Polynomial-time solvability of extensions of the trust-region subproblem

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Motivation: polynomial relaxations of discrete optimization problems

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\begin{aligned}
& \min \quad c^T x \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \ \geq \ b_i \quad 1 \leq i \leq m \\
& \qquad\quad x \ \in \ \{0,1\}^n
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Sherali-Adams, Lovász-Schrijver:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad a_{ik} x_k + \sum_{j \neq k} a_{ij} x_j x_k \ &\geq \ b_i x_k \quad 1 \leq k \leq n, \quad 1 \leq i \leq m \\ \sum_{j \neq k} a_{ij} x_j - \sum_{j \neq k} a_{ij} x_j x_k \ &\geq \ b_i (1 - x_k) \quad 1 \leq k \leq n, \quad 1 \leq i \leq m \\ x \ &\in \ [0,1]^n \end{aligned}$$

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Semidefinite relaxation:

replace $x_j x_k$ with X_{jk}, $x_j = X_{jj}$ (all j), $X \succeq 0$.

Motivation: continuous problems with combinatorial structure

→ Convex objective, cardinality-constrained optimization problems, e.g.

$$\begin{aligned} \min \quad & x^T M x + c^T x \\ \text{s.t.} \quad Ax &= b, \quad l_j \le x_j \le u_j, \quad 1 \le j \le n \\ \|x\|_0 &\le K. \end{aligned}$$

$M \succeq 0$, $\|\boldsymbol{x}\|_{\mathbf{0}}$ = number of nonzero entries in x.

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Relaxation: Let $x^* = \text{argmin}\{x^T M x + c^T x \, : \, Ax = b, \; l_j \le x_j \le u_j\}$

Suppose $\|\boldsymbol{x}^*\|_0 > \boldsymbol{K}$ $\rightarrow$ can compute **ball** $B \subseteq \mathbb{R}^n$ with

$$\boldsymbol{x}^* \in \textbf{int}\,(\boldsymbol{B}) \text{ and } \|\boldsymbol{x}\| > \boldsymbol{K} \quad \forall\, \boldsymbol{x} \in \textbf{int}\,(\boldsymbol{B})$$

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Better relaxation:

$$min\{x^T M x + c^T x \,:\, Ax = b,\ l_j \le x_j \le u_j,\ x \notin \text{int}\,(B)\}$$

After some iterations

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\min \quad x^T M x \;&+\; c^T x \\
\text{s.t.} \quad Ax \;&=\; b, \quad l_j \leq x_j \leq u_j, \quad 1 \leq j \leq n \\
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(a) How to solve?

(b) Experimental observation – the relaxation becomes much stronger after a small number of iterations.

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Quadratically constrained, quadratic programs

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \; \leq \; 0, \quad 1 \leq i \leq m \end{aligned}$$

Here,

$$f_i(x) \; = \; x^T M_i x + c_i^T x + d_i$$

is a general quadratic

Well-known result

$$\begin{aligned} \min \quad & x^T Q x + c^T x \\ \text{s.t.} \quad & Ax \; \leq \; b \end{aligned}$$

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Positive results?

$\rightarrow$ Polynomial optimization polynomially equivalent with QCQP

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Positive results?

→ Polynomial optimization polynomially equivalent with QCQP

→ Cucker and Bürgisser (STOC 2010):

A solution to a system of **complex** polynomial equations can be computed in *near* polynomial time.

→ A near answer to Smale's 17^{th} problem.

How about over the reals? Let's start with easy results.

Simplest example: S-Lemma (abridged)

Let $\boldsymbol{f}, \boldsymbol{g} : \mathbb{R}^{\boldsymbol{n}} \to \mathbb{R}$ be quadratic functions (degree ≤ 2 polynomials).

Suppose there exists $\bar{\boldsymbol{x}} \in \mathbb{R}^{\boldsymbol{n}}$ such that $\boldsymbol{g}(\bar{\boldsymbol{x}}) > \mathbf{0}$. Then

$$\boldsymbol{f}(\boldsymbol{x}) \geq \mathbf{0} \quad \text{whenever} \quad \boldsymbol{g}(\boldsymbol{x}) \geq \mathbf{0}$$

if and only if there exists $\boldsymbol{\gamma} \geq \mathbf{0}$ such that

$$\boldsymbol{f}(\boldsymbol{x}) \geq \boldsymbol{\gamma} \boldsymbol{g}(\boldsymbol{x}) \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^{\boldsymbol{n}}.$$

Yakubovich (1971), also much earlier, related work

Corollary: Can solve

$$\min\{\boldsymbol{f}(\boldsymbol{x}) : \boldsymbol{g}(\boldsymbol{x}) \geq \mathbf{0}\}$$

in polynomial time (using semidefinite programming)

Note: duality may not hold if there is more than one quadratic constraint

An application: the trust-region subproblem

$$\min\{f(x) \; : \; g(x) \le 0\}$$

can be solved in polynomial time, where f, g quadratics, g convex

Scale, rotate, translate:

$$\min\{f(x) \; : \; \|x\| \le 1\}$$

can be solved in poly time $\rightarrow$ $\log \epsilon^{-1}$

Y. Ye (1992) $\rightarrow$ $\log \log \epsilon^{-1}$

How about *extensions* of the trust-region subproblem?

Sturm-Zhang (2003)

Where $f(x)$ is a quadratic,

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad \|x\| &\leq 1 \\ a^T x &\leq b \qquad (\textbf{one} \text{ linear side constraint}) \end{aligned}$$

can be solved in polynomial time, as can

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad \|x\| &\leq 1 \\ \|x - x^0\| &\leq r_0 \qquad (\textbf{one} \text{ additional convex ball constraint}) \end{aligned}$$

Ye-Zhang (2003)

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad \|x\| &\leq 1 \\ a_i^T x &\leq b_i \quad i = 1, 2 \\ (a_1^T x - b_1)(a_2^T x - b_2) &= 0 \end{aligned}$$

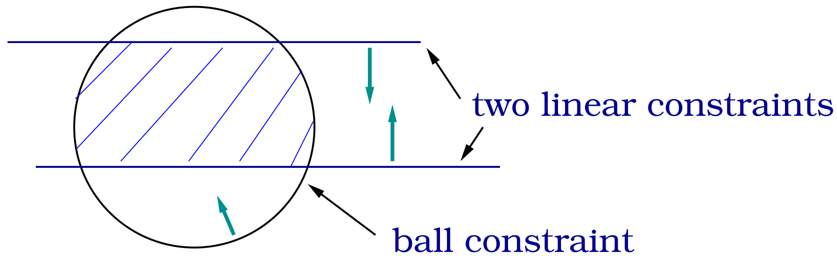
(two linear side constraints, but at least one binding)

Anstreicher-Burer (2012)

In polynomial time, one can solve a problem of the form

$$\begin{aligned} \min \quad & x^T Q x + c^T x \\ \text{s.t.} \quad & \|x\| \leq 1 \\ & a_i^T x \leq b_i \quad i = 1, 2 \end{aligned}$$

provided the two linear constraints are parallel:



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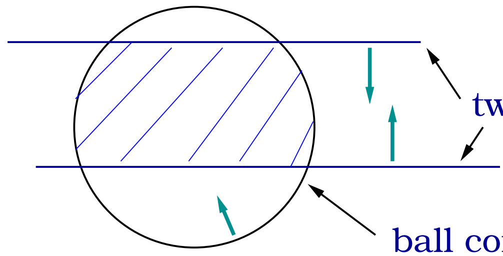

$$\rightarrow \min \{ \, x^T Q x + c^T x \; : \; l \le x_1 \le u, \; \|x\| \le 1 \, \}$$

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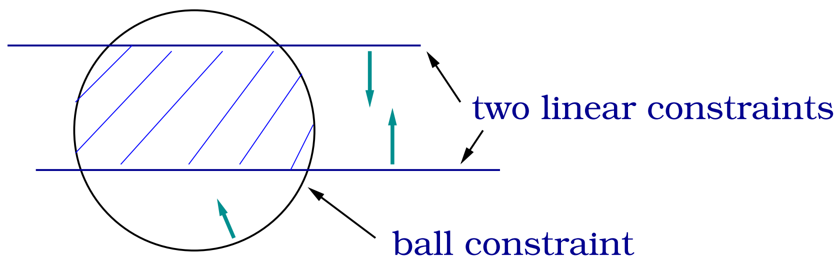

$$\rightarrow \min \{ x^T Q x + c^T x \ : \ l \leq x_1 \leq u, \ \|x\| \leq 1 \}$$

restate as:

$$\begin{aligned} \min \quad & \sum_{i,j} q_{ij} X_{ij} \ + \ c^T x \\ \text{s.t.} \quad & X_{11} + lu \leq (l+u) x_1 \\ & \|X_{.1} - l x\| \leq x_1 - l \\ & \|u x - X_{.1}\| \leq u - x_1 \\ & \sum_j X_{jj} \leq 1 \quad , \quad X \succeq x x^T \end{aligned}$$

Lemma: This problem has an optimal solution with $X = xx^T$. Also: Ye-Zhang

Burer-Yang (2012)

In polynomial time, one can solve a problem of the form

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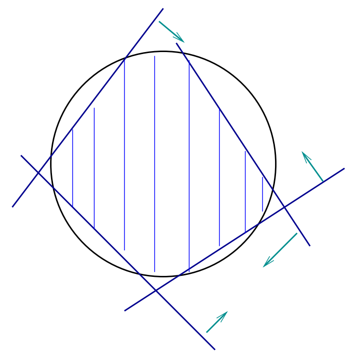

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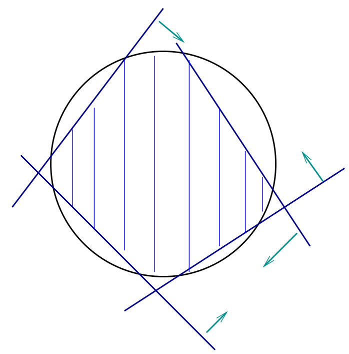

Lemma: the following problem has an optimal solution with $X = xx^T$.

$$\begin{aligned} \min \quad & \sum_{i,j} q_{ij} X_{ij} \; + \; c^T x \\ \text{s.t.} \quad X_{11} + lu \; & \leq \; (l+u)x_1 \\ \|b_i x - X a_i\| \; & \leq \; b_i - a_i^T x \qquad i \leq m \\ b_i b_j - b_j a_i^T x - b_i a_j^T x + a_i^T X a_j \; & \leq \; 0 \qquad i < j \leq m \\ \sum_j X_{jj} \leq 1 \quad & , \quad X \succeq xx^T \end{aligned}$$

This talk

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\text{s.t.} \quad & \|x - \mu_h\| \le r_h, \quad h \in S, \\
& \|x - \mu_h\| \ge r_h, \quad h \in K, \\
& x \in P \doteq \{\, x \in \mathbb{R}^n \; : \; Ax \le b \,\}
\end{aligned}
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Theorem.
For each fixed $|S|$, $|K|$ can be solved in polynomial time if either

(1) $|S| \geq 1$ and polynomially large number of faces of P intersect

$$\bigcap_{h \in S} \{x \in \mathbb{R}^n \; : \; \|x - \mu_h\| \le r_h\},$$

or

(2) $|S| = 0$ and the number of rows of A is bounded.

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Anstreicher-Burer: Case (1) with 3 faces of P meeting the feasible region.

Burer-Yang: Case (1) with $m + 1$ faces of P meeting the feasible region.

More precise statement for case (1)

$$\begin{aligned} \min \quad & x^T Q x + c^T x \\ \text{s.t.} \quad & \|x - \mu_h\| \le r_h, \quad h \in S, \\ & \|x - \mu_h\| \ge r_h, \quad h \in K, \\ & x \in P \doteq \{\, x \in \mathbb{R}^n \,:\, Ax \le b \,\} \end{aligned}$$

Theorem.
For each fixed $|S| \ge 1$, $|K|$ there is an algorithm that solves the problem, to tolerance $0 < \epsilon < 1$ in time

(a) Polynomial in the number of bits in the data and $\log \epsilon^{-1}$

(b) Linear in the number of faces of P that intersect

$$\bigcap_{h \in S} \{x \in \mathbb{R}^n \,:\, \|x - \mu_h\| \le r_h\}.$$

For fixed $|S| \geq 1$, $|K|$ **how to test** for feasibility of

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in time polynomial in the size of the data,

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1. If $K = \emptyset$, a convex optimization problem:

$$\begin{aligned} \min \quad & \|x - \mu_i\|, \quad \text{any given } i \in S \\ \text{s.t.} \quad & \|x - \mu_h\| \leq r_h, \quad h \in S - i, \\ & x \in P \doteq \{\, x \in \mathbb{R}^n \,:\, Ax \leq b \,\} \end{aligned}$$

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2. Otherwise, pick any $i \in K$, and solve

$$\begin{aligned} \min \quad & -\|x - \mu_i\| \\ \text{s.t.} \quad & \|x - \mu_h\| \leq r_h, \quad h \in S, \\ & \|x - \mu_h\| \geq r_h, \quad h \in K - i, \\ & Ax \leq b. \end{aligned}$$

Corollary (but more than we need):

Given a collection of balls $B_h \subset \mathbb{R}^n$ $(h \in S)$

and a polyhedron

$$P = \{x \in \mathbb{R}^n : Ax \leq b\},$$

there is an algorithm that lists the faces of P that intersect $\bigcap_{h \in S} B_h$

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Proof sketch. Use e.g. breadth-first search on the faces of P, starting with P itself.

Basic step:

- Pick a row $a_i^T x \leq b_i$ of $Ax \leq b$.
- Impose $a_i^T x = b_i$.
- Test for feasibility. If feasible, found a new face.

Basic Idea

$$\min\{x^TQx+c^Tx \;:\; \|x-\mu_h\| \leq r_h,\ h \in S, \quad \|x-\mu_h\| \geq r_h,\ h \in K, \quad Ax \leq b\}$$

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Let $\boldsymbol{x^*}$ be optimal. Trivial: there exist (possibly empty) subsets

$\boldsymbol{S^=}$ of $\boldsymbol{S}$, $\boldsymbol{K^=}$ of $\boldsymbol{K}$, and $\boldsymbol{I^=}$ of the rows of $\boldsymbol{Ax \le b}$, such that

$$\|x^* - \mu_h\| = r_h \quad \forall\, h \in S^= \cup K^=, \quad a_i^T x^* = b_i \quad \forall\, i \in I^=$$

$$\|x^* - \mu_h\| < r_h \quad \forall\, h \in S - S^=, \quad \|x^* - \mu_h\| > r_h \quad \forall\, h \in K - K^=$$

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$\boldsymbol{(S^=, K^=, I^=)}$: an **optimal** triple.

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Algorithm will **guess** $\boldsymbol{(S^=,\ K^=,\ I^=)}$ (actually, **compute** $\boldsymbol{I^=}$).

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Algorithm will **guess** $\boldsymbol{(S^=,\ K^=,\ I^=)}$ (actually, **compute** $\boldsymbol{I^=}$).

For each enumerated triple $\boldsymbol{(\hat{S}, \hat{K}, \hat{I})}$, it will (in polynomial time) either

(a) Compute a finite set of vectors tight for $(\hat{S}, \hat{K}, \hat{I})$, one of which must be $\boldsymbol{x^*}$ if the guess is right, **or**

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$\min\{x^TQx+c^Tx \;:\; \|x-\mu_h\| \le r_h,\ h \in S, \quad \|x-\mu_h\| \ge r_h,\ h \in K, \quad Ax \le b\}$

Let $\boldsymbol{x}^*$ be optimal. Trivial: there exist (possibly empty) subsets

$\boldsymbol{S}^=$ of $\boldsymbol{S}$, $\boldsymbol{K}^=$ of $\boldsymbol{K}$, and $\boldsymbol{I}^=$ of the rows of $\boldsymbol{Ax} \le \boldsymbol{b}$, such that

$$\|x^* - \mu_h\| = r_h \quad \forall\, h \in S^= \cup K^=, \quad a_i^T x^* = b_i \quad \forall\, i \in I^=$$

$$\|x^* - \mu_h\| < r_h \quad \forall\, h \in S - S^=, \quad \|x^* - \mu_h\| > r_h \quad \forall\, h \in K - K^=$$

$$a_i^T x^* < b_i \quad \forall\, i \notin I^=.$$

$(\boldsymbol{S}^=,\ \boldsymbol{K}^=,\ \boldsymbol{I}^=)$: an **optimal** triple. x^*: **tight** for $(S^=,\ K^=,\ I^=)$

Algorithm will **guess** $(\boldsymbol{S}^=,\ \boldsymbol{K}^=,\ \boldsymbol{I}^=)$ (actually, **compute** $\boldsymbol{I}^=$).

For each enumerated triple $(\hat{\boldsymbol{S}}, \hat{\boldsymbol{K}}, \hat{\boldsymbol{I}})$, it will (in polynomial time) either

(a) Compute a finite set of vectors tight for $(\hat{S}, \hat{K}, \hat{I})$, one of which must be $\boldsymbol{x}^*$ if the guess is right, **or**

(b) Prove that if $(\hat{S}, \hat{K}, \hat{I})$ is optimal, there is a different **optimal** triple $(\tilde{\boldsymbol{S}}, \tilde{\boldsymbol{K}}, \tilde{\boldsymbol{I}})$ with

$$\tilde{\boldsymbol{S}} \supseteq \hat{\boldsymbol{S}},\ \ \tilde{\boldsymbol{K}} \supseteq \hat{\boldsymbol{K}},\ \ \tilde{\boldsymbol{I}} \supseteq \hat{\boldsymbol{K}} \quad \text{and} \quad |\tilde{\boldsymbol{S}}| + |\tilde{\boldsymbol{K}}| + |\tilde{\boldsymbol{I}}| > |\hat{\boldsymbol{S}}| + |\hat{\boldsymbol{K}}| + |\hat{\boldsymbol{I}}|.$$

Geometry, 1

Notation. Given a ball $B = \{x \in \mathbb{R}^n \,:\, \|x - \hat{\mu}_i\| \le \hat{r}\}$,

$$\boldsymbol{\partial B \doteq \{x \in \mathbb{R}^n \,:\, \|x - \hat{\mu}_i\| = \hat{r}\}}$$

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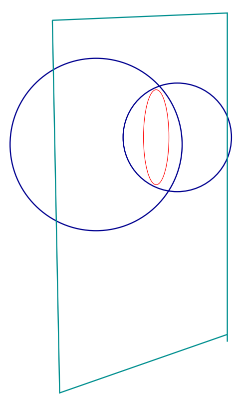
Lemma. Let $B_i = \{x \in \mathbb{R}^n : \|x - \mu_i\| \leq r_i\}$, $i = 1, 2$, be **distinct** and **intersecting**.

There exists an $(n-1)$-dim hyperplane $\boldsymbol{H}$, a point $\boldsymbol{v} \in \boldsymbol{H}$, and $\boldsymbol{r} \geq \boldsymbol{0}$ such that

$$\boldsymbol{\partial B_1 \cap \partial B_2 = \{x \in H : \|x - v\| = r\}}$$

and

$$\boldsymbol{\partial B_i \cap H = \{x \in H : \|x - v\| = r\}, \quad i = 1, 2}$$



Geometry, 1

Corollary Given balls B_i, $i \in I$, not all equal, with

$$\bigcap_{i \in I} B_i \neq \emptyset,$$

there exists an $(n-t)$-dim hyperplane $\boldsymbol{H}$ ($\boldsymbol{t} \geq \mathbf{1}$), $\boldsymbol{v} \in \boldsymbol{H}$ and $\boldsymbol{r} \geq \mathbf{0}$ s.t.

$$\bigcap_{i \in I} \boldsymbol{\partial B_i} = \{\boldsymbol{x} \in \boldsymbol{H} : \|\boldsymbol{x} - \boldsymbol{v}\| = \boldsymbol{r}\}$$

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Implication: When guessing an optimal triple $(\boldsymbol{S}^{=}, \boldsymbol{K}^{=}, \boldsymbol{I}^{=})$

$$\|x^* - \mu_h\| = r_h \quad \forall\, h \in S^= \cup K^=, \quad a_i^T x^* = b_i \quad \forall\, i \in I^=$$

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we

(1) Restrict to a lower dimensional space

(2) Obtain a single, binding, ball constraint

The original problem:

$$
\begin{aligned}
\min \quad & x^T Q x + c^T x \\
s.t. \quad \|x - \mu_h\| &\le r_h, \quad h \in S, \\
\|x - \mu_h\| &\ge r_h, \quad h \in K, \\
a_i^T x &\le b_i, \quad i \in I
\end{aligned}
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Given a guess, this becomes (ignoring the non-binding constraints):

$$
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$$\begin{aligned} \min \quad & x^T Q x + c^T x \\ s.t. \quad \|x - \mu_h\| &\leq r_h, \quad h \in S, \\ \|x - \mu_h\| &\geq r_h, \quad h \in K, \\ a_i^T x &\leq b_i, \quad i \in I \end{aligned}$$

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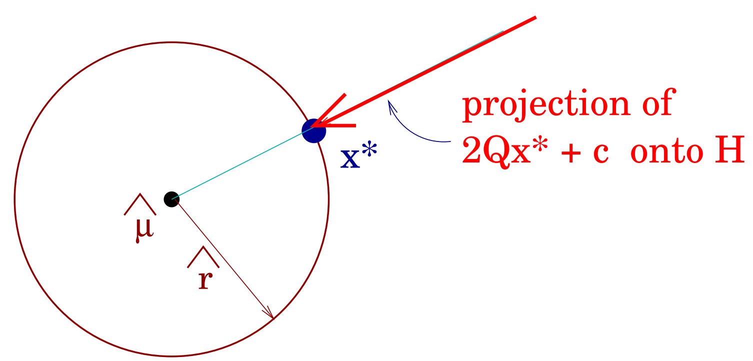
$$\begin{aligned} \min \quad & x^T Q x + c^T x \\ s.t. \quad \|x - \hat{\mu}\| &= \hat{r}, \\ x &\in H \end{aligned}$$

Almost correct: first-order condition restricted to H



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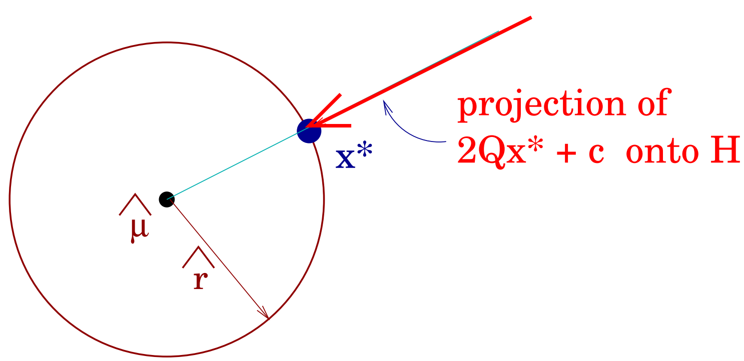

Better: Use projected quadratic representation

Theorem (abridged).

Given a triple $\boldsymbol{(\hat{S}, \hat{K}, \hat{I})}$ there is polynomially computable list of points $\boldsymbol{x^j}$, $\boldsymbol{(j \in J)}$ tight for the triple, such that if $(\hat{S}, \hat{K}, \hat{I})$ is optimal, then either

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(1) $\boldsymbol{x^*} = \boldsymbol{x^j}$ for some $j \in J$, **or**

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$$\boldsymbol{z^T Q z} + \boldsymbol{c^T z} = \boldsymbol{x^{*T} Q x^*} + \boldsymbol{c^T x^*} \quad \forall\, \boldsymbol{z} \in \boldsymbol{\Theta}$$

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Algorithm: Record the minimum-objective $\boldsymbol{x}^j$.

Generalization.

$$\begin{aligned} \min \quad & x^T Q x + c^T x \\ \text{s.t.} \quad & \|x - \mu_h\| \leq r_h, \quad h \in S, \\ & \|x - \mu_h\| \geq r_h, \quad h \in K, \\ & x \in P \doteq \{\, x \in \mathbb{R}^n \,:\, Ax \leq b \,\} \end{aligned}$$

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CDT problem

$$\begin{aligned} \min \quad & x^T Q_0 x + c_0^T x \\ \text{s.t.} \quad & x^T Q_1 x + c_1^T x + d_1 \le 0 \\ & x^T Q_2 x + c_2^T x + d_2 \le 0 \end{aligned}$$

where $\boldsymbol{Q_1 \succ 0}$, $\boldsymbol{Q_2 \succ 0}$

A blast from the past.

Barvinok (STOC 1992):

For each fixed $\boldsymbol{p \geq 1}$, there is a polynomial-time algorithm for deciding feasibility of a system

$$\begin{aligned} x^T M_i \, x \;&=\; 0, \quad 1 \leq i \leq p, \\ \|x\| \;&=\; 1, \end{aligned}$$

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Connection with discrete geometry:

$\rightarrow$ J. Canny, The complexity of robot motion planning (1987)

$\rightarrow$ Connectivity queries in algebraic sets

Theorem.

For each fixed $m \geq 1$ there is a polynomial-time algorithm that, given an optimization problem

$$\begin{aligned} \min \quad & f_0(x) \doteq x^T Q_0 x + c_0^T x \\ \text{s.t.} \quad & x^T Q_i x + c_i^T x + d_i \leq 0 \quad 1 \leq i \leq m, \end{aligned}$$

where $Q_1 \succ 0$, and $0 < \epsilon < 1$, either

(1) proves that the problem is infeasible,

or

(2) computes an ϵ-feasible vector $\hat{x}$ such that there exists no feasible $x \in \mathbb{R}^n$ with $f_0(x) < f(\hat{x}) - \epsilon$.

The complexity of the algorithm is polynomial in the number of bits in the data and in $\log \epsilon^{-1}$