Optimizing Convex Functions over Non-Convex Domains

Daniel Bienstock and Alexander Michalka

Columbia University

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Introduction

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- F is constrained in a nasty way, e.g.

$$x_1 \;-\; 3\sin(x_2) + 2\cos(x_3) \;=\; 4$$

Or,

Convex constraint:

$$Q(x) \ \le q, \quad \text{and} \quad x \in F,$$

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3. Add to the formulation an inequality $\mathbf{az} \; + \; \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{x} \; \geq \; \boldsymbol{\alpha_0}$ valid for

$$\{ \, (x, z) \; : \; x \in \mathbb{R}^n - S, \;\; z \geq Q(x) \, \}$$

but violated by (x^*, z^*).

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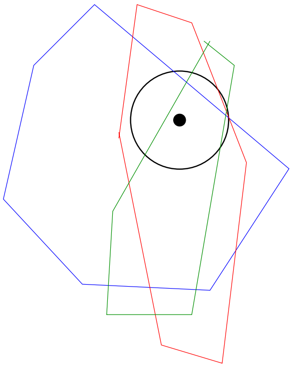
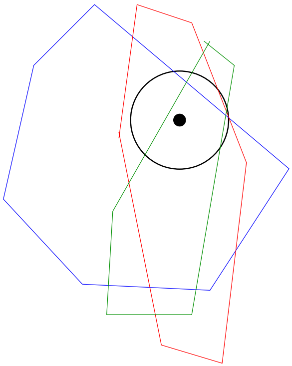
(application: X-ray lythography; see Ahmadia (2010))

(yes, it is NP-hard)



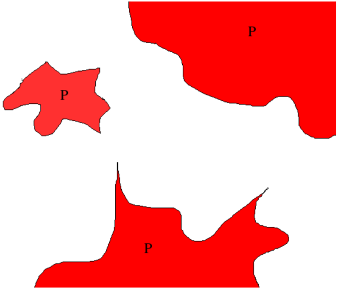


- Typical values for d (dimension): less than 20; usually even smaller
- Typical values for K (number of polyhedra): possibly hundreds, but often less than 50
- Very hard problem

First problem setting

- Let $Q(x)$ be a **strongly convex** function on $\mathbb{R}^d$,
- Let $P \subset \mathbb{R}^d$ be such that each connected component is
  - homeomorphic to a (positive radius) ball or a half-plane
  - so, closed and nonempty interior

We also assume that $\mathbb{R}^d - P \neq \emptyset$.



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$$\left\{(x, q) \in \mathbb{R}^{d+1} \, : \, Q(x) \leq q, \quad x \in \mathbb{R}^d - \mathrm{int}(P)\right\}.$$

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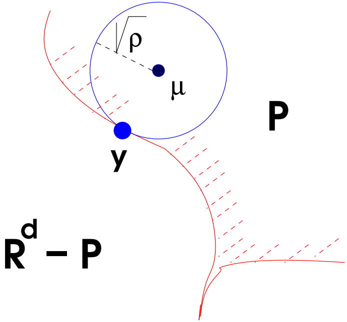
How can we use the structure of P to strengthen the inequality?

Definition:

Given $y \in \partial P$, say P is **locally flat** at y if
$\exists\ \mathcal{B}(\mu, \sqrt{\rho}) \subseteq P$ with $\|\mu - y\|^2 = \rho$ and $\rho > 0$.



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- Inequality is tight at $(y, Q(y))$, and cuts-off points $(x, Q(x))$ and $x \in \mathrm{int}(P)$.
- Largest possible $\boldsymbol{\alpha}$: "lifted first-order inequality".

$$\left\{(x, q) \in \mathbb{R}^{d+1} \,:\, Q(x) \leq q, \quad x \in \mathbb{R}^d - \mathrm{int}(P)\right\}$$

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Theorem.

Any linear inequality valid for S is dominated by a lifted first-order inequality. More precisely,

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How do we make this computationally practicable?

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change in coordinates $\longrightarrow$

$$S \doteq \left\{(x,q) \in \mathbb{R}^{d+1} \,:\, \sum_{j=1}^{d} x_j^2 \le q, \;\; x \in \mathbb{R}^d - \mathrm{int}(P)\right\}.$$

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$$q \ \ge \ 2y^T x \ - \ \|y\|^2 \ - \ \alpha(a_i^T x - b_i)$$

for $\alpha > 0$ appropriately chosen.

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- **Theorem:** Let $(\hat{x}, \hat{q}) \in \mathbb{R}^{d+1}$ with $\hat{v} \in \mathrm{int}(P)$.

  We can compute a lifted first-order inequality maximally violated by $(\hat{x}, \hat{q})$, by solving m linearly constrained convex quadratic programs on $O(d)$ variables.

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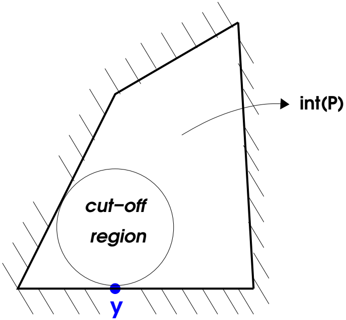
This describes the interior of a **ball**, which must be contained in $\mathrm{int}(P)$

Geometrical characterization



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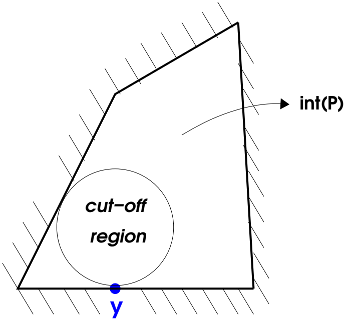

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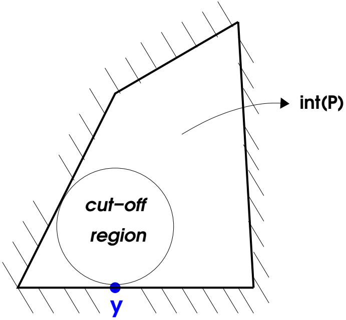

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Separation:
solve SOCP, use SOCP "Farkas Lemma", get linear cut

Second setting: separating across a quadratic set

For $\mathbf{A} \succ \mathbf{0}$, polynomially separable linear inequality description for:

$$\{(x, q) \in \mathbb{R}^{d+1} \,:\, \sum_{j=1}^{d} x_j^2 \leq q, \;\; x^T A x - 2c^T x + b \geq 0\}$$

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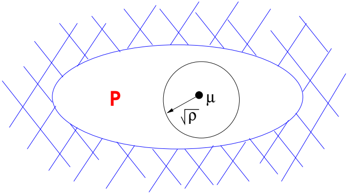

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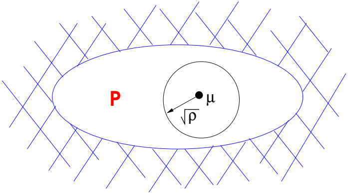

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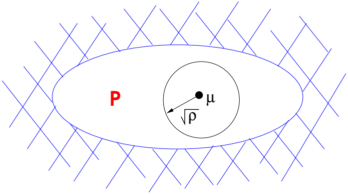

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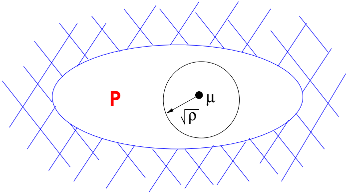

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$$\min_{\mu,\rho} \quad \|\mu\|^2 - \rho - 2\bar{x}^T\mu$$

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Theorem:
Optimal choices for μ and ρ are given by:

$$\hat{\mu} = \hat{\theta}b + (I - \hat{\theta}A)\bar{x}$$

and

$$\hat{\rho} = \|\hat{\mu}\|^2 - 2\bar{x}^T\hat{\mu} + \|\bar{x}\|^2 - \hat{\theta}(\bar{x}^T A\bar{x} - 2b^T\bar{x} + c).$$

Here, $\hat{\theta} \;=\; \frac{1}{\lambda_{\max} A}$.

Separating accross general quadratics

$$\Pi \ \doteq \ \{ (x, w, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \ : \ z \ge x^T Q x + q^T x, \quad w \le x^T A x \}$$

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Linear transformation $\rightarrow$ Π is the set of points $\in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$ s.t.

$$z \geq \|x\|^2 + q^T x, \quad w \leq x^T \Lambda x \quad (\Lambda > 0).$$

Write $P \doteq \{(x, w) \in \mathbb{R}^d \times \mathbb{R} \; : \; x^T \Lambda x - w \leq 0\}$, and for $\mu \in \mathbb{R}^d$, $\nu \in \mathbb{R}$,

$$M(\mu, \nu) \doteq \{(x, w) \in \mathbb{R}^d \times \mathbb{R} \; : \; \lambda_{max} \|x - \mu\|^2 + (\nu - w) \leq 0\}.$$

Then

$x \in \mathbb{R}^d - \mathrm{int}(P)$ iff $x \in \mathbb{R}^d - \mathrm{int}(M(\mu, \nu))$, for all μ, ν with $M(\mu, \nu) \subseteq P$.

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So, valid inequality for any μ, ν with $M(\mu,\nu) \subseteq P$:

$$\lambda_{max}\|\mu\|^2 - \lambda_{max}(2\mu + q)^T x + (\nu - w) + \lambda_{max} z \geq 0$$

Separation problem, given $(\bar{x}, \bar{w}) \in \mathrm{int}(P)$

$$\bar{w} - \nu - \lambda_{max}\|\mu\|^2 + 2\lambda_{max}\bar{x}^T\mu + 2\lambda_{max} q^T \bar{x}$$

subject to: $\nu + \min_x \{ \lambda_{max} \|x - \mu\|^2 - x^T \Lambda x \} \geq 0.$

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Lemma: This problem can be explicitly solved in polynomial time.

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Theorem. Eigenspace not necessary for poly-time separation (only max eigenvalue of A).

Example: $f(x) \doteq 2(x_1x_2 + x_1x_3 + x_2x_3)$ over $[0,1]^3$.

McCormick relaxation gives zero lower bound on $f(\bar{x})$, where

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Above techniques, plus (for example) one disjunction prove

$$f(\bar{x}) \geq .43599,$$

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Vielen Dank!