New results on nonconvex optimization

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Three problems

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   - given full-dimensional polyhedra $P^1, \ldots, P^K$ in $\mathbb{R}^d$,
   - find a point closest to the origin *not* contained inside any of the P^h.

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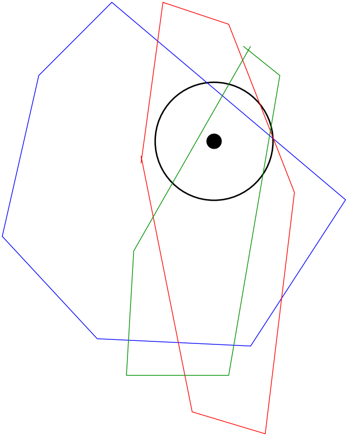
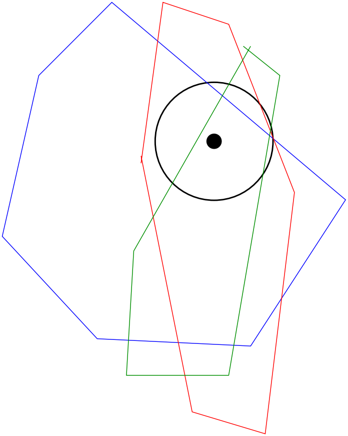
(application: X-ray lythography)





- Typical values for d (dimension): less than 20; usually even smaller
- Typical values for K (number of polyhedra): possibly hundreds, but often less than 50

2.

Cardinality constrained, convex quadratic programming.

$$\begin{aligned} \min \;& x^T Q x + c^T x \\ \text{s.t.} \;& Ax \leq b \\ & x \geq 0, \quad \|x\|_0 \leq k \end{aligned}$$

$\|x\|_0$ = number of nonzero entries in x.

- $Q \succeq 0$
- $x \in \mathbb{R}^n$ for n possibly large
- k relatively small, e.g. $k = 100$ for $n = 10000$
- VERY hard problem – just getting good bounds is tough

2b.

Sparse vector in column space (Spielwan, Wang, Wright '12)

Given a vector $y \in \mathbb{R}^n$ (n large)

$$\begin{aligned} \min \quad & \|y - Ax\|_2 \\ \text{s.t.} \quad & A \in \mathbb{R}^{n\times n},\ x \in \mathbb{R}^n \\ & \|x\|_0 \le k \end{aligned}$$

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- Both A and x are variables
- Usual "convexification" approach may not work
- Again, looks VERY hard

3. AC-OPF problem in rectangular coordinates

Given a power grid, determine voltages at every node so as to minimize a convex objective

$$\begin{aligned} \min\ & v^T A v \\ \text{s.t.}\ & L_k \le v^T F_k v \le U_k, \quad k = 1, \ldots, K \\ & v \in \mathbb{R}^{2n}, \quad (n = \text{number of nodes}) \end{aligned}$$

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- voltages are complex numbers; v is the vector of voltages in rectangular coordinates (real and imaginary parts)
- $A \succeq 0$
- n could be in the tens of thousands, or more
- the F_k are very sparse (neighborhood structure for every node)
- Problem HARD when grid under distress and $L_k \approx U_k$.

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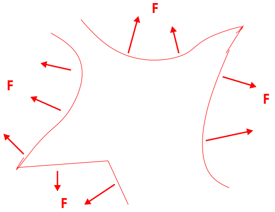

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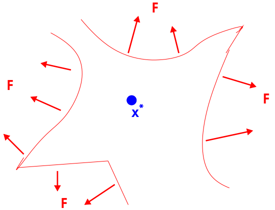
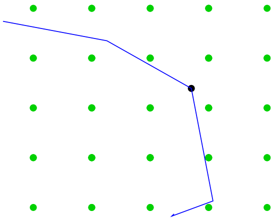


x^* solves $\min \left\{ Q(x), \quad : \quad x \in \hat{F} \right\}$ where $F \subset \hat{F}$ and $\hat{F}$ convex

Lattice-free cuts for **linear** integer programming

Generic problem: $\min \; c^T x, \quad s.t. \quad Ax \leq b, \quad z \in Z^n$



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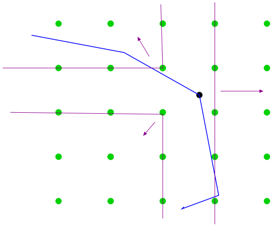

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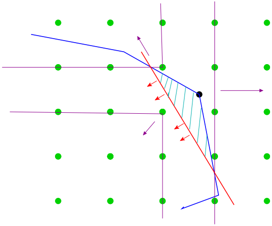

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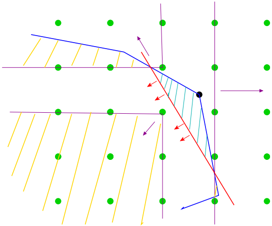

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- Optimal solution at **extreme point** (x^*, z^*) of $\mathrm{conv}\,\{(x,z) \,:\, z \geq Q(x),\ x \in F\}$
- So $x^* \in F$

Exclude-and-cut

$$\min z, \quad s.t. \;\; z \geq Q(x), \;\; x \in F$$

0. $\hat{F}$: a **convex relaxation** of $\mathrm{conv}\left\{(x, z)\ :\ z \geq Q(x),\ \ x \in F\right\}$

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2. Find an **open** set S s.t. $x^* \in S$ and $S \cap F = \emptyset$.
Examples: lattice-free sets, geometry

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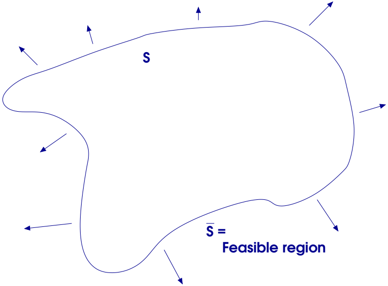
3. Add to the formulation an inequality $\mathbf{az} + \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{x} \geq \boldsymbol{\alpha}_{\mathbf{0}}$ valid for

$$\{ (x,z) \,:\, x \in \overline{S}, \;\; z \geq Q(x) \}$$

but violated by (x^*, z^*).

Valid **linear** inequalities for $\{ (x, z) \, : \, x \in \overline{S}, \; z \geq Q(x) \}$.



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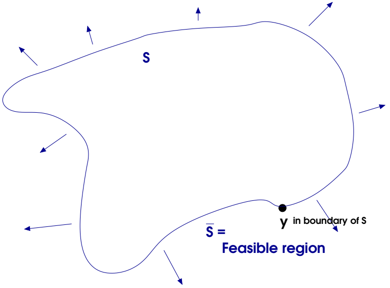

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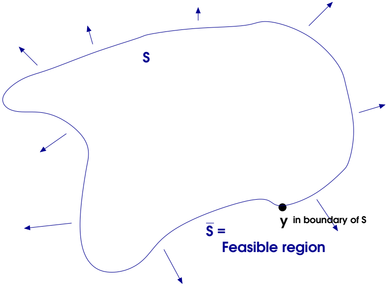

First order inequality:

$$z \ \geq\ [\nabla Q(y)]^T (x - y) + Q(y)$$

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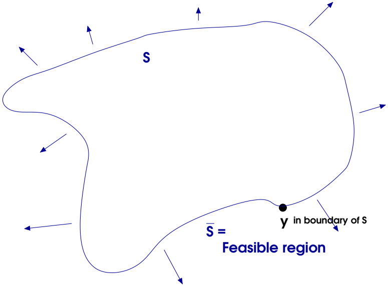

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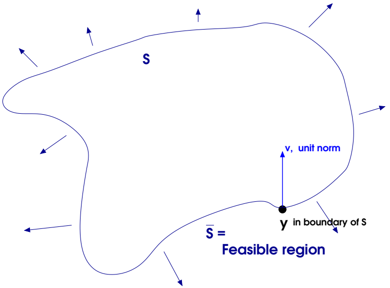

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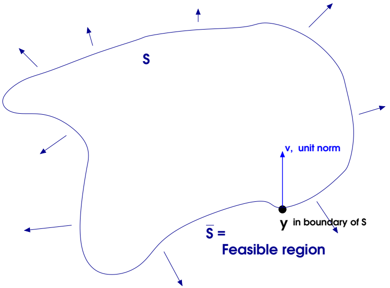

Lifted first order inequality, for $\alpha \geq 0$:

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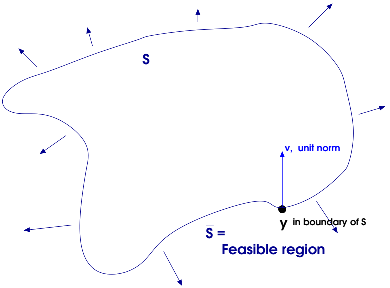

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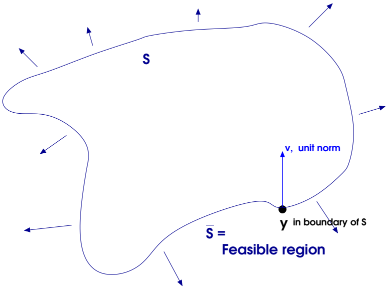

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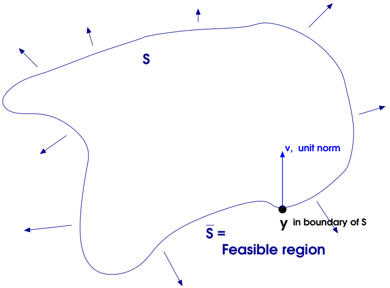

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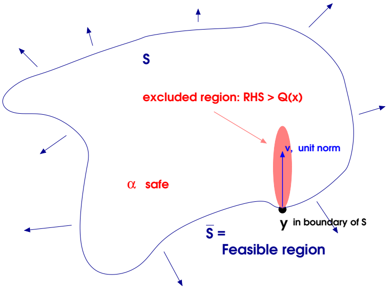

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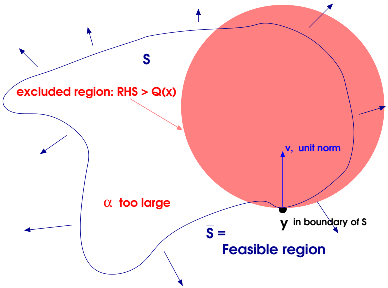

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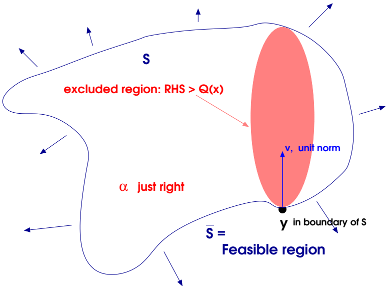

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Theorem. If Q is convex and differentiable, then $conv(\mathcal{F})$ is given by

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(abridged)

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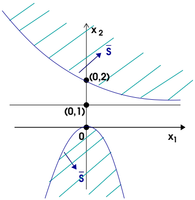
- As a result, poly time separation from $\mathrm{conv}(\mathcal{F})$ is equivalent to poly time separation of lifted first-order inequalities.

Suppose $S = \{x \in \mathbb{R}^2 \,:\, -x_1^2 \leq x_2 \leq 1 + e^{-x_1}\}$, and $Q(x) = x_2 + e^{-x_2} - 1$.



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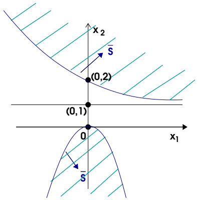

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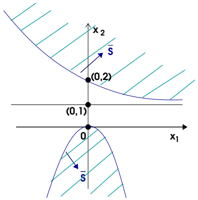

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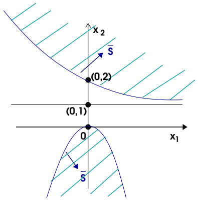

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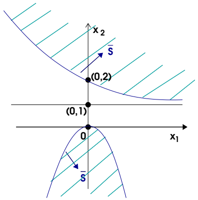

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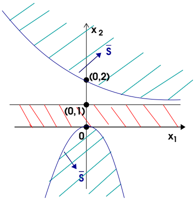

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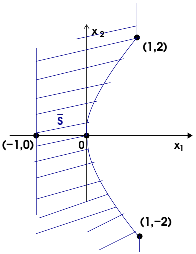

With $v = (1,0)^T$, the lifted first-order inequality at $(0,0)$ is $\mathbf{z} \geq \boldsymbol{\alpha}^* \mathbf{x_1} \ \Rightarrow \ \alpha^* = 2$. **Why?**

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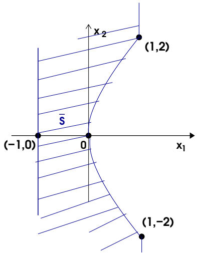

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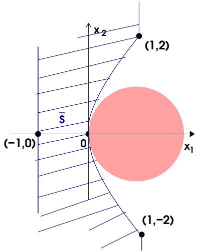

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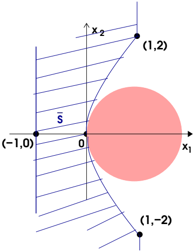

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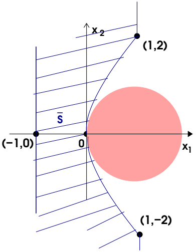

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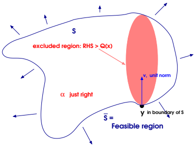
But **fails to hold** for $R > 1$ and $x_1 \approx 0$!

Lifted first-order inequality at $y \in \partial S$, in the direction of v: $Q(x) \ \geq \ [\nabla Q(y)]^T (x - y) + Q(y) + \alpha^* v^T (x - y)$



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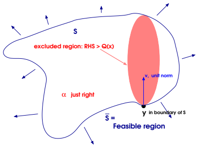

Theorem. If

- $Q(x)$ grows faster than linearly in every direction, and
- There is a ball with interior in the infeasible region, but containing y at its boundary

then the quantity α^* is a "max" and not just a "sup", i.e. the lifted inequality is tight at some point other than y

Quadratics

Valid **linear** inequalities for $\mathcal{F} \ = \ \{ (x, z) \ : \ x \in \overline{S}, \ z \ge Q(x) \}$.

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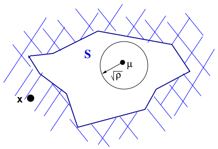

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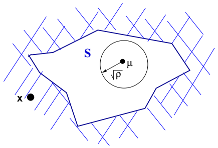

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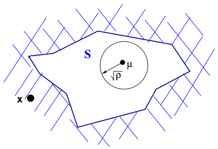

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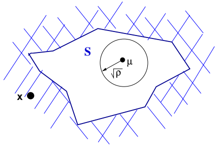

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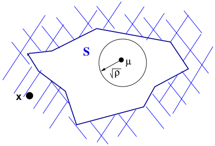

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Theorem: the undominated ball inequalities, and the lifted first-order inequalities, are the same.

Quadratics in action

Lifted first-order inequalities for $\mathcal{F} \; = \; \{\, (x,z) \, : \, x \in \overline{S}, \; z \geq \|x\|^2 \,\}$.

Separation problem. Given $(x^*, z^*) \in \mathbb{R}^n \times \mathbb{R}$, find a lifted-first order inequality maximally violated by (x^*, z^*) (if any)

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Special case: complement of an ellipsoid

Lifted first-order inequalities for $\mathcal{F} = \{ (x, z) : x^T A x - 2b^T x + c \geq 0,\ z \geq \|x\|^2 \}$. Here, $A \succ 0$.

Special case: complement of an ellipsoid

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The right-hand side is the **first-order** (tangent), at $\bar{x}$, for the convex quadratic

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The right-hand side is the **first-order** (tangent), at $\bar{x}$, for the convex quadratic

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Corollary: $\mathrm{conv}(\mathcal{F}) = \{(x,z)\;:\; z \geq x(I - \lambda^{-1}A)x + 2\lambda^{-1}b^Tx - \lambda^{-1}c,\;\, z \geq \|x\|^2\}$.

Also obtained by J.P. Vielma (2013)

But ... Exclude-and-cut, again

$$\min z, \quad s.t. \;\; z \geq Q(x), \;\; x \in F$$

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Examples: lattice-free sets, geometry

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3. Add to the formulation an inequality $\mathbf{az} + \boldsymbol{\alpha}^{\mathbf{T}}\mathbf{x} \geq \boldsymbol{\alpha}_{\mathbf{0}}$ valid for

$$\{\,(x,z)\,:\,x \in \overline{S},\;\; z \geq Q(x)\,\}$$

but violated by (x^*, z^*).

A classical problem: the trust-region subproblem

$$\begin{aligned} \min \quad & x^T A x + b^T x + c \\ \text{s.t.} \quad & \|x\|^2 \leq 1 \\ & x \in \mathbb{R}^n \end{aligned}$$

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Sturm and Zhang (2000): two extensions are polynomially solvable:

$$\begin{aligned} \min \quad & x^T A x + b^T x + c \\ \text{s.t.} \quad & \|x\|^2 \leq 1 \\ & \|x - x^0\|^2 \leq r \end{aligned}$$

(one additional ball inequality), and

$$\begin{aligned} \min \quad & x^T A x + b^T x + c \\ \text{s.t.} \quad & \|x\|^2 \leq 1 \\ & c^T x \leq c^0 \end{aligned}$$

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Note: Results leave open the general case with $m = 2$

A generalization

$$
\textbf{(TLIN):} \quad \begin{aligned} \min \quad & x^T A x + b^T x + c \\ \text{s.t.} \quad & \|x\|^2 \leq 1 \\ & a_i^T x \leq b_i \qquad i = 1, \ldots, m \\ & x \in \mathbb{R}^n. \end{aligned}
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- $P = \{x \,:\, a_i^T x \;\leq\; b_i \quad i = 1, \ldots, m\}$

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- $F^* =$ the number of **faces** of P that intersect the unit ball

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- $F^* =$ the number of **faces** of P that intersect the unit ball
- Ye-Zhang (or Anstreicher-Burer) case: $F^* = 3$.
- Burer-Yang case: $F^* = m + 1$

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Theorem: Problem **TLIN** can be solved in time polynomial in the problem size and F^*.

A stronger generalization

$$
\textbf{(TGEN):} \quad \begin{aligned}
\min \quad & x^T A x + b^T x + c \\
\text{s.t.} \quad & \|x - x^k\|^2 \ \leq \ f_k \qquad k = 1, \ldots, L_k \\
& \|x - y^k\|^2 \ \geq \ g_k \qquad k = 1, \ldots, M_k \\
& \|x - z^k\|^2 \ = \ h_k \qquad k = 1, \ldots, E_k \\
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Theorem: For every **fixed** $L_k \geq 1$, $M_k \geq 0$, $E_k \geq 0$, problem **TGEN** can be solved in time polynomial in the problem size and F^*.

(SODA 2014)