Experiments with Robust Optimization

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Robust Optimization

- Optimization under parameter (data) uncertainty
- Ben-Tal and Nemirovsky, El Ghaoui et al
- Bertsimas et al
- Uncertainty is modeled by assuming that data is not known precisely, and will instead lie in known sets.
- Example: a coefficient a_i is uncertain. We allow $a_i \in [l_i, u_i]$.
- Typically, a **minimization** problem becomes a **min-max** problem.

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Example: Linear Programs with Row-Wise uncertainty

Ben-Tal and Nemirovsky, 1999 (also: Soyster (1973))

$\min c^t x$
Subject to:
$Ax \geq b$ **for all** $A \in \mathcal{U}$
$x \in X$

$\mathcal{U} =$ **uncertainty set**
$\longrightarrow$ **the** i^{th} **row of A belongs to an** ellipsoidal **set** $\mathcal{E}_i$
e.g. $\sum_j \alpha_{ij}^2 (a_{ij} - \bar{a}_{ij})^2 \leq 1$

$\longrightarrow$ can be solved using SOCP techniques

Other forms of optimization under uncertainty

- Stochastic programming
- Adversarial queueing, online optimization
- “Risk-aware” optimization
- Optimization of utility functions as a substitute for handling infeasibilities

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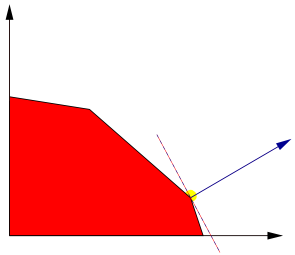
Scenario I: Stability

Data is fairly accurate, though possibly noisy – small errors are possible



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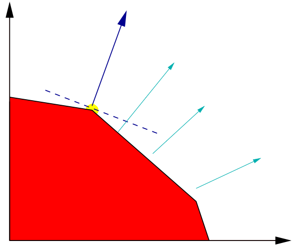

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Scenario II: Hedging

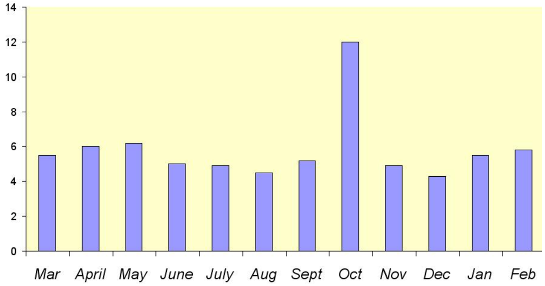
Significant, but within order-of-magnitude, data uncertainty

Example:
A certain parameter, α, is volatile. Its long-term average is **1.5** but it we could expect changes of the order of **.3**.

- Possibly more than just noise
- Could use deviations to our advantage, especially if there are several uncertain parameters that act "correlated"
- Are we guarding against risk or are we hedging?

Scenario III: Insurance

Real world data can exhibit undesirable and unexpected behavior



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- Need to clearly spell out desired tradeoff between risk and performance
- Magnitude and geometry of risk are not the same

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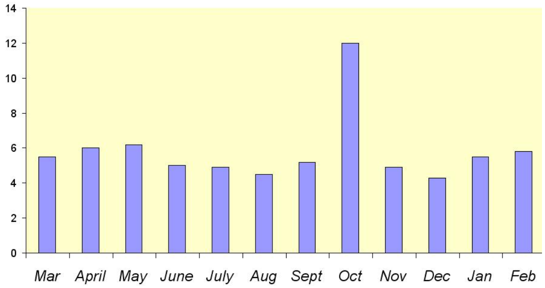

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Application: Portfolio Optimization

- $$\min \lambda x^T Qx \; - \; \mu^T x$$

  Subject to:

  $$Ax \geq b$$

- μ = vector of “returns”, Q = “covariance” matrix
- x = vector of “asset weights”
- $Ax \geq b$: general linear constraints
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Robust Portfolio Optimization

Goldfarb and Iyengar, 2001

$\rightarrow$ $\boldsymbol{Q}$ and μ are uncertain

Robust Problem

$$\min_x \left\{ \max_{Q \in \mathcal{Q}} \lambda x^T Q x \; - \; \min_{\mu \in \mathcal{E}} \mu^T x \right\}$$

Subject to:

$$\sum_j x_j = 1, \quad x \geq 0$$

$\rightarrow$ When $\mathcal{Q}$ is an ellipsoid and $\mathcal{E}$ is a product of intervals the robust problem can be solved as an SOCP

Robust Portfolio Optimization

A different uncertainty model

$\longrightarrow$ Want to model that deviations of the returns μ_j from their nominal values are rare but could be significant

A simple example

- Parameters: $0 \leq \gamma \leq 1$, integer $N \geq 0$,
  for each asset j:
  $\bar{\mu}_j =$ expected return, $0 \leq \delta_j$ small (possibly zero)
- **Well-behaved asset j:** $\bar{\mu}_j - \delta_j \leq \mu_j \leq \bar{\mu}_j + \delta_j$
- **Misbehaving asset j:** $(1 - \gamma)\bar{\mu}_j \leq \mu_j \leq \bar{\mu}_j$
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A more comprehensive setting

- Parameters: $\mathbf{0} \le \gamma_{\mathbf{1}} \le \gamma_{\mathbf{2}} \le \ldots \le \gamma_{\boldsymbol{K}} \le \mathbf{1}$,
  integers $\mathbf{0} \le \boldsymbol{n_i} \le \boldsymbol{N_i}$, $\mathbf{1} \le \boldsymbol{i} \le \boldsymbol{K}$

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- $\sum_j \mu_j \geq \Gamma \sum_j \bar{\mu}_j$; $\Gamma > 0$ a parameter

- (R. Tütüncü) For $1 \leq h \leq H$,

  - a **set** ("tier") T_h of assets, and a parameter $\Gamma_h > 0$

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Note: only downwards changes are modeled

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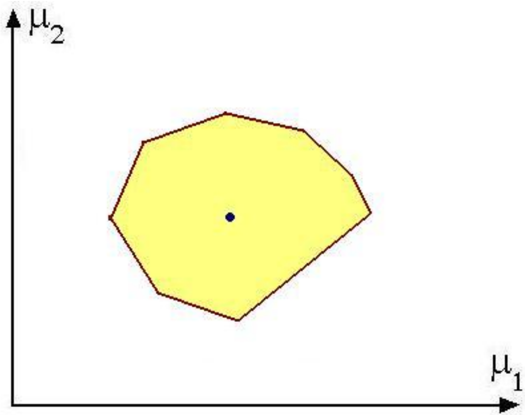
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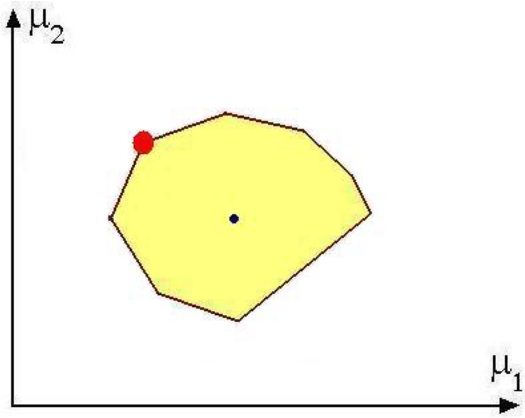
μ_1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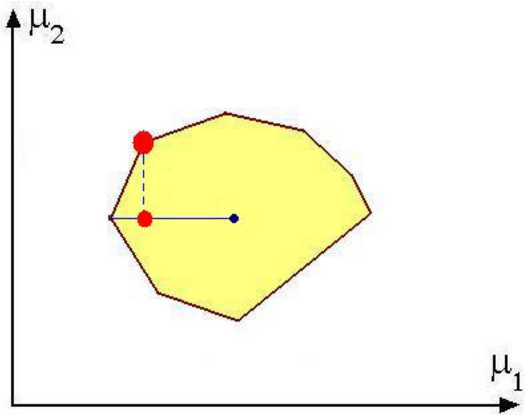

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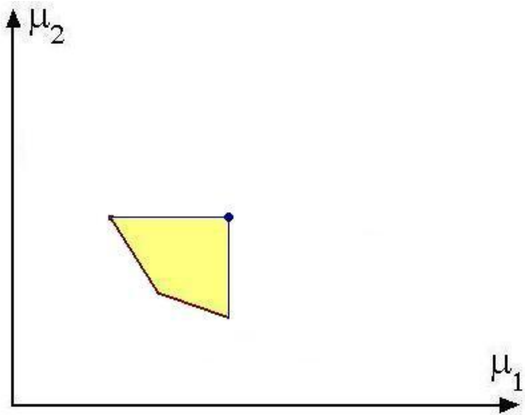

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General methodology:

Benders' decomposition (= cutting-plane algorithm)

Generic problem: $\min_{x \in X} \max_{d \in \mathcal{D}} f(x, d)$

$\longrightarrow$ Maintain a **finite subset** $\tilde{D}$ of $\mathcal{D}$ (a **"model"**)

GAME

1. **Implementor:** solve $\min_{x \in X} \max_{d \in \tilde{D}} f(x, d)$, with solution x^*
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3. **Add** $\tilde{d}$ to $\tilde{D}$, and go to 1.

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- Decoupling of implementor and adversary yields considerably simpler, and smaller, problems
- Decoupling allows us to use more sophisticated uncertainty models
- If number of iterations is small, implementor's problem is a small "convex" problem
- Most progress will be achieved in initial iterations – permits "soft" termination criteria

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Implementor's problem

A convex quadratic program

At iteration m, solve

$$\min \ \lambda x^T Q x \ - \ r$$

Subject to:

$$Ax \geq b$$

$$r \leq \mu_{(i)}^T x, \quad i = 1, \ldots, m$$

Here, $\mu_{(1)}, \ldots, \mu_{(m)}$ are given return vectors

Adversarial problem: A mixed-integer program

$x^* =$ given asset weights

$$\min \ \sum_j x_j^* \mu_j$$

Subject to:

$$\bar{\mu}_j (1 - \sum_i \gamma_{i-1} y_{ij}) \leq \mu_j \leq \bar{\mu}_j (1 - \sum_i \gamma_i y_{ij}) \quad \forall i \geq 1$$

$\sum_i y_{ij} \leq 1, \quad \forall j$ (each asset in at most one segment)

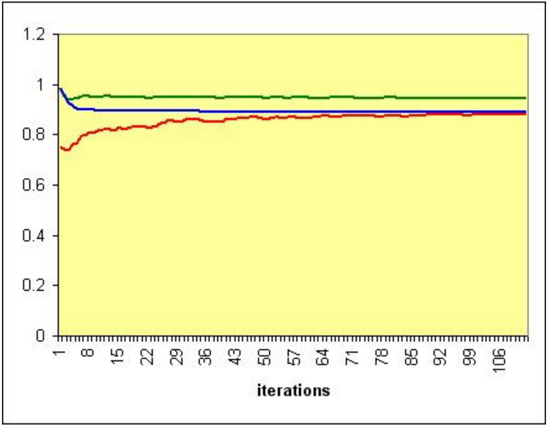
$n_i \leq \sum_j y_{ij} \leq N_i, \quad 1 \leq i \leq K$ (segment cardinalities)

$\sum_{j \in T_h} \mu_j \geq \Gamma_h \sum_{j \in T_h} \bar{\mu}_j, \quad 1 \leq h \leq H$ (tier ineqs.)

μ_j free, $y_{ij} = 0$ or 1, all i, j

2464 assets, 152 factors; total CPU time = 93.27 sec.

4 segments: (400, 0.01), (50, 0.05), (30, 0.2), (10, 0.3)
3 tiers: the three top deciles lose at most **10%** each



green = nominal return, blue = estimate, red = adversarial

Same run

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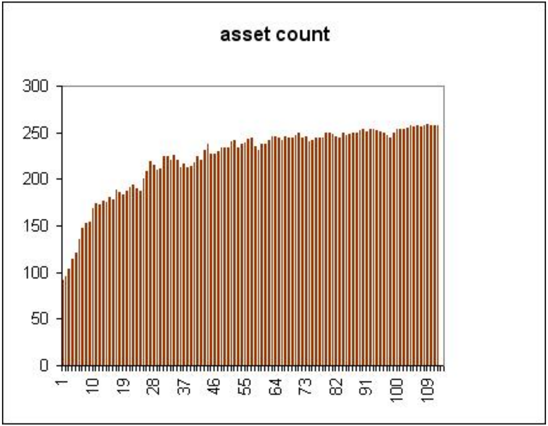

Summary

	columns	rows	iterations	time (sec.)	imp. time	adv. time
1	500	20	47	1.85	1.34	0.46
2	500	20	3	0.09	0.01	0.03
3	703	108	1	0.29	0.13	0.04
4	499	140	3	3.12	2.65	0.05
5	499	20	19	0.42	0.21	0.17
6	1338	81	7	0.45	0.17	0.08
7	2019	140	8	41.53	39.6	0.36
8	2443	153	2	12.32	9.91	0.07
9	2464	153	111	100.81	60.93	36.78

Why the adversarial problem is “easy”

($x^* =$ given asset weights)

$$\min \ \sum_j x_j^* \mu_j$$

Subject to:

$$\bar{\mu}_j (1 - \sum_i \gamma_{i-1} y_{ij}) \leq \mu_j \leq \bar{\mu}_j (1 - \sum_i \gamma_i y_{ij})$$

$\sum_i y_{ij} \leq 1, \ \forall j$ (each asset in at most one segment)

$n_i \leq \sum_j y_{ij} \leq N_i, \ \forall i$ (segment cardinalities)

$\sum_{j \in T_h} \mu_j \geq \Gamma_h \, (\sum_{j \in T_h} \bar{\mu}_j), \ \forall h$ (tier inequalities)

μ_j free, $y_{ij} = 0$ or 1, all i, j

Why the adversarial problem is “easy”

(K = no. of segments, H = no. of tiers)

Theorem. For every fixed K and H, and for every $\epsilon > 0$, there is an algorithm that finds a solution to the adversarial problem with optimality relative error $\leq \epsilon$, in time polynomial in ϵ^{-1} and n (= no. of assets).

The simplest case

$$\max \ \sum_j x_j^* \delta_j$$

Subject to:

$$\sum_j \delta_j \ \leq \ \Gamma$$

$$0 \ \leq \ \delta_j \ \leq \ u_j y_j, \ \ y_j \ = \ 0 \text{ or } 1, \ \ \text{all j}$$

$$\sum_j y_j \ \leq \ N$$

$\cdots$ a *cardinality constrained knapsack problem*
B. (1995), DeFarias and Nemhauser (2004)

What is the impact of the uncertainty model

All runs on the same data set with 1338 columns and 81 rows

- 1 segment: $(200, 0.5)$
  robust random return = **4.57**, **157** assets
- 2 segments: $(200, 0.25)$, $(100, 0.5)$
  robust random return = **4.57**, **186** assets
- 2 segments: $(200, 0.2)$, $(100, 0.6)$
  robust random return = **3.25**, **213** assets
- 2 segments: $(200, 0.1)$, $(100, 0.8)$
  robust random return = **1.50**, **256** assets
- 1 segment: $(100, 1.0)$
  robust random return = **1.24**, **281** assets

Ambiguous chance-constrained models

1. The implementor chooses a vector x^* of assets
2. The adversary chooses a *probability distribution* P for the returns vector
3. A random returns vector μ is drawn from P

$\rightarrow$ Implementor wants to choose x^* so as to minimize **value-at-risk** (conditional value at risk, etc.)

Erdogan and Iyengar (2004), Calafiore and Campi (2004)

$\rightarrow$ We want to model *correlated* errors in the returns

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Uncertainty set

Given a vector x^* of assets, the adversary

1. Chooses a vector $w \in R^n$ (n = no. of assets) with $0 \leq w_j \leq 1$ for all j.
2. Chooses a random variable $0 \leq \delta \leq 1$

$\rightarrow$ Random return: $\mu_j = \bar{\mu}_j (1 - \delta w_j)$ ($\bar{\mu}$ = nominal returns).

Definition: Given reals ν and $0 \leq \theta \leq 1$ the *value-at-risk* of x^* is the real $\rho \geq 0$ such that

$$\text{Prob}(\nu - \mu^T x^* \geq \rho) \geq \theta$$

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OR ...

Given a vector x^* of assets, the adversary

1. Chooses a vector $w \in R^n$ (n = no. of assets) with $0 \leq w_j \leq W$ for all j.
2. Chooses a random variable $0 \leq \delta \leq 1$

$\rightarrow$ Random return: $\mu_j = \bar{\mu}_j - \delta w_j$ ($\bar{\mu}$ = nominal returns).

Definition: Given reals ν and $0 \leq \theta \leq 1$ the *value-at-risk* of x^* is the real $\rho \geq 0$ such that

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$\longrightarrow$ The adversary wants to maximize VAR

The classical factor model for returns

$$\mu = \bar{\mu} + \boldsymbol{V}^{\boldsymbol{T}}\boldsymbol{f} + \epsilon$$

where

- $\bar{\mu}$ = expected return,
- $\boldsymbol{V}$ = "factor exposure matrix",
- $\boldsymbol{f}$ = a bounded random variable,
- ϵ = residual errors

V is $r \times n$ with $r << n$.

$\rightarrow$ Random return$_j$ $= \bar{\mu}_j(1 - \delta w_j)$ where $0 \leq w_j \leq 1 \;\; \forall j$, and $0 \leq \delta \leq 1$ is a random variable.

A *discrete distribution:*

- We are given **fixed** values $0 = \delta_0 \leq \delta_2 \leq \ldots \leq \delta_K = 1$ example: $\delta_i = \frac{i}{K}$
- Adversary *chooses* $\pi_i = \text{Prob}(\delta = \delta_i)$, $0 \leq i \leq K$
- The π_i are *constrained*: we have fixed bounds, $\pi_i^l \leq \pi_i \leq \pi_i^u$ (and possibly other constraints)
- Tier constraints: for sets ("tiers") T_h of assets, $1 \leq h \leq H$, we require:

  $E(\delta \sum_{j \in T_h} w_j) \leq \Gamma_h$ (given)

  or, $(\sum_i \delta_i \pi_i) \sum_{j \in T_h} w_j \leq \Gamma_h$
- Cardinality constraint: $w_j > 0$ for at most N indices j

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The adversarial problem is "easy"

K = no. of points in discrete distribution, H = no. of tiers

Theorem

- Without the cardinality constraint, for each fixed K and H the adversarial problem can be solved as a polynomial number of linear programs.
- With the cardinality constraint, for each fixed K and H the adversarial problem can be solved as a polynomial number of knapsack problems.

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Adversarial problem as an MIP

Recall: random return$_j$ $\mu_j = \bar{\mu}_j(1 - \delta w_j)$
where $\delta = \delta_i$ (given) with probability π_i (chosen by adversary),
$0 \leq \delta_0 \leq \delta_1 \leq \ldots \leq \delta_K = 1$ and $0 \leq w$

$\min_{\pi,w,V} \ \min_{1 \leq i \leq k} V_i$

Subject to

$0 \leq w_j \leq 1$, all j, $\pi_i^l \leq \pi_i \leq \pi_i^u$, all i,
$\sum_i \pi_i = 1$,

$V_i = \sum_j \bar{\mu}_j(1 - \delta_i w_j)x_j^*$, if $\pi_i + \pi_{i+1} + \ldots + \pi_K \geq 1 - \theta$
$V_i = M$ (large), otherwise

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Approximation

$\left(\sum_i \delta_i \pi_i\right) \sum_{j \in T_h} w_j \leq \Gamma_h,$ for each tier h $(*)$

Let $N > 0$ be an integer. For $1 \leq k \leq N$, write

$\frac{k}{N} \sum_{j \in T_h} w_j \leq \Gamma_h + M\,(1 - z_{hk}),$ where

$z_{hk} = 1$ if $\frac{k-1}{N} < \sum_i \delta_i \pi_i \leq \frac{k}{N}$

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$\sum_k z_{hk} = 1$

and M is large

Lemma. Under reasonable conditions, replacing $(*)$ with this system changes the value of the problem by at most a factor of $(1 + \frac{1}{N})$

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Implementor's problem

Find a near-optimal solution with minimum value-at-risk

Nominal problem:

$$v^* = \min_x \lambda x^T Q x - \mu^T x$$

Subject to:

$$Ax \geq b$$

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Implementor's problem

Find a near-optimal solution with minimum value-at-risk

Given asset weights $\boldsymbol{x}$, we have:

- value-at-risk $\geq \rho$, if the adversary can produce a return vector μ with

  $$\text{Prob}(\nu - \mu^T x \geq \rho) \geq \theta$$

where ν is a fixed reference value.

Implementor's problem

Find a near-optimal solution with minimum value-at-risk

Implementor's problem at iteration r:

$$\min \ V$$

Subject to:

$$\lambda x^T Q x \ - \ \mu^T x \ \leq \ (1+\epsilon) v^*$$

$$Ax \geq b$$

$$V \ \geq \ \nu \ - \ \sum_j \bar{\mu}_j \left(1 - \delta_{i(t)} w_j^{(t)}\right) x_j, \qquad t = 1, 2, \ldots, r-1$$

Here, $\delta_{i(t)}$ and $w^{(t)}$ are the adversary's output at iteration $t < r$.

Implementor's problem

Find a near-optimal solution with minimum value-at-risk

Implementor's problem at iteration r:

$$\min\ V$$

Subject to:

$$\lambda x^T Q x \ - \ \mu^T x \ \leq \ (1+\epsilon) v^*$$

$$Ax \geq b$$

$$V \ \geq \ \nu \ - \ \sum_j \bar{\mu}_j \left(1 - \delta_{i(t)} w_j^{(t)}\right) x_j, \qquad t = 1, 2, \ldots, r-1$$

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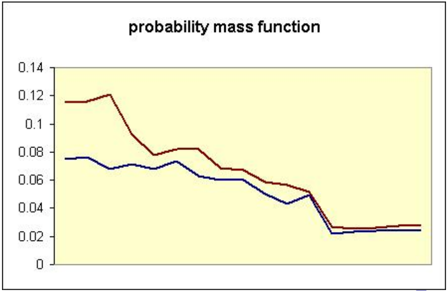
First set of experiments

1338 assets, 41 factors, 81 rows

- problem: find a "near optimal" solution with minimum value-at-risk, for a given threshold probability θ
- experiment: investigate different values of θ
- "near optimal": want solutions that are at most 1% more expensive than optimal
- random variable δ:



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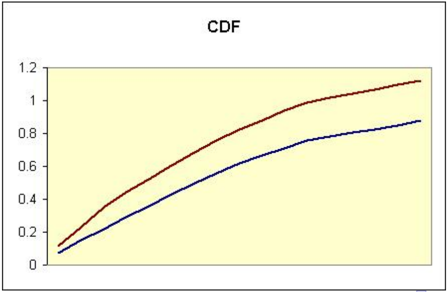
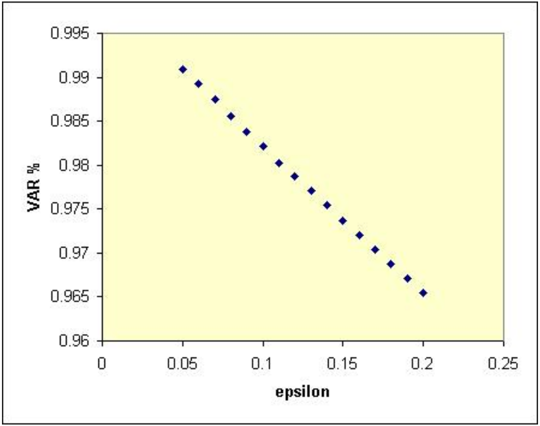


1338 assets, 41 factors, 81 rows, $\leq 1\%$ suboptimality

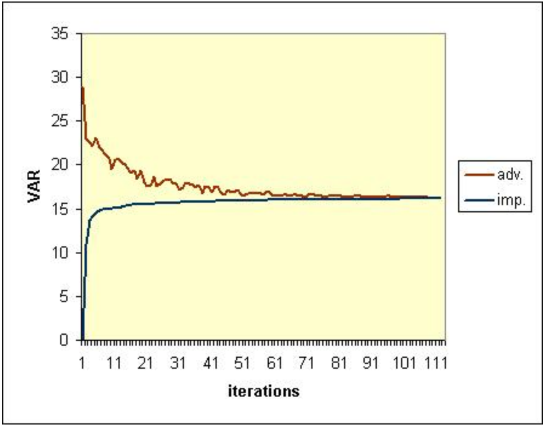
θ	time (sec.)	iters.	VAR	VAR as %
.89	1.18	2	3.43131	71.50
.90	1.42	3	3.74498	78.04
.91	1.42	3	3.74498	78.04
.92	3.47	11	4.05669	84.53
.93	6.35	29	4.05721	84.54
.94	6.37	20	4.05721	84.54
.95	26.59	51	4.35481	90.74
.96	26.25	51	4.35481	90.74
.97	26.20	51	4.35481	90.74
.98	33.07	58	4.63938	96.67
.99	33.11	58	4.63938	96.67

Second set of experiments

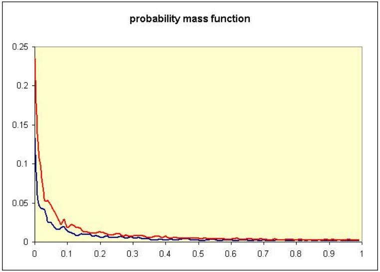
Fix $\theta = 0.90$ but vary suboptimality criterion



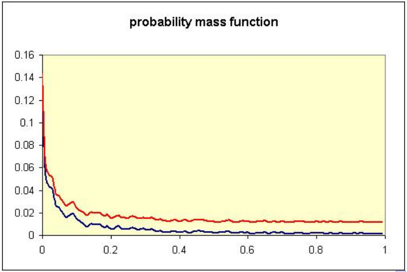
Typical convergence behavior



- Heavy tail, proportional error (100 points):



- Heavy tail, constant error (100 points):



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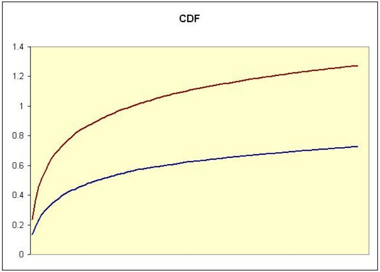

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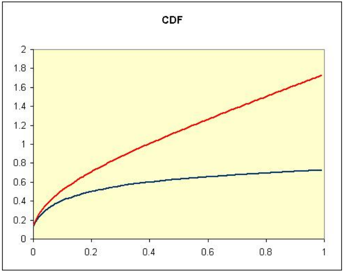

	Proportional		Constant	
θ	Time	Its	Time	Its
.85	4.79	8	11.84	11
.86	2.32	3	8.27	8
.87	6.40	10	9.55	11
.88	4.34	4	18.10	24
.89	8.00	14	5.85	6
.90	2.58	4	13.54	20
.91	4.79	9	16.31	23
.92	7.99	15	13.13	22
.93	13.43	27	22.47	40
.94	10.04	15	21.99	40
.95	9.59	16	11.90	25
.96	6.63	17	29.89	54
.97	48.43	110	16.45	35
.98	20.25	53	20.25	45
.99	22.02	52	21.89	47

A difficult case

- 2464 columns, 152 factors, 3 tiers
- time = 6191 seconds
- 258 iterations
- implementor time = 6123 seconds,
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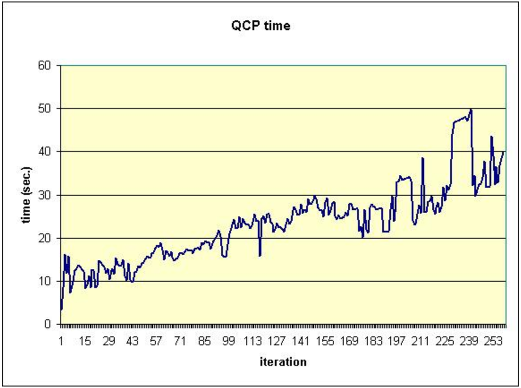
Implementor runtime



Implementor's problem at iteration r

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Subject to:

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Implementor's problem at iteration r

Approximate version

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Subject to:

$$2\lambda\, x_{(k)}^T Q x \;-\; \lambda\, x_{(k)}^T Q x_{(k)} \;-\; \mu^T x \;\leq\; (1+\epsilon)v^*, \quad \forall k < r$$

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$$V \;\geq\; \nu \;-\; \sum_j \bar{\mu}_j \left(1 - \delta_{i(k)} w_j^{(k)}\right) x_j, \quad \forall\, k < r$$

Here, $\delta_{i(k)}$ and $w^{(k)}$ are the adversary's output at iteration $k < r$, and $x_{(k)}$ is the implementor's output at iteration k.

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Other robust models

- Min-max expected *loss* with orthogonal missing factor

  Random return = $\bar{\mu} \bullet (\mathbf{1} - \delta \boldsymbol{w})$ where $-\mathbf{1} \leq \boldsymbol{w_j} \leq \mathbf{1} \;\; \forall \boldsymbol{j}$, and $\mathbf{0} \leq \delta \leq \mathbf{1}$ is a random variable.

  normalization constraints, e.g. $\sum_j w_j = 0$

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  robust problem: $\rightarrow \min_x \; \max_{Q \in \mathcal{Q}} \; \lambda x^T Q x - \mu^T x$

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On-going work: a provably good version of Benders' algorithm