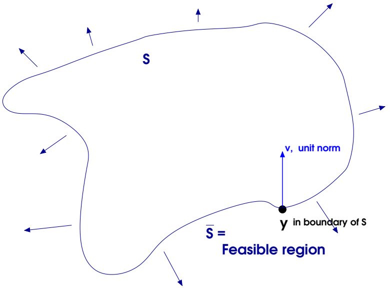

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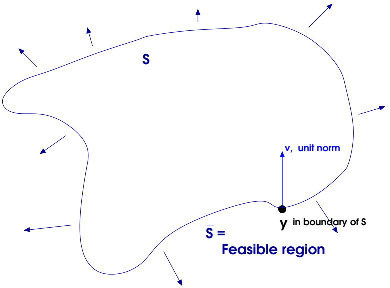

Lifted first order inequality, for $\alpha \geq 0$:

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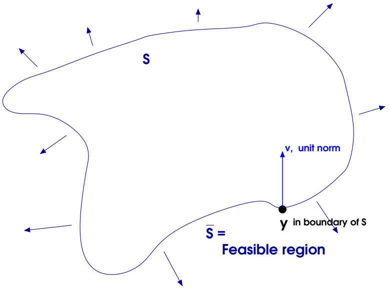

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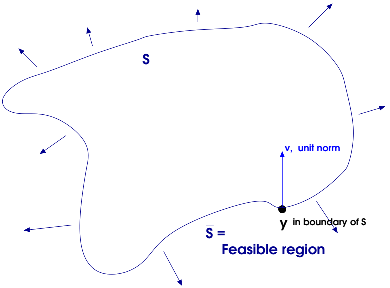

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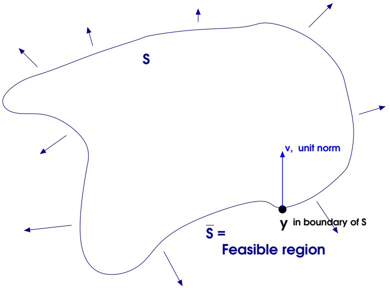

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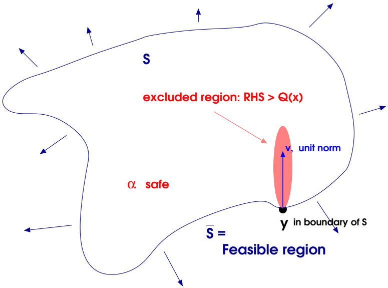

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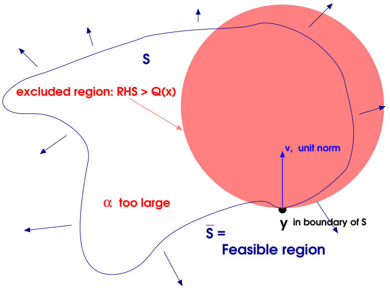

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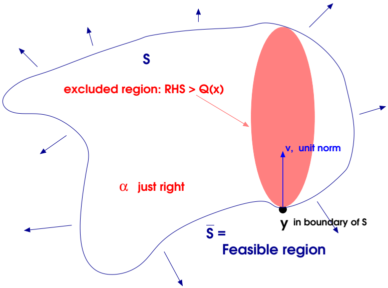

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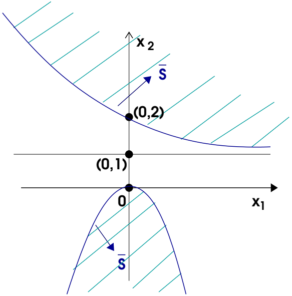
Theorem. If Q is convex and differentiable, then $conv(\mathcal{F})$ is given by

$$\begin{aligned} Q(x) &\geq [\nabla Q(y)]^T (x-y) + Q(y) \qquad \forall y \\ Q(x) &\geq [\nabla Q(y)]^T (x-y) + Q(y) + \alpha^* v^T (x-y) \\ &\qquad \forall v \text{ and } y \in \partial S. \end{aligned}$$

(abridged)

$S = \{x \in \mathbb{R}^2 \,:\, -x_1^2 \leq x_2 \leq 1 + e^{-x_1}\}$, $Q(x) = x_2 + e^{-x_2} - 1$.



With $v = (0, 1)^T$, the lifted first-order inequality at $(0, 0)$ is $\mathbf{z} \geq \alpha^* \mathbf{x_2}$
$\Rightarrow \alpha^* = e^{-1}$.

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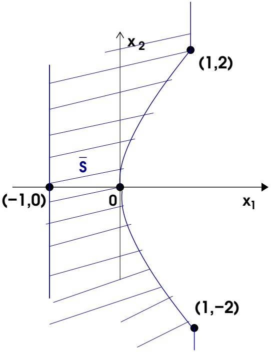

If $v = (1,0)^T$, lifted first-order inequality at 0 is $\mathbf{z} \geq \boldsymbol{\alpha}^* \mathbf{x_1}$ $\Rightarrow$ $\alpha^* = 2$.

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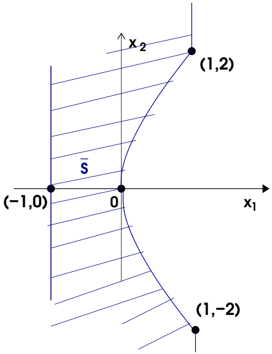

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Because for $\alpha^* = 2R$, $Q(x) \leq \alpha^* x_1$ iff $|x_2| \leq (2Rx_1 - x_1^2)^{1/2}$

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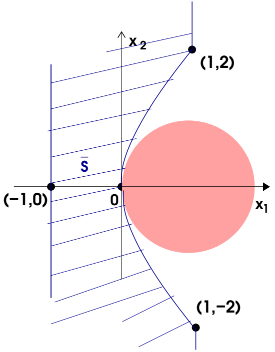

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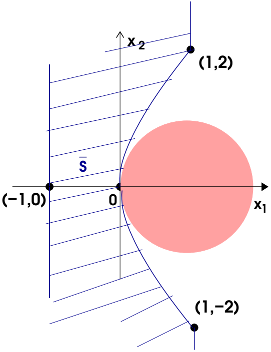

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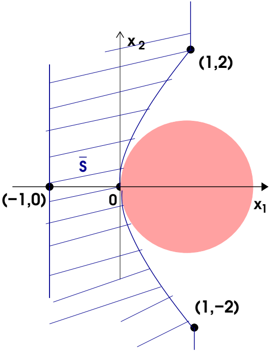

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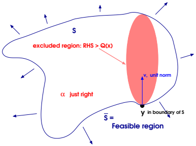
But **fails to hold** for $R > 1$ and $x_1 \approx 0$!

Lifted first-order inequality at $y \in \partial S$, in the direction of v: $Q(x) \ \geq \ [\nabla Q(y)]^T(x-y) + Q(y) + \alpha^* v^T(x-y)$



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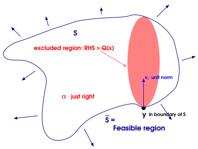

Theorem. If

- $Q(x)$ grows faster than linearly in every direction, and
- There is a ball with interior in the infeasible region, but containing y at its boundary

then the quantity α^* is a "max" and not just a "sup", i.e. the lifted inequality is tight at some point other than y

Quadratics in action

Lifted first-order inequalities for $\mathcal{F} \; = \; \{ (x, z) \, : \, x \in \overline{S}, \; z \geq \|x\|^2 \}$.

Separation problem. Given $(x^*, z^*) \in \mathbb{R}^n \times \mathbb{R}$, find a lifted-first order inequality maximally violated by (x^*, z^*) (if any)

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Theorem: We can separate in polynomial time when:

- $\bar{S}$ (or S) is a union of polyhedra
- S is a convex ellipsoid or paraboloid (many cases)

Special case: complement of an ellipsoid

Lifted first-order inequalities for
$\mathcal{F} \;=\; \{\,(x,z)\,:\, x^T A x - 2b^T x + c \geq 0,\; z \geq \|x\|^2\,\}.$ Here, $A \succ 0$.

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Theorem. The **strongest** lifted first-order inequality at $\bar{x} \in \mathbb{R}^n$ is:

$$z \ \geq \ 2[(I - \lambda^{-1} A)\bar{x} + \lambda^{-1} b]^T (x - \bar{x}) \ + \ \bar{x}(I - \lambda^{-1} A)\bar{x} + 2\lambda^{-1} b^T \bar{x} - \lambda^{-1} c$$

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The right-hand side is the **first-order** (tangent), at $\bar{x}$, for the convex quadratic

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Corollary:
$\mathrm{conv}(\mathcal{F}) = \{(x,z) \ : \ z \geq x(I - \lambda^{-1}A)x + 2\lambda^{-1}b^T x - \lambda^{-1}c, \ z \geq \|x\|^2\}$.

Obtained by Modaresi and Vielma (2013)

But ... Exclude-and-cut, again

$$\min z, \quad s.t. \;\; z \geq Q(x), \;\; x \in F$$

0. $\hat{F}$: a **convex relaxation** of $\mathrm{conv}\left\{(x, z) \, : \, z \geq Q(x), \;\; x \in F\right\}$

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Examples: lattice-free sets, geometry

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3. Add to the formulation an inequality $\mathbf{az} \; + \; \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{x} \; \geq \; \boldsymbol{\alpha_0}$
valid for

$$\{\,(x,z) \,:\, x \in \overline{S}, \;\; z \geq Q(x)\,\}$$

but violated by (x^*, z^*).

A classical problem: the trust-region subproblem

$$\begin{aligned} \min \quad & x^T A x + b^T x + c \\ \text{s.t.} \quad & \|x\|^2 \ \leq \ 1, \qquad x \in \mathbb{R}^n \end{aligned}$$

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Sturm and Zhang (2000): two extensions are polynomially solvable:

$$\begin{aligned} \min \quad & x^T A x + b^T x + c \\ \text{s.t.} \quad & \|x\|^2 \ \leq \ 1, \quad \|x - x^0\|^2 \ \leq \ r \end{aligned}$$

(one additional ball inequality), and

$$\begin{aligned} \min \quad & x^T A x + b^T x + c \\ \text{s.t.} \quad & \|x\|^2 \ \leq \ 1, \qquad c^T x \ \leq \ c^0 \end{aligned}$$

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Ye and Zhang (2003): two **parallel** linear inequalities are added:

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poly-time solvable if no two linear constraints intersect within unit ball

$$\forall i \neq j, \quad \{x \, : \, a_i^T x = b_i\} \cap \{x \, : \, a_j^T x = b_j\} \cap \{x \, : \|x\|^2 \le 1\} \; = \; \emptyset$$

A generalization

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\textbf{(TLIN):} \quad \begin{aligned} \min \quad & x^T A x + b^T x + c \\ \text{s.t.} \quad & \|x\|^2 \leq 1 \\ & a_i^T x \leq b_i \qquad i = 1, \ldots, m \\ & x \in \mathbb{R}^n. \end{aligned}
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Theorem: Problem **TLIN** can be solved in time polynomial in the problem size and F^*.

A stronger generalization

TGEN(S,K):

$$\begin{aligned} \min \quad & x^T A x + b^T x + c \\ \text{s.t.} \quad & \|x - x^k\|^2 \leq f_k \qquad k \in S \\ & \|x - y^k\|^2 \geq g_k \qquad k \in K \\ & a_i^T x \leq b_i \qquad i = 1, \ldots, m \end{aligned}$$

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2. For every **fixed** $|K| \geq 0$, and $m \geq 0$, problem **TGEN($\emptyset$, K)** polytime solvable.

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(SODA 2014)