LP formulations for mixed-integer polynomial optimization problems

Daniel Bienstock and Gonzalo Muñoz, Columbia University

An application: the Optimal Power Flow problem (ACOPF)

Input: an undirected graph G.

- For every vertex i, **two** variables: $\boldsymbol{e_i}$ and $\boldsymbol{f_i}$
- For every edge $\{k, m\}$, **four** (specific) quadratics:

$$H^P_{k,m}(e_k, f_k, e_m, f_m), \quad H^Q_{k,m}(e_k, f_k, e_m, f_m)$$

$$H^P_{m,k}(e_k, f_k, e_m, f_m), \quad H^Q_{m,k}(e_k, f_k, e_m, f_m)$$

$$\textbf{min} \quad \sum_k F_k \left(\sum_{\{k,m\} \in \delta(k)} H^P_{k,m}(e_k, f_k, e_m, f_m) \right)$$

$$\text{s.t.} \quad L^P_k \leq \sum_{\{k,m\} \in \delta(k)} H^P_{k,m}(e_k, f_k, e_m, f_m) \leq U^P_k \quad \forall k$$

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$$V^L_k \leq \|(e_k, f_k)\| \leq V^U_k \quad \forall k.$$

Function F_k in the objective: convex quadratic

Complexity

Theorem (2011) Lavaei and Low: OPF is (weakly) NP-hard on trees.

Theorem (2014) van Hentenryck et al: OPF is NP-hard on trees.

Theorem (2007) B. and Verma (2009): OPF is strongly NP-hard on general graphs.

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Recent insight: use the SDP relaxation (Lavaei and Low, 2009 + many others)

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Reformulation of ACOPF:

$$\begin{aligned} \min \quad & F \bullet W \\ \text{s.t.} \quad & A_i \bullet W \ \leq \ b_i \quad i = 1, 2, \ldots \\ & W \succeq 0, \quad W \text{ of rank } 1. \end{aligned}$$

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SDP Relaxation of OPF:

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Fact: The SDP relaxation is often good! ("near" rank 1 solution).

But: the SDP relaxation is always slow on large graphs

- Real-life grids $\longrightarrow > 10^4$ vertices
- SDP relaxation of OPF does not terminate

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Fact? Real-life grids have **small tree-width**

Definition 1: A graph has treewidth $\leq \boldsymbol{w}$ if it has a chordal supergraph with clique number $\leq \boldsymbol{w+1}$

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(Seymour and Robertson, late 1980s)

Tree-width

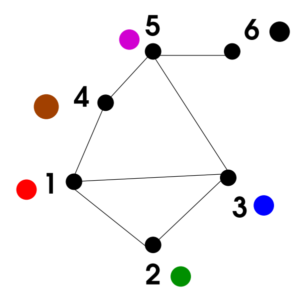
Let G be an undirected graph with vertices $V(G)$ and edges $E(G)$.

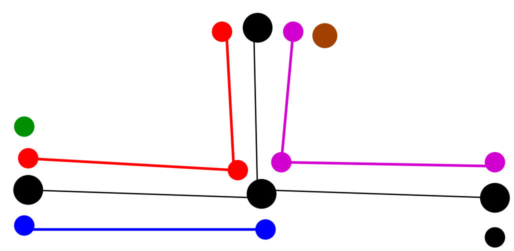
A tree-decomposition of G is a pair (T, Q) where:

- T is a tree. **Not** a subtree of G, just a tree
- For each vertex t of T, Q_t is a subset of $V(G)$. These subsets satisfy the two properties:

  (1) For each vertex v of G, the set $\{t \in V(T) : v \in Q_t\}$ is a **subtree** of T, denoted T_v.

  (2) For each edge $\{u, v\}$ of G, the two subtrees T_u and T_v **intersect**.
- The **width** of (T, Q) is $\max_{t \in T} |Q_t| - 1$.





$\rightarrow$ two subtrees T_u, T_v may overlap even if $\{u, v\}$ is **not** an edge of G

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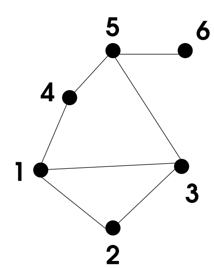

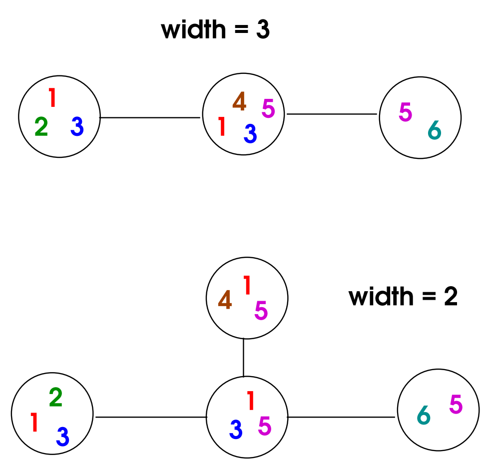

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Matrix-completion Theorem

gives fast SDP implementations:

Real-life grids with $\approx 3 \times 10^3$ vertices: $\rightarrow$ 20 minutes runtime

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$\rightarrow$ Perhaps low tree-width yields **direct** algorithms for ACOPF itself?

That is to say, not for a relaxation?

Much previous work using structured sparsity

- Bienstock and Özbay
- Wainwright and Jordan
- Grimm, Netzer, Schweighofer
- Laurent
- Lasserre et al
- Waki, Kim, Kojima, Muramatsu

older work ...

- Lauritzen (1996): tree-junction theorem
- Bertele and Brioschi (1972): nonserial dynamic programming
- Bounded tree-width in combinatorial optimization (too many authors)
- Fulkerson and Gross (1965): matrices with consecutive ones

ACOPF, again

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$$\textbf{min} \quad \sum_k w_k$$

$$\text{s.t.} \quad L^P_k \;\le\; \sum_{\{k,m\}\in\delta(k)} H^P_{k,m}(e_k, f_k, e_m, f_m) \;\le\; U^P_k \quad \forall k$$

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$$w_k \;=\; F_k\left(\sum_{\{k,m\}\in\delta(k)} H^P_{k,m}(e_k, f_k, e_m, f_m)\right) \quad \forall k$$

Graphical QCQP

Input: an undirected graph G.

- For every vertex k, a set of variables: $\{x_j \ : \ j \in I(k)\}$
- For every edge $e = \{k, m\}$, a quadratic

$$H_e(x) \quad = \quad H_e\left(\boldsymbol{\{x_j \ : \ j \in I(k) \cup I(m)}\}\right).$$

- For now, the sets $I(k)$ are disjoint

$$\min \quad \sum_k \sum_{j \in I(k)} c_{k,j} x_j$$

$$\text{s.t.} \quad \sum_{e \in \delta(k)} H_e(x) \ \leq \ b_k \quad \forall k$$

$$0 \leq x_j \leq 1, \quad \forall j$$

→ Easy to solve if graph has small tree-width?

Subset-sum problem

Input: positive integers $p_1, p_2, \ldots, p_n$.

Problem: find a solution to:

$$\sum_{j=1}^{n} p_j x_j \ = \ \frac{1}{2}\sum_{j=1}^{n} p_j$$

$$x_j \in \{0,1\}, \quad \forall j$$

(weakly) NP-hard (well...)

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$$x_j(1 - x_j) \ = \ 0, \quad \forall j$$

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This is a graphical QCQP on a **star** – so treewidth 1.

(Perhaps) approximate solutions?

$\mathbf{\{0,1\}}$ solutions with error $\left(\frac{1}{2}\sum_{j=1}^{n} p_j\right) \epsilon$ in time polynomial in ϵ^{-1}?

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$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & \sum_{e \in \delta(k)} H_e(x) \ \leq \ b_k \quad \forall k \\ & 0 \leq x_j \leq 1, \quad \forall j \end{aligned}$$

Graphical PCLP

Input: an undirected graph G.

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- For every edge $e = \{k, m\}$, a **polynomial**

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Density of a problem: size of **largest** set $\boldsymbol{I(k)}$
Density of ACOPF problems: 3

Graphical, mixed-integer PCLP – or **GMIPCLP**

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- **Density** = size of largest $\boldsymbol{I(k)}$

Theorem 3

For any instance of **GMIPCLP** on a graph with **treewidth** $\boldsymbol{w}$, **density** $\boldsymbol{d}$, **max. degree** $\boldsymbol{\pi}$, and any fixed $\boldsymbol{0 < \epsilon < 1}$, there is a **linear program** of size (rows + columns) $\boldsymbol{O^*(\pi^{wd}\epsilon^{-w}\, n)}$ whose feasibility and optimality error is $\boldsymbol{O(\epsilon)}$
(abridged).

More general: MIPCLP (Basic polynomially-constrained mixed-integer LP)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & P_i(x) \ \le \ b_i \quad 1 \le i \le m \\ & x_j \in \{0,1\} \quad \forall j \in I, \quad 0 \le x_j \le 1, \quad \text{otherwise} \end{aligned}$$

Each $\boldsymbol{P_i(x)}$ is a polynomial.

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Theorem 2

For any instance of **MIPCLP** whose **intersection graph** has treewidth $\boldsymbol{w}$, **max. degree** $\boldsymbol{\pi}$, and any fixed $\boldsymbol{0 < \epsilon < 1}$, there is a **linear program** of size (rows + columns) $\boldsymbol{O^*(\pi^w \epsilon^{-w-1}\, n)}$ whose feasibility and optimality error is $\boldsymbol{O(\epsilon)}$ (abridged).

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Intersection graph of a constraint system: (Fulkerson? (1962?))

- Has a **vertex** for every variably $\boldsymbol{x_j}$
- Has an **edge** $\boldsymbol{\{x_i, x_j\}}$ whenever x_i and x_j appear in the same constraint

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Two graphs:

- G, the graph of the instance
- H, the intersection graph of the constraints

$\rightarrow$ Even if G has small treewidth, H might not

Example: subset sum problem. G is a **star**, H is a **clique**.

Theorem 0

Given an instance of graphical mixed-integer PCLP

- On a graph G of treewidth $\boldsymbol{w}$,
- with density $\boldsymbol{d}$,
- For every vertex k, a set of variables: $\{x_j \, : \, j \in I(k)\}$, for every edge $e = \{k, m\}$, a **polynomial**

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There is an **equivalent**

mixed-integer polynomial optimization problem

whose **intersection graph** has tree-width $\boldsymbol{O(wd)}$.

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There is an **equivalent** **mixed-integer polynomial optimization problem** whose **intersection graph** has tree-width $\boldsymbol{O(wd)}$.

ACOPF problem on small treewidth graph $\longrightarrow$ (generalize)

Graphical QCQP on small treewidth graph and small density $\longrightarrow$ (generalize)

GMIPCLP on small treewidth graph and small density $\longrightarrow$ (generalize, reduce)

Mixed-integer PCLP with small treewidth intersection graph

Basic theorem:

There is a polynomial-time ϵ-approximate algorithm for such problems

Main technique: approximation through pure-binary problems

Glover, 1975 (abridged)

Let $\boldsymbol{x}$ be a variable, with bounds $\mathbf{0} \leq \boldsymbol{x} \leq \mathbf{1}$. Let $\mathbf{0} < \boldsymbol{\gamma} < \mathbf{1}$. Then we can approximate

$$\boldsymbol{x} \approx \sum_{h=1}^{L} 2^{-h} y_h$$

where each $\boldsymbol{y_h}$ is a **binary variable**. In fact, choosing $\boldsymbol{L} = \lceil \log_2 \epsilon^{-1} \rceil$, we have

$$\boldsymbol{x} \leq \sum_{h=1}^{L} 2^{-h} y_h \leq \boldsymbol{x} + \boldsymbol{\epsilon}.$$

$\rightarrow$ Given a mixed-integer polynomially constrained LP (MIPCLP), apply this technique to each continuous variable x_j

Mixed-integer polynomially-constrained LP:

$$\textbf{(P)} \quad \min \quad c^T x$$
$$\text{s.t.} \quad P_i(x) \ \le \ b_i \quad 1 \le i \le m$$
$$x_j \in \{0,1\} \quad \forall j \in I, \quad 0 \le x_j \le 1, \quad \text{otherwise}$$

substitute: $\forall j \notin I$, $\boldsymbol{x_j} \ \rightarrow \ \sum_{h=1}^{L} \mathbf{2}^{-h} \boldsymbol{y_{h,j}}$, where each $\boldsymbol{y_{h,j}} \in \{\mathbf{0}, \mathbf{1}\}$

$$\boldsymbol{L \approx \log_2 \epsilon^{-1}}$$

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$\boldsymbol{L \approx \log_2 \epsilon^{-1}}$

obtain **pure binary problem**:

$$\textbf{(Q)} \quad \min \quad \hat{c}^T z$$
$$\text{s.t.} \quad \hat{P}_i(z) \ \le \ \hat{b}_i \quad 1 \le i \le m$$
$$z_k \in \{0,1\} \quad \forall k$$

If **(P)** has intersection graph of treewidth $\boldsymbol{w}$,
then **(Q)** has intersection graph of treewidth $\boldsymbol{Lw}$.

Mixed-integer polynomially-constrained LP:

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If **(P)** has intersection graph of treewidth $\boldsymbol{w}$,
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Theorem

Consider a pure-binary PCLP with $\boldsymbol{n}$ variables.
If the intersection graph has treewidth $\le \boldsymbol{W}$ then there is an **exact** linear programming formulation with

$$\boldsymbol{O(2^W n)} \quad \text{variables and constraints.}$$

Conclusion

Given an ACOPF problem on a graph of treewidth $\leq w$ and n edges, and $0 \leq \epsilon \leq 1$ there is an LP formulation with the following properties:

- It has $O(\text{poly}(\epsilon^{-1}) 2^{O(w)} n)$ variables and constraints
- It produces ϵ-optimal and -feasible solutions.

Talk on Friday by Gonzalo on the pure-binary problems.