New results on nonconvex optimization

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ORC 2013
Three problems

1. The “SUV” problem
   - given full-dimensional polyhedra $P^1, \ldots, P^K$ in $\mathbb{R}^d$,
   - find a point closest to the origin $not$ contained inside any of the $P^h$. 
Three problems

1. The “SUV” problem
   - given full-dimensional polyhedra $P^1, \ldots, P^K$ in $\mathbb{R}^d$,
   - find a point closest to the origin \textit{not} contained inside any of the $P^h$.

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   \min \| x \|^2 \\
   s.t. \quad x \in \mathbb{R}^d - \bigcup_{h=1}^{K} \text{int}(P^h),
   \]
Three problems

1. The "SUV" problem
   - given full-dimensional polyhedra $P^1, \ldots, P^K$ in $\mathbb{R}^d$,
   - find a point closest to the origin not contained inside any of the $P^h$.

   $$\min \ |x|^2$$

   $$s.t. \quad x \in \mathbb{R}^d - \bigcup_{h=1}^{K} \text{int}(P^h),$$

   (application: X-ray lithography)
Typical values for $d$ (dimension): less than 20; usually even smaller

Typical values for $K$ (number of polyhedra): possibly hundreds, but often less than 50

Very hard problem
• Typical values for \( d \) (dimension): less than 20; usually even smaller
• Typical values for \( K \) (number of polyhedra): possibly hundreds, but often less than 50
2.

Cardinality constrained, convex quadratic programming.

$$\min \ x^T Q x + c^T x$$

s.t. $A x \leq b$

$x \geq 0, \quad \|x\|_0 \leq k$

$\|x\|_0 = \text{number of nonzero entries in } x.$

- $Q \succeq 0$
- $x \in \mathbb{R}^n$ for $n$ possibly large
- $k$ relatively small, e.g. $k = 100$ for $n = 10000$
- VERY hard problem – just getting good bounds is tough
Sparse vector in column space (Spielwan, Wang, Wright '12)

Given a vector $y \in \mathbb{R}^n$ (n large)

$$\min \|y - Ax\|_2$$

s.t. $A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$

$$\|x\|_0 \leq k$$
Sparse vector in column space (Spielwan, Wang, Wright ’12)

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- Both $A$ and $x$ are variables
- Usual “convexification” approach may not work
- Again, looks VERY hard
3. AC-OPF problem in rectangular coordinates

Given a power grid, determine voltages at every node so as to minimize a convex objective

\[
\begin{align*}
\min & \quad v^T A v \\
\text{s.t.} & \quad L_k \leq v^T F_k v \leq U_k, \quad k = 1, \ldots, K \\
& \quad v \in \mathbb{R}^{2n}, \quad (n = \text{number of nodes})
\end{align*}
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s.t. \( L_k \leq v^T F_k v \leq U_k, \quad k = 1, \ldots, K \)

\( v \in \mathbb{R}^{2n}, \quad (n = \text{number of nodes}) \)

- Voltages are complex numbers; \( v \) is the vector of voltages in rectangular coordinates (real and imaginary parts)
- \( A \succeq 0 \)
- \( n \) could be in the tens of thousands, or more
- The \( F_k \) are very sparse (neighborhood structure for every node)
- Problem HARD when grid under distress and \( L_k \approx U_k \).
Why are these problems so hard

Generic problem: \( \min Q(x), \ s.t. \ x \in F, \)
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- \(F\) nonconvex
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Generic problem: \( \min Q(x), \quad s.t. \quad x \in F, \)

- \( Q(x) \) (strongly) convex, especially: positive-definite quadratic
- \( F \) nonconvex

\( x^* \) solves \( \min \left\{ Q(x), \quad : \quad x \in \hat{F} \right\} \) where \( F \subset \hat{F} \) and \( \hat{F} \) convex
Lattice-free cuts for linear integer programming

Generic problem: \( \min c^T x, \quad s.t. \quad Ax \leq b, \quad z \in Z^n \)
Lattice-free cuts for **linear** integer programming

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An old trick

Don’t solve

\[ \min Q(x), \quad \text{over } x \in F \]
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Do solve

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\min z, \quad \text{over } \text{conv}\{(x, z) : z \geq Q(x), \ x \in F\}
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Do solve

\[ \min z, \quad \text{over } \text{conv}\left\{ (x, z) : z \geq Q(x), \ x \in F \right\} \]

- Optimal solution at extreme point \((x^*, z^*)\) of \(\text{conv}\left\{ (x, z) : z \geq Q(x), \ x \in F \right\}\)

- So \(x^* \in F\)
Exclude-and-cut

\[ \min z, \quad s.t. \quad z \geq Q(x), \quad x \in F \]

0. \( \hat{F} \): a **convex relaxation** of \( \text{conv} \{ (x, z) : z \geq Q(x), \ x \in F \} \)
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0. \( \hat{F} \): a **convex relaxation** of \( \text{conv} \{ (x, z) : z \geq Q(x), \ x \in F \} \)

1. Let \( (x^*, z^*) = \text{argmin}\{ z : (x, z) \in \hat{F} \} \)
Exclude-and-cut

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\min z, \quad s.t. \quad z \geq Q(x), \quad x \in F
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0. \( \hat{F} \): a **convex relaxation** of \( \text{conv} \left\{ (x, z) : z \geq Q(x), \ x \in F \right\} \)

1. Let \( (x^*, z^*) = \arg\min \{ z : (x, z) \in \hat{F} \} \)

2. Find an **open** set \( S \) s.t. \( x^* \in S \) and \( S \cap F = \emptyset \). Examples: lattice-free sets, geometry
Exclude-and-cut

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3. Add to the formulation an inequality \(az + \alpha^Tx \geq \alpha_0\)
   valid for \(\{(x, z) : x \in \overline{S}, \ z \geq Q(x)\}\)
   but violated by \((x^*, z^*)\).
Valid **linear** inequalities for \( \{ (x, z) : x \in \overline{S}, \ z \geq Q(x) \} \).
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First order inequality:

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z \geq [\nabla Q(y)]^T (x - y) + Q(y)
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is valid EVERYWHERE
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**Lifted** first order inequality, for \( \alpha \geq 0 \):

\[
z \geq [\nabla Q(y)]^T(x - y) + Q(y) + \alpha v^T(x - y)
\]

- first-order term \( \approx Q(x) \)
- lifting

\( \overline{S} = \text{Feasible region} \)

NOT valid EVERYWHERE: RHS \( > Q(x) \) for \( \alpha > 0 \), \( v^T(x - y) > 0 \) and \( x \approx y \).
Valid **linear** inequalities for \( \{ (x, z) : x \in \bar{S}, \; z \geq Q(x) \} \).

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**Lifted** first order inequality, for \( \alpha \geq 0 \):

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z \geq \underbrace{[\nabla Q(y)]^T (x - y) + Q(y)}_{\text{first-order term} \approx Q(x)} + \alpha v^T (x - y)
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Valid **linear** inequalities for \( \{ (x,z) : x \in \overline{S}, \ z \geq Q(x) \} \).

\[ z \geq \left[ \nabla Q(y) \right]^T (x - y) + Q(y) + \alpha v^T (x - y) \]

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Want \( \text{RHS} \leq Q(x) \) for \( x \in \overline{S} \) (\( \alpha = 0 \) always OK)
Valid **linear** inequalities for $\mathcal{F} = \{ (x, z) : x \in \bar{S}, \ z \geq Q(x) \}$.

**Lifted** first order inequality, for $\alpha \geq 0$:

$$z \geq \left[ \nabla Q(y) \right]^T (x - y) + Q(y) + \alpha v^T (x - y)$$

where $\alpha$ is a scalar, $v$ is a unit vector, $Q(y)$ is the value of the quadratic function at $y$, and $\nabla Q(y)$ is the gradient of $Q$ at $y$. The term $\left[ \nabla Q(y) \right]^T (x - y)$ is the first-order term, which approximates $Q(x)$ for $x$ close to $y$.

Lifting the inequality involves adding the term $\alpha v^T (x - y)$ to the right-hand side (RHS) of the inequality. The resulting inequality is not valid **EVERYWHERE**: RHS $> Q(x)$ for $\alpha > 0$, $v^T (x - y) > 0$ and $x \approx y$.

Want $RHS \leq Q(x)$ for $x \in \bar{S}$ ($\alpha = 0$ always OK).
Valid linear inequalities for $\mathcal{F} \doteq \{(x, z) \in \mathbb{R}^n \times \mathbb{R} : x \in \overline{S}, z \geq Q(x)\}$. 

Given $y \in \partial \overline{S}$, let $\alpha^* = \sup\{\alpha \geq 0 : Q(x) \geq \nabla Q(y)^T (x - y) + Q(y) + \alpha v^T (x - y)\}$ valid for $\mathcal{F}$. 

Note: $\alpha^* = \alpha^*(v, y)$.

Theorem. If $Q$ is convex and differentiable, then $\text{conv} (\mathcal{F})$ is given by $Q(x) \geq \nabla Q(y)^T (x - y) + Q(y) \forall y \in \partial \overline{S}$ and $Q(x) \geq \nabla Q(y)^T (x - y) + Q(y) + \alpha^* v^T (x - y) \forall v$ and $y \in \partial \overline{S}$. 

(abridged)
Valid **linear** inequalities for \( \mathcal{F} \doteq \{ (x, z) \in \mathbb{R}^n \times \mathbb{R} : x \in \overline{S}, \ z \geq Q(x) \} \).

Given \( y \in \partial S \), let

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\alpha^* \doteq \sup \{ \alpha \geq 0 : Q(x) \geq [\nabla Q(y)]^T (x - y) + Q(y) + \alpha v^T (x - y) \}
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valid for \( \mathcal{F} \).
Valid **linear** inequalities for \( \mathcal{F} \triangleq \{ (x, z) \in \mathbb{R}^n \times \mathbb{R} : x \in \overline{S}, \ z \geq Q(x) \} \).

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**Theorem.** If $Q$ is convex and differentiable, then $\text{conv}(\mathcal{F})$ is given by

$$Q(x) \geq [\nabla Q(y)]^T(x-y) + Q(y) \quad \forall y$$

$$Q(x) \geq [\nabla Q(y)]^T(x-y) + Q(y) + \alpha^* v^T(x-y) \quad \forall v \text{ and } y \in \partial S.$$

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Valid linear inequalities for $\mathcal{F} \doteq \{ (x, z) \in \mathbb{R}^n \times \mathbb{R} : x \in \overline{S}, z \geq Q(x) \}$. 

**Theorem.** If $Q$ is convex and differentiable, then $\text{conv}(\mathcal{F})$ is given by

(first-order ineqs) \[ Q(x) \geq [\nabla Q(y)]^T (x - y) + Q(y) \quad \forall y \]

(lifted first-order ineqs) \[ Q(x) \geq [\nabla Q(y)]^T (x - y) + Q(y) + \alpha^* v^T (x - y) \quad \forall v \text{ and } y \in \partial S. \]
Valid **linear** inequalities for $\mathcal{F} \doteq \{ (x, z) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \overline{S}, \ z \geq Q(x) \}$.

**Theorem.** If $Q$ is convex and differentiable, then $\text{conv}(\mathcal{F})$ is given by

1. (first-order ineqs) $Q(x) \geq [\nabla Q(y)]^T (x - y) + Q(y) \quad \forall y$
2. (lifted first-order ineqs) $Q(x) \geq [\nabla Q(y)]^T (x - y) + Q(y) + \alpha^* v^T (x - y) \quad \forall v$ and $y \in \partial S$.

Given $(x^*, z^*) \in \mathbb{R}^n \times \mathbb{R}$, how do we separate it from $\text{conv}(\mathcal{F})$?
Valid **linear** inequalities for $\mathcal{F} \doteq \{ (x, z) \in \mathbb{R}^n \times \mathbb{R} : x \in \overline{S}, \ z \geq Q(x) \}$. 

**Theorem.** If $Q$ is convex and differentiable, then $\text{conv}(\mathcal{F})$ is given by

- **(first-order ineqs)** $Q(x) \geq [\nabla Q(y)]^T (x - y) + Q(y) \quad \forall y$
- **(lifted first-order ineqs)** $Q(x) \geq [\nabla Q(y)]^T (x - y) + Q(y) + \alpha^* v^T (x - y) \quad \forall v$ and $y \in \partial S$. 

Given $(x^*, z^*) \in \mathbb{R}^n \times \mathbb{R}$, how do we separate it from $\text{conv}(\mathcal{F})$?

- Convexity $\Rightarrow$ strongest first-order inequality at $x^*$ is

$$Q(x) \geq [\nabla Q(x^*)]^T (x - x^*) + Q(x^*)$$
Valid **linear** inequalities for $\mathcal{F} = \{ (x, z) \in \mathbb{R}^n \times \mathbb{R} : x \in \overline{S}, \ z \geq Q(x) \}$. 

**Theorem.** If $Q$ is convex and differentiable, then $\text{conv}(\mathcal{F})$ is given by

- **(first-order ineqs)** $Q(x) \geq [\nabla Q(y)]^T (x - y) + Q(y) \quad \forall y$
- **(lifted first-order ineqs)** $Q(x) \geq [\nabla Q(y)]^T (x - y) + Q(y) + \alpha^* \psi^T (x - y) \quad \forall \psi$ and $y \in \partial S$.

Given $(x^*, z^*) \in \mathbb{R}^n \times \mathbb{R}$, how do we separate it from $\text{conv}(\mathcal{F})$?

- Convexity $\Rightarrow$ strongest first-order inequality at $x^*$ is

  $$Q(x) \geq [\nabla Q(x^*)]^T (x - x^*) + Q(x^*)$$

- As a result, poly time separation from $\text{conv}(\mathcal{F})$ is equivalent to poly time separation of lifted first-order inequalities.
Suppose $S = \{x \in \mathbb{R}^2 : -x_1^2 \leq x_2 \leq 1 + e^{-x_1}\}$, and $Q(x) = x_2 + e^{-x_2} - 1$.

With $v = (0, 1)^T$, the lifted first-order inequality at $(0, 0)$ is $z \geq \alpha^* x_2$.
Suppose $S = \{ x \in \mathbb{R}^2 : -x_1^2 \leq x_2 \leq 1 + e^{-x_1} \}$, and $Q(x) = x_2 + e^{-x_2} - 1$.

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With $v = (0, 1)^T$, the lifted first-order inequality at $(0, 0)$ is $z \geq \alpha^* x_2 \Rightarrow \alpha^* = e^{-1}$.
Suppose $S = \{x \in \mathbb{R}^2 : -x_1^2 \leq x_2 \leq 1 + e^{-x_1}\}$, and $Q(x) = x_2 + e^{-x_2} - 1$.

With $v = (0, 1)^T$, the lifted first-order inequality at $(0, 0)$ is $z \geq \alpha^* x_2 \Rightarrow \alpha^* = e^{-1}$. Why?
Suppose \( S = \{ x \in \mathbb{R}^2 : -x_1^2 \leq x_2 \leq 1 + e^{-x_1} \} \), and \( Q(x) = x_2 + e^{-x_2} - 1 \).

With \( v = (0, 1)^T \), the lifted first-order inequality at \((0, 0)\) is \( z \geq \alpha^* x_2 \Rightarrow \alpha^* = e^{-1} \). Why?

Because when \( x_2 = 1 \), \( x_2 + e^{-x_2} - 1 = e^{-1} = e^{-1}x_2 \), but any larger value for \( \alpha^* \) will result with \( x_2 + e^{-x_2} - 1 < \alpha^* x_2 \) for some \( x_2 > 1 \).
Suppose $S = \{ x \in \mathbb{R}^2 : -x_1^2 \leq x_2 \leq 1 + e^{-x_1} \}$, and $Q(x) = x_2 + e^{-x_2} - 1$.

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Because when $x_2 = 1$, $x_2 + e^{-x_2} - 1 = e^{-1} = e^{-1}x_2$, but any larger value for $\alpha^*$ will result with $x_2 + e^{-x_2} - 1 < \alpha^* x_2$ for some $x_2 > 1$. 
Suppose \( S = \{ x \in \mathbb{R}^2 : x_1 \geq 1 \} \cup \{ x \in \mathbb{R}^2 : 0 \leq x_1 \leq 1 \text{ and } |x_2| \leq (2x_1 - x_1^2)^{1/2} + x_1 \}, \) and \( Q(x) = \|x\|^2 \)

With \( v = (1, 0)^T \), the lifted first-order inequality at \((0, 0)\) is \( z \geq \alpha^* x_1 \Rightarrow \alpha^* = 2 \). Why?
Suppose $S = \{x \in \mathbb{R}^2 : x_1 \geq 1\} \cup \{x \in \mathbb{R}^2 : 0 \leq x_1 \leq 1 \text{ and } |x_2| \leq (2x_1 - x_1^2)^{1/2} + x_1\}$, and $Q(x) = ||x||^2$.

With $\nu = (1, 0)^T$, the lifted first-order inequality at $(0, 0)$ is $z \geq \alpha^* x_1 \Rightarrow \alpha^* = 2$. Why?

Because for $\alpha^* = 2R$, $Q(x) \leq \alpha^* x_1 \text{ iff } |x_2| \leq (2Rx_1 - x_1^2)^{1/2}$.
Suppose $S = \{ x \in \mathbb{R}^2 : x_1 \geq 1 \} \cup \{ x \in \mathbb{R}^2 : 0 \leq x_1 \leq 1 \text{ and } |x_2| \leq (2x_1 - x_1^2)^{1/2} + x_1 \}$, and $Q(x) = \|x\|^2$.

With $v = (1, 0)^T$, the lifted first-order inequality at $(0, 0)$ is $z \geq \alpha^* x_1 \Rightarrow \alpha^* = 2$. Why?

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Suppose $S = \{ x \in \mathbb{R}^2 : x_1 \geq 1 \} \cup \{ x \in \mathbb{R}^2 : 0 \leq x_1 \leq 1 \text{ and } |x_2| \leq (2x_1 - x_1^2)^{1/2} + x_1 \}$, and $Q(x) = \|x\|^2$

With $v = (1, 0)^T$, the lifted first-order inequality at $(0, 0)$ is $z \geq \alpha^* x_1 \Rightarrow \alpha^* = 2$. Why?

Because for $\alpha^* = 2R$, $Q(x) \leq \alpha^* x_1$ iff $|x_2| \leq (2Rx_1 - x_1^2)^{1/2} \leq (2x_1 - x_1^2)^{1/2} + x_1$ : true for $R \leq 1$.

But fails to hold for $R > 1$ and $x_1 \approx 0$!
Lifted first-order inequality at $y \in \partial S$, in the direction of $v$: $Q(x) \geq [\nabla Q(y)]^T(x - y) + Q(y) + \alpha^* v^T(x - y)$
Lifted first-order inequality at \( y \in \partial S \), in the direction of \( v \):

\[
Q(x) \geq [\nabla Q(y)]^T (x - y) + Q(y) + \alpha^* v^T (x - y)
\]

**Theorem.** If

- \( Q(x) \) grows faster than linearly in every direction, and
- There is a ball with interior in the infeasible region, but containing \( y \) at its boundary

then the quantity \( \alpha^* \) is a “max” and not just a “sup”, i.e. the lifted inequality is tight at some point other than \( y \)
Valid linear inequalities for $\mathcal{F} = \{ (x, z) : x \in \overline{S}, \ z \geq Q(x) \}$.

Special case $Q(x)$ a positive definite quadratic.
Valid **linear** inequalities for \( \mathcal{F} = \{ (x, z) : x \in \overline{S}, \ z \geq Q(x) \} \).

Special case \( Q(x) \) a **positive definite** quadratic. Change of coordinates \( \rightarrow Q(x) = \|x\|^2 \).
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![Diagram](image)

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**Theorem:** the undominated ball inequalities, and the lifted first-order inequalities, are the same.
Quadratics in action

Lifted first-order inequalities for \( F = \{ (x, z) : x \in \overline{S}, z \geq \|x\|^2 \} \).

**Separation problem.** Given \((x^*, z^*) \in \mathbb{R}^n \times \mathbb{R}\), find a lifted-first order inequality maximally violated by \((x^*, z^*)\) (if any)
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**Theorem:** We can separate in polynomial time when:

- $\overline{S}$ (or $S$) is a union of polyhedra
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Bienstock, Michalka
Convex obj non-convex domain
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Special case: complement of an ellipsoid

Lifted first-order inequalities for \( \mathcal{F} = \{ (x, z) : x^T Ax - 2b^T x + c \geq 0, \quad z \geq \|x\|^2 \} \). Here, \( A > 0 \).
Special case: complement of an ellipsoid

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Let \( \lambda = \text{largest} \) eigenvalue of \( A \). Then:

Theorem. The strongest lifted first-order inequality at \( \bar{x} \in \mathbb{R}^n \) is:

\[
z \geq 2 \left[ \left( I - \frac{\lambda - 1}{\lambda} A \right) \bar{x} + \frac{\lambda - 1}{\lambda} b \right]^T (x - \bar{x}) + \bar{x} \left( I - \frac{\lambda - 1}{\lambda} A \right) \bar{x} + 2 \frac{\lambda - 1}{\lambda} b^T \bar{x} - \frac{\lambda - 1}{\lambda} c
\]

The right-hand side is the first-order (tangent), at \( \bar{x} \), for the convex quadratic \( x \left( I - \frac{\lambda - 1}{\lambda} A \right) x + 2 \frac{\lambda - 1}{\lambda} b^T x - \frac{\lambda - 1}{\lambda} c \).

Corollary: \( \text{conv} (\mathcal{F}) = \{ (x, z) : z \geq x \left( I - \frac{\lambda - 1}{\lambda} A \right) x + 2 \frac{\lambda - 1}{\lambda} b^T x - \frac{\lambda - 1}{\lambda} c, \ z \geq \|x\|^2 \} \).

Also obtained by J.P. Vielma (2013).
Special case: complement of an ellipsoid

Lifted first-order inequalities for \( \mathcal{F} = \{ (x, z) : x^T A x - 2 b^T x + c \geq 0, \ z \geq \|x\|^2 \} \). Here, \( A > 0 \).

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**Theorem.** The strongest lifted first-order inequality at \( \bar{x} \in \mathbb{R}^n \) is:

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z \geq 2[(I - \lambda^{-1} A)\bar{x} + \lambda^{-1} b]^T (x - \bar{x}) + \bar{x}(I - \lambda^{-1} A)\bar{x} + 2\lambda^{-1} b^T \bar{x} - \lambda^{-1} c
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Lifted first-order inequalities for $\mathcal{F} = \{ (x, z) : x^T A x - 2 b^T x + c \geq 0, \ z \geq \|x\|^2 \}$. Here, $A \succ 0$.

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**Corollary:** $\text{conv}(\mathcal{F}) = \{ (x, z) : z \geq x(I - \lambda^{-1}A)x + 2\lambda^{-1}b^T x - \lambda^{-1}c, \ z \geq \|x\|^2 \}$. Also obtained by J.P. Vielma (2013)
But ... Exclude-and-cut, again

\[
\min z, \quad \text{s.t. } z \geq Q(x), \ x \in F
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0. \( \hat{F} \): a **convex relaxation** of \( \text{conv} \ \{(x, z) : z \geq Q(x), \ x \in F\} \)
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1. Let $(x^*, z^*) = \arg\min \{ z : (x, z) \in \hat{F} \}$

2. Find an **open** set $S$ s.t. $x^* \in S$ and $S \cap F = \emptyset$.
   Examples: lattice-free sets, geometry
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3. Add to the formulation an inequality \( az + \alpha^T x \geq \alpha_0 \)
   valid for
   \[ \{(x, z) : x \in \overline{S}, \ z \geq Q(x)\} \]
   but violated by \( (x^*, z^*) \).
A classical problem: the trust-region subproblem

\[
\begin{align*}
\min & \quad x^T Ax + b^T x + c \\
\text{s.t.} & \quad \|x\|^2 \leq 1 \\
& \quad x \in \mathbb{R}^n
\end{align*}
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Sturm and Zhang (2000): two extensions are polynomially solvable:

\[
\begin{align*}
\min & \quad x^T A x + b^T x + c \\
\text{s.t.} & \quad \|x\|^2 \leq 1 \\
& \quad \|x - x^0\|^2 \leq r
\end{align*}
\]

(one additional ball inequality), and

\[
\begin{align*}
\min & \quad x^T A x + b^T x + c \\
\text{s.t.} & \quad \|x\|^2 \leq 1 \\
& \quad c^T x \leq c^0
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\begin{align*}
\min & \quad x^T Ax + b^T x + c \\
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\end{align*}
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can be solved in polynomial time if

\[\forall i \neq j, \{x: a_i^T x = b_i\} \cap \{x: a_j^T x = b_j\} \cap \{x: \|x\|^2 \leq 1\} = \emptyset\]

Note: Results leave open the general case with \(m = 2\)
A classical problem: the trust-region subproblem Ye and Zhang (2003): two parallel linear inequalities are added:

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Note: Results leave open the general case with \( m = 2 \)
A generalization

(TLIN): \[ \min \ x^T A x + b^T x + c \]
\[ \text{s.t.} \quad ||x||^2 \leq 1 \]
\[ a_i^T x \leq b_i \quad i = 1, \ldots, m \]
\[ x \in \mathbb{R}^n. \]

\[ P = \{ x : a_i^T x \leq b_i \quad i = 1, \ldots, m \} \]
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\[x \in \mathbb{R}^n. \]

- \( P = \{x : a_i^T x \leq b_i \mid i = 1, \ldots, m\} \)
- \( F^* = \text{the number of faces of } P \text{ that intersect the unit ball} \)
A generalization

(TLIN): \[ \begin{align*} 
\min & \quad x^T A x + b^T x + c \\
\text{s.t.} & \quad \|x\|^2 \leq 1 \\
& \quad a_i^T x \leq b_i \quad i = 1, \ldots, m \\
& \quad x \in \mathbb{R}^n. 
\end{align*} \]

- \( P = \{x : a_i^T x \leq b_i \quad i = 1, \ldots, m\} \)
- \( F^* = \) the number of faces of \( P \) that intersect the unit ball
- Ye-Zhang (or Anstreicher-Burer) case: \( F^* = 3 \).
- Burer-Yang case: \( F^* = m + 1 \)
A generalization

\[ \begin{align*}
\text{(TLIN):} && \quad \min \quad & x^T A x + b^T x + c \\
\text{s.t.} \quad & ||x||^2 \leq 1 \\
& a_i^T x \leq b_i \quad i = 1, \ldots, m \\
& x \in \mathbb{R}^n.
\end{align*} \]

- \( P = \{ x : a_i^T x \leq b_i \quad i = 1, \ldots, m \} \)
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- Ye-Zhang (or Anstreicher-Burer) case: \( F^* = 3 \).
- Burer-Yang case: \( F^* = m + 1 \)

**Theorem:** Problem \textbf{TLIN} can be solved in time polynomial in the problem size and \( F^* \).
A stronger generalization

(TGEN): \[
\begin{align*}
\min & \quad x^T A x + b^T x + c \\
\text{s.t.} & \quad \|x - x^k\|^2 \leq f_k \quad k = 1, \ldots, L_k \\
& \quad \|x - y^k\|^2 \geq g_k \quad k = 1, \ldots, M_k \\
& \quad \|x - z^k\|^2 = h_k \quad k = 1, \ldots, E_k \\
& \quad a^T_i x \leq b_i \quad i = 1, \ldots, m \\
x & \in \mathbb{R}^n.
\end{align*}
\]
A stronger generalization

(TGEN): \[ \min x^T A x + b^T x + c \]
\[ \text{s.t.} \quad \|x - x^k\|^2 \leq f_k \quad k = 1, \ldots, L_k \]
\[ \|x - y^k\|^2 \geq g_k \quad k = 1, \ldots, M_k \]
\[ \|x - z^k\|^2 = h_k \quad k = 1, \ldots, E_k \]
\[ a_i^T x \leq b_i \quad i = 1, \ldots, m \]
\[ x \in \mathbb{R}^n. \]

\[ P = \{x : a_i^T x \leq b_i \quad i = 1, \ldots, m\} \]
A stronger generalization

(TGEN): \[
\begin{align*}
\min & \quad x^T A x + b^T x + c \\
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& \quad \|x - y^k\|^2 \geq g_k \quad k = 1, \ldots, M_k \\
& \quad \|x - z^k\|^2 = h_k \quad k = 1, \ldots, E_k \\
& \quad a_i^T x \leq b_i \quad i = 1, \ldots, m \\
& \quad x \in \mathbb{R}^n.
\end{align*}
\]

- \( P = \{x : a_i^T x \leq b_i \quad i = 1, \ldots, m\} \)
- \( F^* = \text{the number of faces of } P \text{ that intersect } \bigcap_k \{x : \|x - x^k\| \leq f_k\}. \)
A stronger generalization

\[(TGEN)\]: \[\min \ x^T A x + b^T x + c\]
\[\text{s.t.} \quad \|x - x^k\|^2 \leq f_k \quad k = 1, \ldots, L_k\]
\[\|x - y^k\|^2 \geq g_k \quad k = 1, \ldots, M_k\]
\[\|x - z^k\|^2 = h_k \quad k = 1, \ldots, E_k\]
\[a_i^T x \leq b_i \quad i = 1, \ldots, m\]
\[x \in \mathbb{R}^n.\]

- \( P = \{x : a_i^T x \leq b_i \quad i = 1, \ldots, m\}\)
- \( F^* = \) the number of faces of \(P\) that intersect \(\bigcap_k \{x : \|x - x^k\| \leq f_k\}\).

**Theorem:** For every fixed \(L_k \geq 1, M_k \geq 0, E_k \geq 0\), problem \(TGEN\) can be solved in time polynomial in the problem size and \(F^*\).

(SODA 2014)