# $S$-free Sets for Polynomial Optimization and Oracle-Based Cuts

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May, 2017

# Background

Consider a mathematical program of the following form

$$\min c^T x$$

subject to $x \in S \cap P$.

$P := \{x \in \mathbb{R}^n | Ax \leq b\}$ is a polyhedral set, and $S \subset \mathbb{R}^n$ is a closed set.

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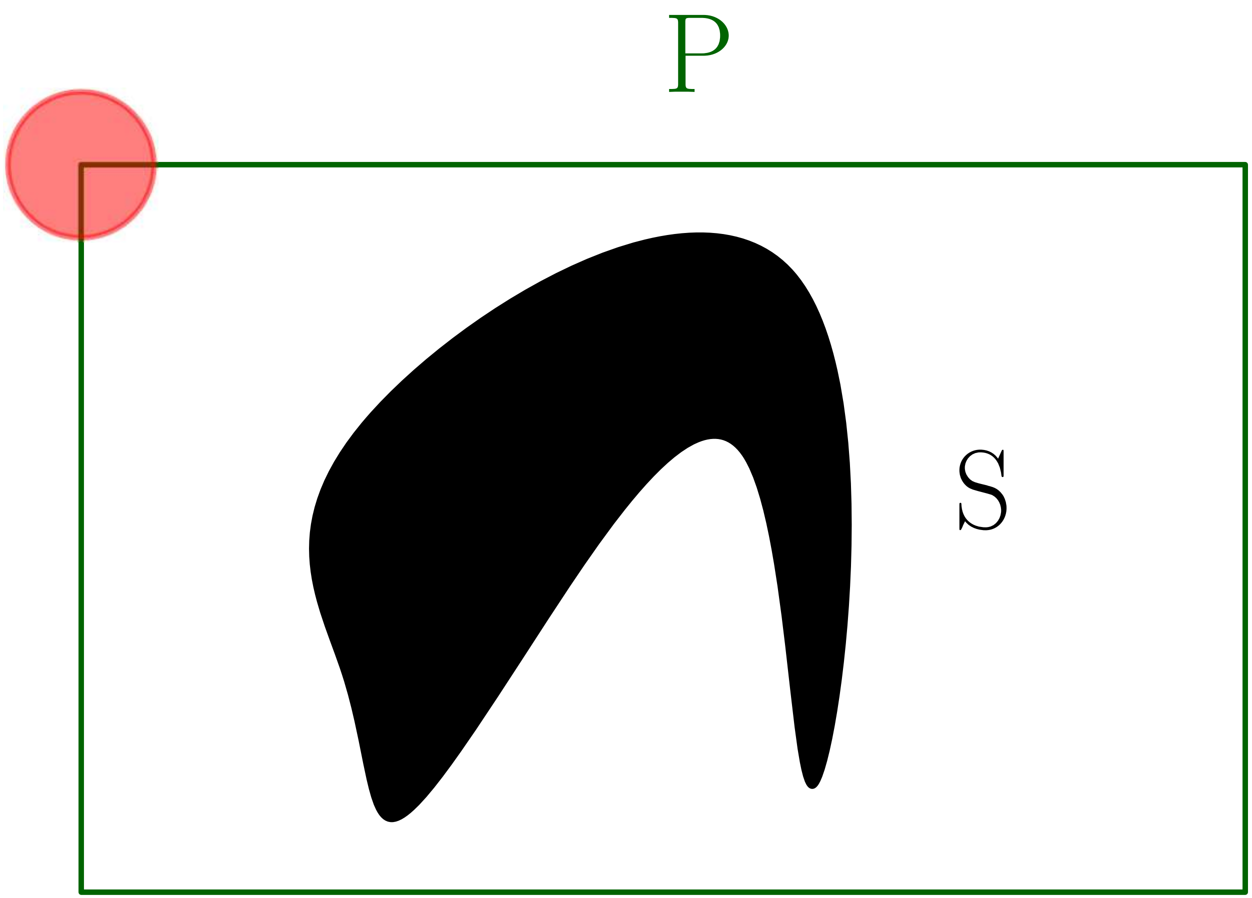
Can we strengthen $P$ with cuts?

We shall focus on the geometric approach: cuts via $S$-free sets.

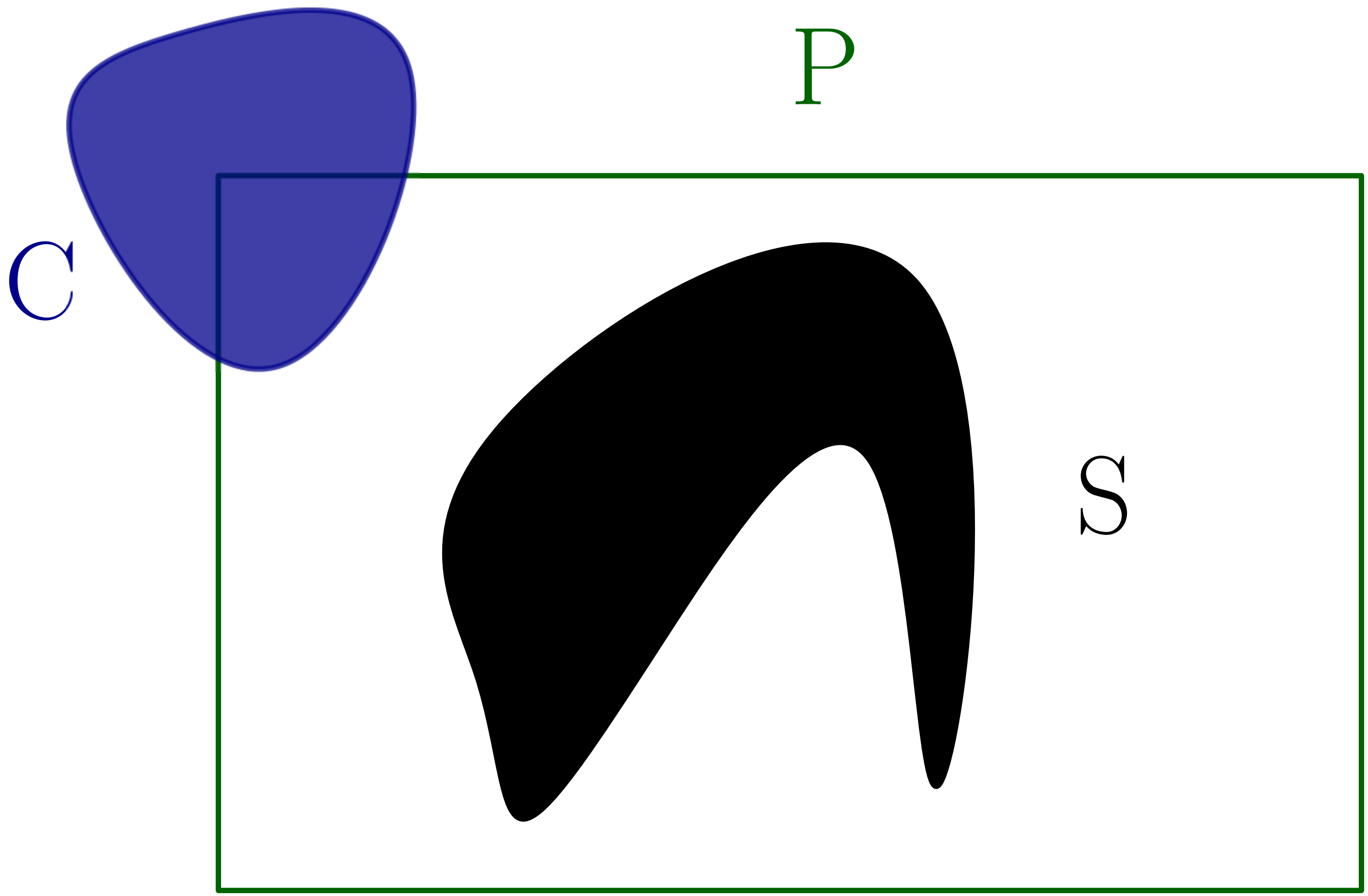
(Many other ways to generate cuts, e.g. disjunctions, algebraic arguments, combinatorics, convex cuts, etc.)



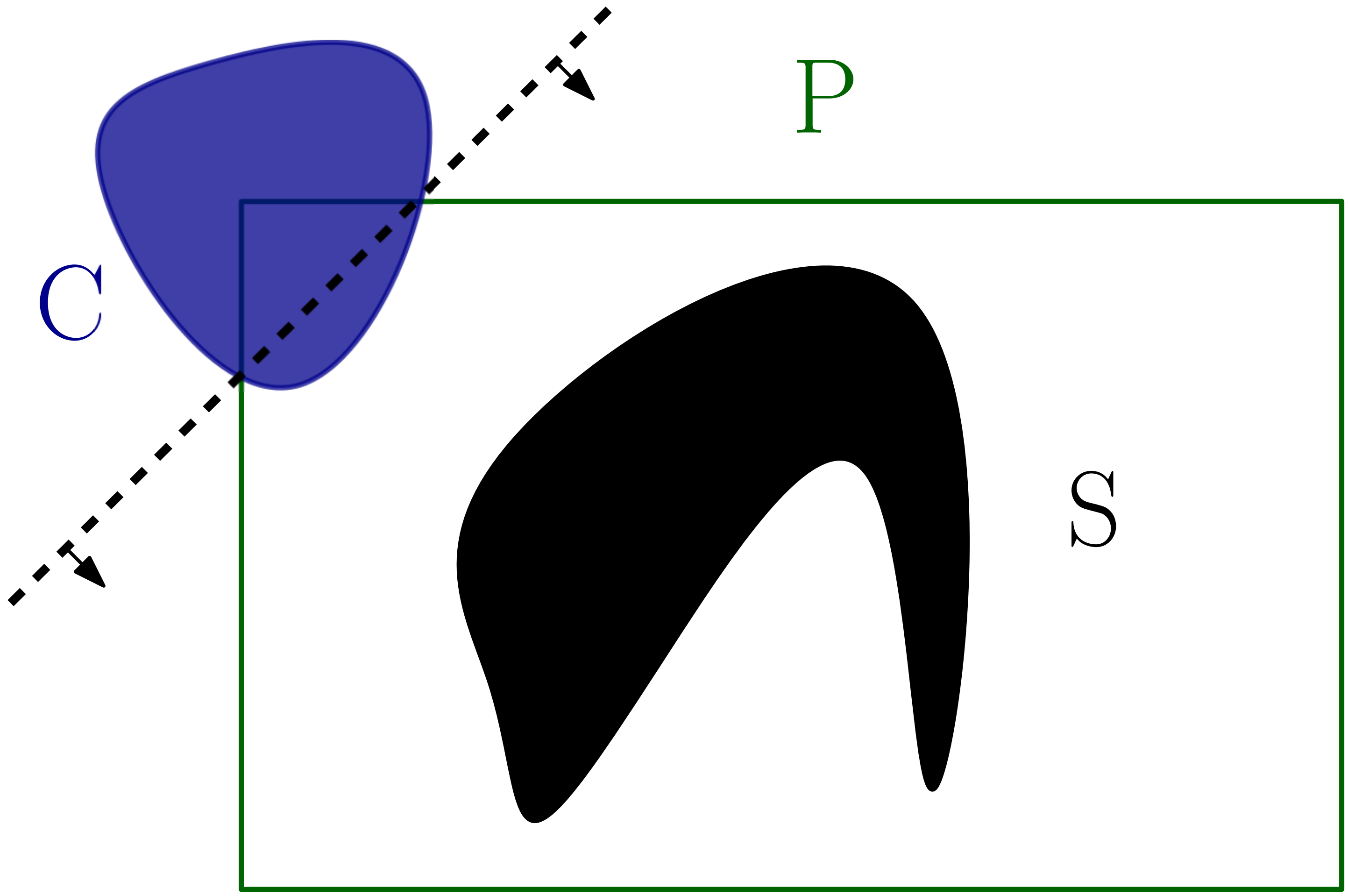

Let $S$ be some closed set. Example: a rectangle $P$. Want to separate the extreme point marked with a red circle from $S$.

# Tightening $P$ with an $S$-free set $C$

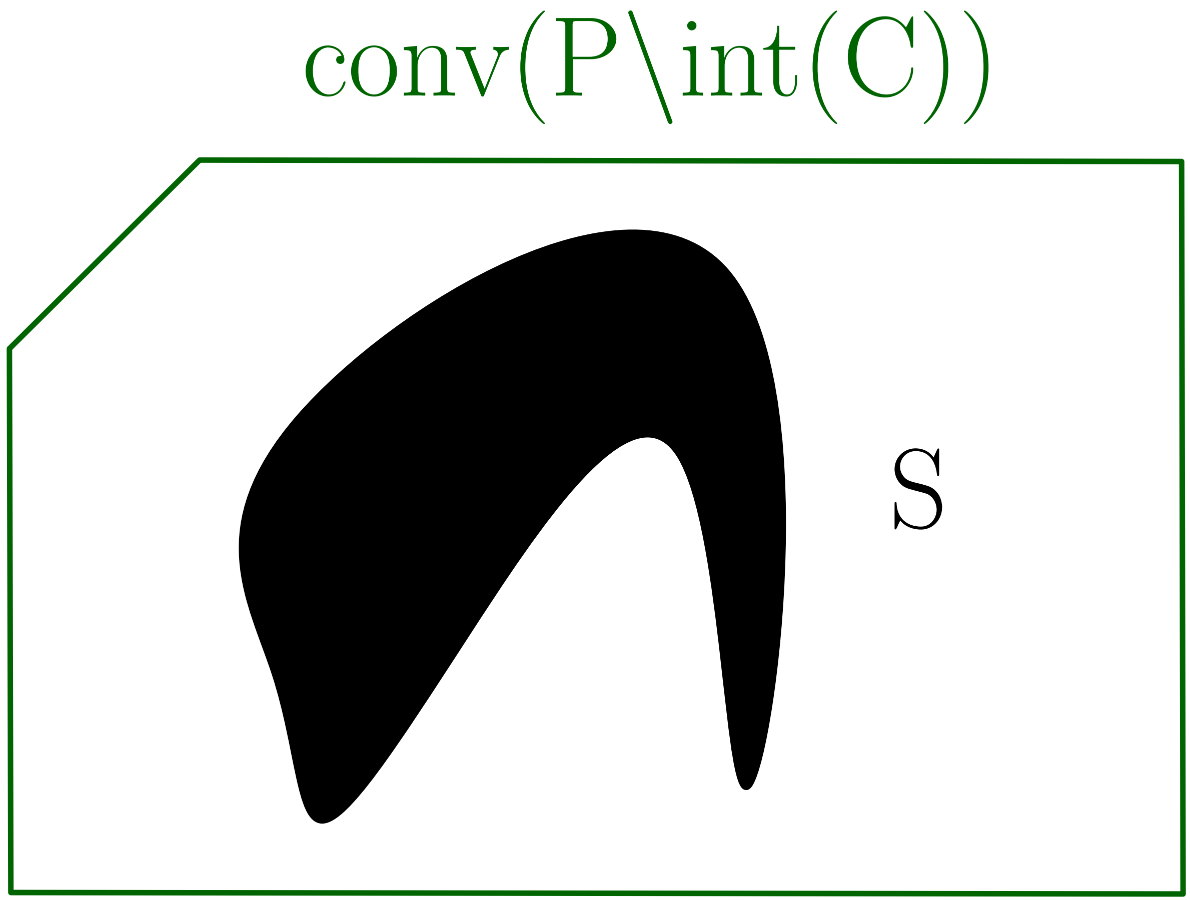




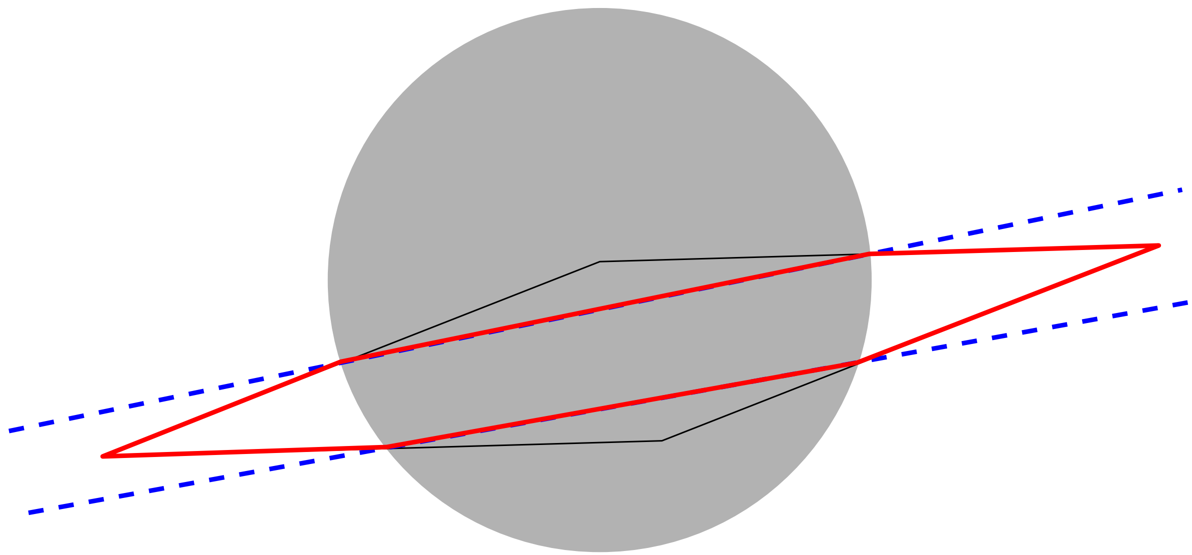

$C$ is an $S$-free set [Dey and Wolsey 2010]: a closed convex set with an interior that does not intersect with $S$.




We obtain a cut by subtracting $C$ from $P$.




Applying the single cut gives us conv($P \setminus C$).



More than one cut, possibly an infinite number are needed.
Separation is generally NP-Hard, e.g. $P$ a polytope, $C$ a ball models an NP-complete set containment problem.

# conv($P \setminus$ int$(C)$) is tricky

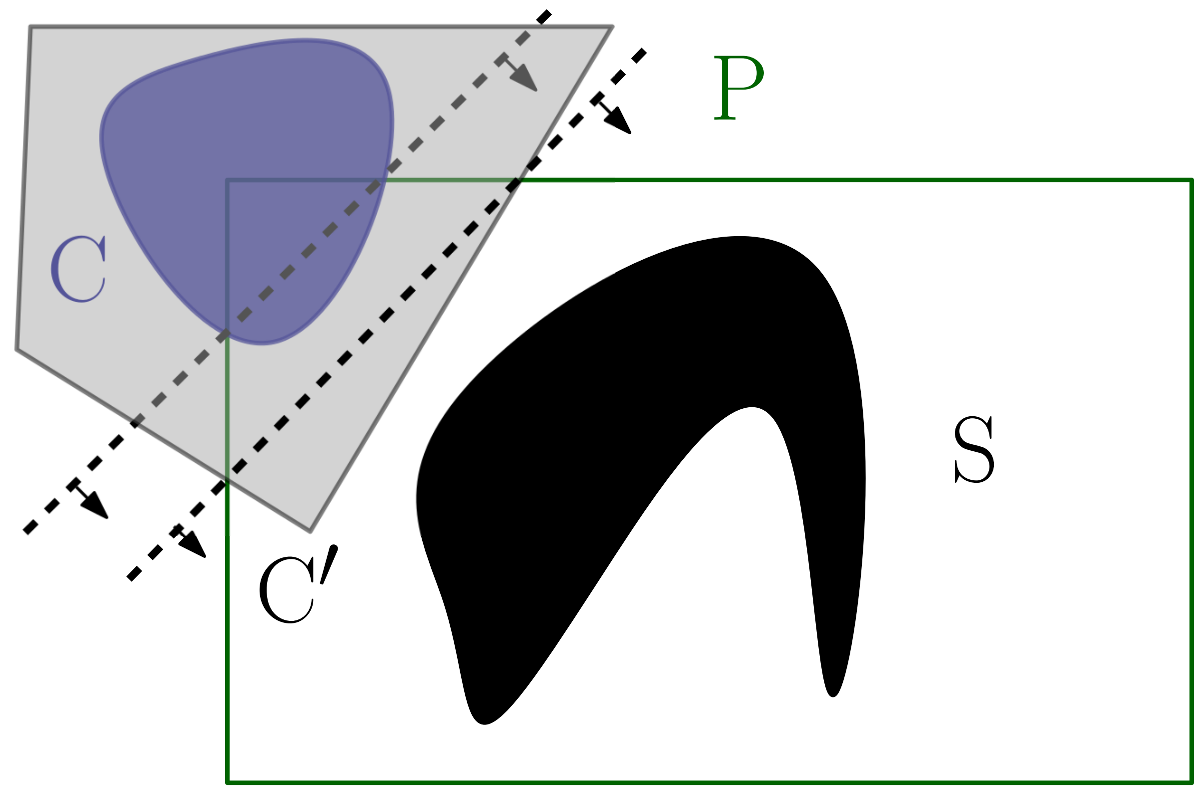


Balas, 1971 (see also Tuy, 1964): If $P$ is a simplicial cone then the intersection cut guarantees separation over conv($P \setminus$ int$(C)$)
Simplicial cone: $n$ linearly independent linear inequalities
Simplicial conic relaxation $P' \supseteq P$ is easily obtained from a basic solution of $P$
With less ambition we go for conv($P' \setminus$ int$(C)$)
Intersection cut is described in closed form $\rightarrow$ fast separation of extreme points of $P$ using $P'$

# Intersection cuts and maximal $S$-free sets



Bigger $C$, deeper cuts.




An $S$-free set $C$ is *maximal* $S$-free if it is not contained in another $S$-free set.

# (Non-Exhaustive) Additional Literature



Maximal $S$-free sets and minimal valid inequalities: [Basu et al. 2010], [Conforti et al. 2014], [Cornuejols, Wolsey, Yildiz, 2015], [Kilinc-Karzan 2015], etc.

Intersection cuts and for mixed-integer conic programs programming: [Atamturk and Narayanan 2010], [Belotti et al., 2013], [Andersen and Jensen, 2013], [Dadush, Dey, Vielma 2011], [Modaresi, Kilinc, Vielma 2015/2016], etc.

Intersection cuts for bilevel optimization: [Fischetti, Monaci, Sinni, 2016].

Generalized intersection cut procedures: [Balas and Margot, 2013], [Balas, Kazachkov, Margot 2016]

# Our Contributions



1) A simple, generic way to generate $S$-free sets that ensures separation. Also, a corresponding cutting plane method for arbitrary closed sets, guaranteed to converge on bounded problems.
2) A study of maximal $S$-free sets for polynomial optimization

# Oracle-Based Cuts

# Distance Oracle



Suppose we have an oracle for a closed set $S$ that gives us the distance $d(x, S)$ from any point $x \in \mathbb{R}^n$ to the nearest point in $S$.

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Integer programming: if $S$ is the lattice, then one can round.
Polynomial optimization: distance can be calculated to arbitrary accuracy in polynomial time
Cardinality constraints: nearest vector of card $(k)$ can be obtained by rounding.

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Observation. The ball centered around $x$ with radius $d(x, S)$ is $S$-free. Call it $\mathcal{B}(x, d(x, S))$.
We shall call the corresponding intersection cut an *oracle ball cut*.

# Cut Closure Convergence



Start with a polytope $P_0$. Define $P_{k+1}$ as $P_k$ intersected with conv($P_k \setminus$ int($\mathcal{B}(x, d(x, S))$)) for every extreme point $x$. This is the rank $k$ *oracle cut closure*.

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Theorem: $\lim_{k\to\infty} P_k =$ conv$(S \cap P)$.

Corollary: given an inexact but arbitrarily accurate distance oracle, we can obtain arbitrarily close (in terms of Hausdorff distance) polyhedral approximation to conv$(S \cap P)$ in finite time.

Borrows from proof technique used in [Averkov 2011].

# Polynomial Optimization

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$$z^* := \inf_{x \in S} p_0(x)$$

$$S := \{x \in \mathbb{R}^n | p_1(x) \geq 0, ..., p_m(x) \geq 0\}$$

# Cuts for Polynomial Optimization



[Saxena, Bonami, Lee 2010/2011] Disjunctive cuts from MILP inner-approximation + convex cuts
Applies to bounded polynomial optimization
[Ghaddar, Vera, Anjos 2011] Projections of moment relaxations.
Generalizes Balas, Ceria, Cornuejols lifting. Separation not guaranteed in general.
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Older literature on convex envelopes of functions, e.g. multilinear.
Our intersection cuts (using e.g. the ball) guarantee polynomial-time separation without boundedness assumptions.

# Lifted Polynomial Representation



[Shor 1987], [Lovasz and Schrijver 1991] Define a vector of monomials, $m = [1, x_1, ..., x_n, x_1x_2, x_1x_3, ..., x_n^k]$. Let $M = mm^T$.

Polynomial optimization can be formulated as

$$\min \langle A_0, M \rangle$$
$$\text{s.t. } \langle A_i, M \rangle \leq b_i, i = 1, ..., m.$$

This is a linear programming relaxation with respect to $M$. $\langle A_i, M \rangle := \sum a_{ij} m_{ij}$ is the inner product.

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Equivalency when $M \succeq 0$, rank$(M) = 1$ and consistency constraints. Dropping the rank constraint gives the moment relaxation [Lasserre, *SIAMOPT* 2001].

# $S$-free sets for Polynomial Optimization



Geometric notions are with respect to a vectorized space, e.g.

$$M \in \mathbb{S}^{2\times 2} \to \{M_{11}, M_{12}, M_{22}\} \in \mathbb{R}^3$$

A convex set (in the appropriate vectorized space) is *outer-product-free* (OPF) if no point in the interior corresponds to a matrix that can be represented as an outer-product.
A set is maximal OPF if no OPF set strictly contains it.

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It turns out OPF sets can have nice structure and lead to easily generated intersection cuts.

# Violation Ball



Using a modification by Dax (2016) of the Eckart-Young-Mirksy theorem, we can find the nearest (by Frobenius norm) outer-product to a given matrix. This uses eigenvalues, and hence can be generated (to specified precision) in polynomial time.

# Max Full-Dim OPF Sets are Cones



Theorem: Let $C \subset \mathbb{S}^{n \times n}$ be an outer-product-free set with full dimension. Then clcone$(C)$ is outer-product-free.

# Negative Semidefinite $P_i$



$\langle P_i, M \rangle \geq 0$

Theorem: Every such halfspace with $P_i$ NSD is maximal outer-product-free.

These halfspaces yield the standard outer approximation cut for SDP: $M \succeq 0 \iff c^T M c \geq 0 \forall c \in \mathbb{R}^n$.

Define $\mathcal{C}_{i,j} := \{M \in \mathbb{S}^{n\times n} | M_{[i,j]} \succeq 0\}$
Theorem: $\mathcal{C}_{i,j}$ is max OPF for $1 \le i \neq j \le n$.

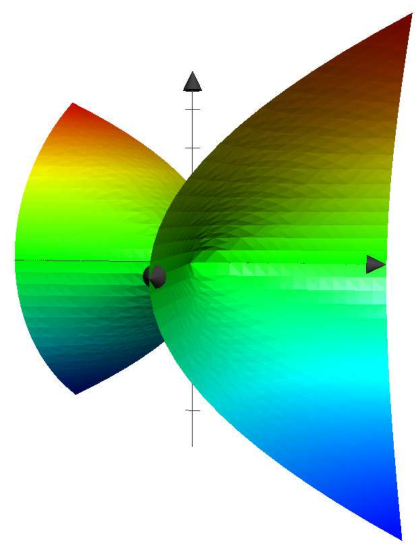
# Characterization for $2 \times 2$



The 3-dimensional case.

Theorem: For $n = 2$, the maximal OPF set are $\mathcal{C}_{1,2}$ and halfspaces of the form $\langle P, M \rangle \geq 0$, where $P$ is NSD.

i.e. the PSD cone and halfspaces with boundaries that support the cone



# Cuts

Recall: we want every extreme point of our relaxation to be an outer-product, i.e. PSD and rank one.
Any non-PSD extreme point can be separated by the outer-approximation cuts $c^T M c \geq 0$.
Any PSD extreme point with rank greater than 1 can be separated by the intersection cut given by $\mathcal{C}_{i,j}$ for some $i, j$ [Chen, Oren, Atamturk 2016].
The oracle cut can be strengthened by recentering the ball and taking the conic hull (complicated expression but computationally fast).

# Numerical Experiments: Setup



Pure cutting plane algorithm implemented in Python using combinations of the following cuts:

OB (Oracle Ball Cuts), SO (Strengthened Oracle Cuts)

OA (Outer Approximation Cuts), 2x2 (2x2 Principal Minor Cuts)

LP solver: Gurobi 7.0.1

Hardware: 20-core server, Intel Xeon 3.10GHz CPU, 264 GB RAM

26 QCQP problems from GLOBALLib (6-63 variables)

99 BoxQP instances (21-126 variables)

# Numerical Experiments: Results



| Cut Family | Initial Gap | End Gap | Closed Gap | # Cuts | Iters | Time (s) | LPTime (%) |
|---|---|---|---|---|---|---|---|
| OB | 1387.92% | 1387.85% | 1.00% | 16.48 | 17.20 | 2.59 | 2.06% |
| SO | | 1387.83% | 8.77% | 18.56 | 19.52 | 4.14 | 2.29% |
| OA | | 1001.81% | 8.61% | 353.40 | 83.76 | 33.25 | 7.51% |
| 2x2 + OA | | 1003.33% | 32.61% | 284.98 | 118.08 | 30.40 | 15.03% |
| SO+2x2+OA | | 1069.59% | 31.91% | 174.79 | 107.16 | 29.55 | 12.56% |

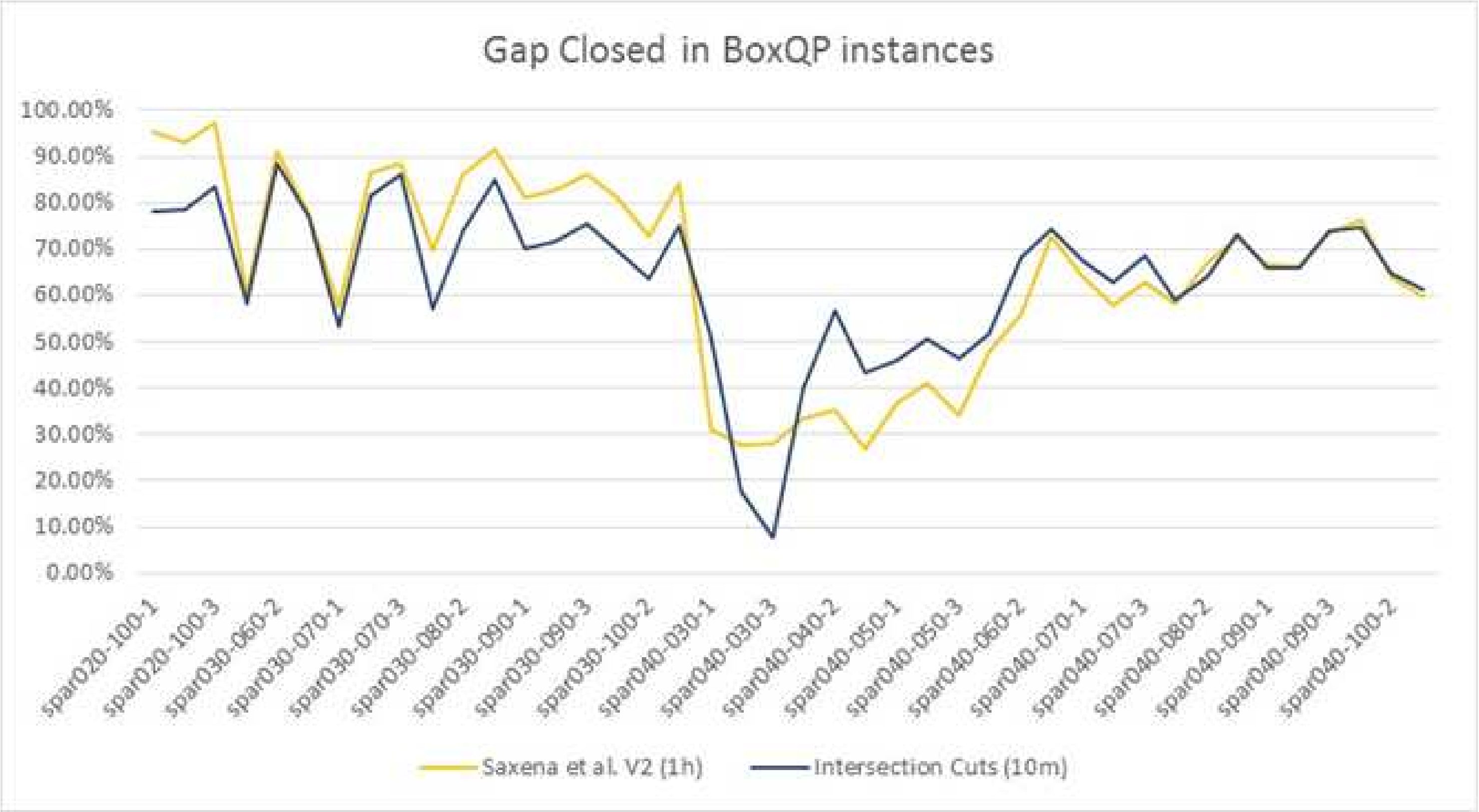
Table 1: Averages for GLOBALLib instances

| Cut Family | Initial Gap | End Gap | Closed Gap | # Cuts | Iters | Time (s) | LPTime (%) |
|---|---|---|---|---|---|---|---|
| OB | 103.59% | 103.56% | 0.04% | 12.84 | 13.62 | 127.15 | 0.40% |
| SO | | 103.33% | 0.34% | 14.34 | 15.45 | 132.07 | 0.49% |
| OA | | 30.88% | 75.55% | 676.90 | 137.52 | 459.28 | 31.80% |
| 2x2 + OA | | 32.84% | 74.52% | 349.21 | 140.40 | 473.18 | 28.76% |
| SO+2x2+OA | | 33.43% | 74.03% | 227.39 | 136.93 | 475.38 | 26.59% |

Table 2: Averages for BoxQP instances

# Comparison with V2: BoxQP

V2: second-order conic outer-approximation of PSD constraint MIP to derive disjunctive cuts



# Comparison with V2: GLOBALLib



| Instance | V2 Gap | V2 Time | Gap Closed | Time |
|---|---|---|---|---|
| Ex2_1_1 | 72.62% | 704.40 | 53.21% | 0.41 |
| Ex2_1_5 | 99.98% | 0.17 | 99.68% | 0.13 |
| Ex2_1_6 | 99.95% | 3397.65 | 93.87% | 0.95 |
| Ex2_1_8 | 84.70% | 3632.28 | 73.23% | 19.13 |
| Ex2_1_9 | 98.79% | 1587.94 | 29.87% | 36.9 |
| Ex3_1_1 | 15.94% | 3600.27 | 0.34% | 0.55 |
| Ex3_1_2 | 99.99% | 0.08 | 99.98% | 0.04 |
| Ex3_1_4 | 86.31% | 21.26 | 29.49% | 0.26 |
| Ex5_2_2_case1 | 0.00% | 0.02 | **2.05%** | 0.47 |
| Ex5_2_2_case2 | 0.00% | 0.05 | 0.00% | 0.26 |
| Ex5_2_2_case3 | 0.36% | 0.36 | 0.00% | 0.16 |
| Ex5_2_4 | 79.31% | 68.93 | 29.04% | 5.69 |

Table 3: Comparison with V2 on GLOBALLib instances

# Comparison with V2: GLOBALLib



| Instance | V2 Gap | V2 Time | Gap Closed | Time |
|---|---|---|---|---|
| Ex5_3_2 | 7.27% | 245.82 | 0.00% | 2.33 |
| Ex5_4_2 | 27.57% | 3614.38 | 0.24% | 0.59 |
| Ex9_1_4 | 0.00% | 0.60 | 0.00% | 0.34 |
| Ex9_2_1 | 60.04% | 2372.64 | 54.17% | 28.37 |
| Ex9_2_2 | 88.29% | 3606.36 | 77.90% | 30.84 |
| Ex9_2_6 | 87.93% | 2619.02 | **90.45%** | 0.12 |
| Ex9_2_8 | - | - | **83.27%** | 0.12 |

# Conclusion

# Take-homes



We introduced an oracle-based intersection cut for closed sets. Furthermore, we constructed a convergent cutting plane algorithm that uses this oracle to 'ping' the set $S$. All of this is done without using any explicit structure about $S$.

Outer-product-free sets provide a new way to generate cuts for polynomial optimization.

# Thanks!

Preprint available:

http://arxiv.org/abs/1610.04604