THE COMPETITIVE FACILITY LOCATION PROBLEM IN A DUOPOLY

Y. GUR*, D. SABAN*, AND N.E. STIER-MOSES*,†

* *Graduate School of Business, Columbia University, New York, USA*

† *Universidad Torcuato Di Tella, School of Business, and CONICET, Buenos Aires, Argentina*

{ygur14,dsaban15,stier}@gsb.columbia.edu

Abstract. We consider a competitive facility location problem on a network in which consumers located on vertices wish to connect to the nearest facility. Knowing this, each competitor locates a facility on a vertex, trying to maximize her market share. We focus on the two-player case and study conditions that guarantee the existence of a pure-strategy Nash equilibrium in this finite non-cooperative game, for progressively more complicated classes of networks. For general graphs, we show that attention can be restricted to a subset of vertices referred to as the central block. By constructing trees of maximal bi-connected components, we obtain sufficient conditions for equilibrium existence. Moreover, when the central block is a vertex or a cycle (e.g., cactus graphs), this provides a complete and efficient characterization of equilibria: at equilibrium both competitors locate their facilities in a solution to the 1-median problem, generalizing a well-known insight arising from Hotelling's model. We further show that an equilibrium must solve the 1-median problem in other classes of graphs, including grids which essentially capture the topology of urban networks. In addition, when both players select a 1-median, the solution must be at equilibrium for strongly-chordal graphs, generalizing a previously known result for trees. We perform an efficiency analysis and show that removing edges from a graph whose central block is a cycle or a vertex increases the consumer cost. This implies that the worst-possible equilibria are achieved in trees. While equilibria can be arbitrarily inefficient relative to centralized solutions, we quantify the inefficiency gap with parametric upper bounds that depend on topological network parameters. Finally, we consider the practical robustness of our results by taking random neighborhoods of a real city and solving for equilibria. We show that the insights derived from theory hold with high probability for urban topologies even when not a grid or when players are not homogeneous.

Keywords: Competitive facility location, Hotelling competition, Nash equilibrium, Price of anarchy.

1. Introduction

Facility location problems study how to best locate facilities under the assumption that consumers will go to the facility that is most convenient to them. A huge amount of work has been done in this area. Most of it considers the perspective of centralized optimization, but a significant part of the existing literature has focused on competitive models where each player chooses a location that maximizes consumer demand. Namely, Hotelling (1929) introduced an influential competitive facility location model where each of two players select a location in a linear segment, and a continuum of uniformly-distributed consumers along that segment select the closest facility. Assuming that the price of consumption is constant (hence, pricing is not part of the strategic considerations of the game), the objective of both players is to maximize their demand. The model described by Hotelling induces a non-cooperative game that results in an equilibrium in which both players locate together in the center of the segment, providing intuition regarding observed behavior in a wide spectrum of competitive situations.

Facility location games have several applications. Besides the most immediate one, which is to use them to locate facilities, there are applications related to voting theory (Taylor 1971, Hansen et al. 1990), rumor dissemination and seeding in social networks (Bharathi et al. 2007, Kostka et al. 2008), and product differentiation models. In the latter, competing firms wish to maximize market share by selecting features

Date: April 2014.

of the product, knowing that its competitors will behave similarly. The analysis of such situation has been the subject of several research articles that focused on spatial competition in a continuous (usually linear) market in which consumers are distributed (in most cases, uniformly) over a low (mostly one) dimensional space (see, e.g., Eaton and Lipsey 1976, Graitson 1982, D'Aspremont et al. 1983, de Palma et al. 1985, Gabszewicz and Thisse 1986). Another traditional model is Salop's circular city (Salop 1979), which is used for location as well as product applications. We note that there have been relatively few attempts to study these problems when the location problem is captured by more general topologies. Below we provide a short overview of several research efforts related to location theory, focusing in those most relevant to our proposed model. For additional details, we refer the reader to surveys (Eiselt and Laporte 1989, Eiselt et al. 1993, Eiselt and Laporte 1996, Plastria 2001, ReVelle and Eiselt 2005, Smith et al. 2009, Dasci 2011, Kress and Pesch 2012) and books (Mirchandani and Francis 1990, Drezner 1995b, Eiselt and Marianov 2011).

The main insight from Hotelling's model is that both competitors choose to locate in the *center of the market*. This manifests itself in practice even for markets that are more complex than the original setting of customers located uniformly in a continuous segment. For instance, it is common to observe in small towns or in neighborhoods of big cities that competing local businesses such as fast food chains, coffee shops or banks are located one next to another. Gas stations tend to be located in opposite corners instead of being uniformly spaced in an area. A similar behavior is observed in product design selection, where many competing products in a market have similar design and comparable features. While other considerations besides spatial competition may contribute to this behavior, these empirical observations suggest that the intuition provided by Hotelling extends to more complex networks. The purpose of this work is to offer a framework to formalize why competitors choose to cluster in close proximity to the median, as opposed to distribute throughout the network to serve the market more efficiently. We do this for a duopoly and progressively consider more complex underlying network topologies to show that these insights are quite robust.[1] More specifically, we characterize the location of equilibria in simple structures such as trees and cycles, and continue to more general topologies, such as graphs whose *central block* is a vertex or cycle, as well as grids, median graphs, and strongly chordal graphs. The facility location game we study is a constant-sum game, and as such, dynamics such as best response and fictitious play converge to a Nash equilibrium. This provides support to the locations predicted by this model because of the robustness of the solution concept.

In our model, vertices represent both the possible locations of facilities and demand. The discrete structure of the graph greatly complicates the analysis of the game compared to the more-studied continuous version of the problem. To the best of our knowledge, Wendell and McKelvey (1981) were the first to extend the basic model of Hotelling to a network context in which vertices represent potential products as well as consumers, and edges succinctly encode the possible assignments of consumers to products. (Slater (1975) and Hakimi (1983) had already considered a network context but for a different model.) While Wendell and McKelvey considered the set of possible locations to be the entire graph, including the interior of edges, we only allow facilities to be located in the vertices. The assumption that players select vertices of the network, which is a common one, makes the game finite, and therefore guaranteeing that an equilibrium in mixed strategies always exists (Nash 1951). Nevertheless, as randomizing over the space of locations or product features is not natural (it is hard to imagine that someone can make an important decision with a big financial impact using a coin flip), we focus on pure-strategy equilibria. The possibility that a pure-strategy Nash equilibrium may not exist motivates us to study conditions that guarantee its existence for progressively more complicated network structures. (Interestingly, Bhadury and Eiselt (1995) propose to smooth the dichotomic equilibrium-existence decision by quantifying how far a given instance is from admitting an equilibrium.)

[1]When there are more than two players, they may not have the incentive to choose the same location. Actually, even for very simple topologies equilibria may fail to exist. We discuss the case of more than two players in Section 7.

Dürr and Thang (2007) prove that deciding if an instance of the game for a general number of players possesses a Nash equilibrium is $\mathcal{NP}$-complete, even for the case of vertices with equal demands.

Main Contributions. Our main contribution lies in putting forward connections between spatial competition and centralized optimization. Motivated by the result of Eiselt and Laporte (1991), our work establishes further links between Nash equilibria of the facility location problem in a duopoly and the 1-median problem for various classes of topologies. To the best of our knowledge, with the exception of that reference, we are not aware of other results of this kind. A 1-median, or simply a median, is a vertex that minimizes the (weighted) distance to all other vertices. Since it is natural for players to choose a central location in the market, 1-medians would seem to be obvious candidates to locate facilities at equilibrium. Nevertheless, this intuition is not always right. There are instances that admit equilibria not located at 1-medians. These differences provide the motivation to understand the circumstances under which the players' incentive is to select solutions to the 1-median problem. In more detail, the contribution of the paper goes along the following lines.

Considering duopolies on cycles with general positive weights, we prove that an equilibrium exists if and only if one of the vertices has a sufficiently big demand so it is convenient to locate a facility there. This relates to Mavronicolas et al. (2008), who studied cycles with the restriction that all vertices have unit demand and characterized all possible equilibria for an arbitrary number of players. To provide results for more general graphs, we rely on Eiselt and Laporte (1991) who showed that an equilibrium of a facility location game with two players on a tree with arbitrary weights is always guaranteed to exist and it consists of facilities located arbitrarily in the 1-medians—also known as centroids—of the tree, of which there are at most two. Using this, we reduce an arbitrary graph to a tree of maximal bi-connected components, usually referred to as *blocks*, and show that the equilibrium must be located in the bi-connected component that corresponds to the 1-median of that tree (referred to as the *central block*). This provides a general characterization of equilibria for graphs whose central block has a topology for which we can describe equilibria (for example, graphs with a central block that is a vertex or a cycle, a property we refer to as having a *simple central block*). The construction of a tree of bi-connected components resembles those of Slater (1975) and Hatzl (2007) for centralized median problems in general graphs.

Our analysis allows one to verify existence of equilibria in linear time with respect to the number of vertices for progressively more complicated topologies (trees, cycles, sparse simple central block graphs), which is at least a cubic speedup compared to a naive approach of computing equilibria by exhaustive search.[2] For general graphs, we reduce the decision problem to that in a bi-connected component that can be identified in linear time with respect to the number of edges. Although the worst-case complexity of the problem is the same as in the general graph, this allows one to focus attention to the "appropriate" part of the graph, which is typically smaller.

In addition, we provide the following broad necessary condition for equilibria of cycles or graphs with a simple central block: a vertex chosen at equilibrium must be a 1-median. For example, a class of graphs with simple central blocks consists of *cacti*, for which edges are contained in at most one cycle. Checking if an instance admits equilibria reduces to checking if its central block admits one itself. To put this in perspective, it is also known that restricting attention to cacti simplifies the facility location problem (e.g., Ben-Moshe et al. (2005) and Hatzl (2007) provide more efficient algorithms for the (centralized) 1-median and 2-median problems on cacti than what is known for general graphs).

[2]Because the facility location game is constant-sum, linear programming can be used to compute (mixed-strategy) Nash equilibria, hence immediately proving a polynomial-time upper bound to the time complexity of finding solutions. Although this is slower than exhaustive search, this approach provides a combinatorial description of equilibria.

Next, we show that for an arbitrary graph topology, when an equilibrium exists, both players select vertices that are *local optima* of the 1-median problem. This result automatically translates to proving that equilibria can only be located at a 1-median for graphs where all local optima are global ones, such as median, quasi-median, and Helly graphs. Importantly, these families of graphs include grids and latices, which capture the topology of many real urban networks, supporting observations of competitors locating close to each other in urban environments. We also provide the first generalization of the result of Eiselt and Laporte (1991) for trees. We prove that when both players select an arbitrary 1-median of a strongly chordal graph automatically results in an equilibrium. That class of graphs includes trees as well as other topologies such as interval graphs and block graphs.

Subsequently, we study the inefficiency introduced by the spatial competition. At equilibrium, the consumer cost is not necessarily optimal because players ignore it when selecting their location. In the late 1990s, Koutsoupias and Papadimitriou (2009) introduced the concept of *price of anarchy* to quantify the gap between the consumer cost at equilibrium and the minimum possible consumer cost that would be attained if players were controlled by a social planner (the term was coined later by Papadimitriou 2001). Considering simple instances of facility location problems such as the traditional one proposed by Hotelling, it is clear that the equilibrium outcome of both facilities placed in the middle of the $(0, 1)$ segment is inefficient compared to the centralized solution, which would place the facilities in positions $1/4$ and $3/4$. Considering the case of cycles with unit demand, Mavronicolas et al. (2008) computed that the price of anarchy equals $9/4$. We provide results on the inefficiency of equilibria for more general topologies. Indeed, we present instances that illustrate that equilibria of the facility location game can be extremely inefficient, even in trees or cycles.

To understand which instances are particularly bad, we study sufficient conditions that guarantee that the game is monotone with respect to edge inclusions. In other words, we study in what cases removing edges cannot increase the total consumer cost at equilibrium, excluding situations that are reminiscent of the *Braess paradox* (1968).[3] We conclude that for monotone instances the worst-possible inefficiencies are achieved in trees. Although the price of anarchy cannot be bounded from above, we provide parametric upper bounds that depend on the size of the network, its diameter, and the variability of demands in the instance. These parameters refer to the instance and are independent of the equilibrium itself; hence, they can be used to provide efficiency results for classes of instances that satisfy certain characteristics *ex-ante*. These bounds are informative when considering realistic applications for which the aforementioned parametric values are typically bounded. While an equilibrium in a duopoly can be inefficient when compared to the centralized 2-median problem, when determining the efficiency by comparing it to the centralized 1-median problem, conclusions change drastically. Because we prove that the equilibrium and the 1-median coincide, both solutions have the same total consumer cost. Hence, a facility location problem in a duopoly at equilibrium is not worse than what the same centrally coordinated problem will achieve if it can only use a single facility. This observation resembles the resource augmentation approach of Roughgarden and Tardos (2002) whereby one quantifies the inefficiency of an equilibrium comparing it to a more costly centralized problem to compensate the coordination effort that an equilibrium must go through.

Finally, we validate the robustness of our insights by relaxing the assumption of player homogeneity and by considering graphs outside predefined families, both on which our theoretical results rely upon. To that end, we test the insights arising from our theoretical results on a set of random experiments using graphs extracted from neighborhoods of the city of Buenos Aires, Argentina, which exhibit a grid-like structure for the most part. Regarding equilibrium existence, although one can construct very simple examples that do not admit equilibria for heterogenous players (in which case the demand to equidistant facilities is not split

[3]This paradox refers to the apparently contradictory behavior of a transportation network for which the consumer cost at equilibrium *increases* when an edge is added to the network.

half-and-half), the experiments provide evidence that equilibria exist with high probability for real-world urban networks and mild to modest departures from players' homogeneity. Equilibria and the solutions to the 1-median problem are located in the same set of vertices for the majority of the simulated instances, although our simulations found a limited number of counterexamples.

Structure of the Paper. We now describe additional connections to the literature. Section 2 introduces the facility location game that we study. In Section 3 we characterize equilibrium locations in progressively more complex topologies, starting with trees and cycles. Using a transformation of general graphs into cycles and trees that allows us to conserve crucial information to characterize equilibria, we present a complete characterization of equilibria on simple central block graphs. In Section 4 we further explore the relations between equilibrium profiles and the solutions of the 1-median problem, while in Section 5 we discuss the efficiency of the equilibria of the game. Section 6 validates the robustness of our results using random neighborhoods from real-world urban networks. Section 7 concludes with directions of future research and final notes. Proofs of some results appear in the Appendix.

Further connections to location theory literature. We do not intend to re-examine the taxonomy of facility location games here because there is extensive work in the area. We would rather refer the readers to the previously-cited surveys and books, and instead comment on additional research related to competitive location theory. Recognizing that the existence of Nash equilibria in location games cannot be guaranteed in general, several research efforts focused on alternative solution concepts, on multiple variants of the game, and on the dynamics of the game when players are allowed to re-optimize. A common equilibrium concept employed by several articles was to consider sequential models where agents locate facilities sequentially (Hay 1976, Prescott and Visscher 1977, Drezner 1982, ReVelle 1986). This leads to Stackelberg games where players anticipate the moves of their opponents and play accordingly. Hakimi (1983, 1986) presented an algorithmic approach to compute the Stackelberg equilibrium. Some of these papers analyze cases when players control several facilities; other variants include using utility models (Leonardi and Tadei 1984, Drezner and Drezner 1996), distributing demand using a logit approach (Drezner et al. 1998), applying a gravity rule (Drezner 1995a, Drezner et al. 2002, Drezner and Drezner 2004), and defining a sphere of influence to capture that demand that is far from facilities may be lost (ReVelle 1986, Drezner et al. 2011).

Voronoi games are continuous versions of a similar competitive facility location problems where facilities can be placed in a domain and players obtain the area corresponding to points in the domain that are closest to the facility (Ahn et al. 2004). Vetta (2002) introduced a model of facility location with competition where agents that place facilities compete on prices for consumer demand. Indeed, as in our case demand will be allocated to the facility that is at a shortest distance but will be sold at a price equal to the production cost of the second most efficient facility that can serve the customer. In that context, Vetta proves that the game always has a pure-strategy Nash equilibrium. Furthermore, in the absence of fixed costs, he proves that the social welfare at equilibrium is never less than half of the maximum social welfare, which would be achieved by a centralized assignment. Mirrokni and Vetta (2004) and Goemans et al. (2005) follow up on Vetta's work and study the speed of convergence of best response dynamics and provide a more detailed analysis of the inefficiency induced in that game. In addition, Jain and Mahdian (2007) provide an overview of related research from the point of view of cooperative game theory, where the cost of opening facilities must be shared by participating agents in the system.

Finally, while in this work we assume that the price of consumption is constant (potentially zero) so pricing is not part of the strategic considerations of the game, one may find a multitude of different assumptions in the literature. For instance, under *mill pricing* all consumers are charged the same price but transport costs are passed to them, under *uniform delivered pricing* consumers pay equal prices regardless of their location, and under *spatial price discrimination* prices are location-specific. Since it is out of scope in this paper to comment on the various alternatives, we refer the reader to the survey papers cited earlier for details.

2. The Competitive Facility Location Game

We consider a finite undirected graph $G(V, E)$ whose vertices represent the locations of consumers and the potential locations of facilities. Each vertex $v \in V$ has an associated weight $w(v) > 0$ that measures the demand level of that location.[4] We define $W(S) := \sum_{v \in S} w(v)$, and refer to the total demand as $W := W(V)$. Edges in E form a connected graph. Every edge $e \in E$ has a unit length. This assumption is made for the sake of simplicity, and we do not lose much generality since we can always subdivide edges (we will further discuss this in Section 7). The game is played between two players $N := \{1, 2\}$ who select a vertex each to locate its facility. We denote the solution of the game by $\overline{x} := (x_1, x_2)$, where $x_i \in V$ is the vertex selected by player $i \in N$.

Given vertices $x, y \in V$, we denote the set of x-y paths by P_{xy}. We refer to the distance between x and y by $d_G(x, y) := \min\{|p| : p \in P_{xy}\}$, or simply by $d(x, y)$ when the graph is understood from the context. We assume that demand is divisible. Then, given a solution $\overline{x}$, each vertex v splits its demand $w(v)$ among facilities $F(v, \overline{x}) := \arg\min_{i \in N} d(v, x_i)$ that are closest to it. If the closest facility is a singleton, all demand goes to it. Otherwise, there is a tie and demand is split equally. Hence, player i will receive demand from vertices in $V_i(\overline{x}) := \{v \in V | d(x_i, v) \leq d(x_{3-i}, v)\}$, where x_{3-i} denotes the location chosen by the other player. The motivation of splitting demand between equidistant facilities equally arises from the assumption that both facilities are equally attractive all else being equal so consumers decide randomly. It is possible to consider alternative splitting proportions; in Section 6 we consider heterogenous players who get unequal demand fractions whenever demand is located equidistantly between them.

The *utility* of player i is given by the market share, defined as the total demand achieved at x_i:

$$u_i(\overline{x}) = \sum_{v \in V : V_i(\overline{x})} \frac{w(v)}{|F(v, \overline{x})|} .$$

Since both utilities must sum up to the total demand W, this is a constant-sum game. According to Nash (1951), a solution $\overline{x}$ is a *pure-strategy Nash equilibrium* of the facility location game if $u_i(x_i, x_{3-i}) \geq u_i(y, x_{3-i})$ for any $y \in V$ and for any $i \in N$. This work focuses on pure-strategy Nash equilibria in the case of duopolies. We use the following basic but key observation that permits a characterization.[5]

Remark 2.1. *In an equilibrium of a facility location game with two players, both of them experience the same utility. This holds because otherwise the player with the lower utility can select the location of the other player and consequently split the market. Hence, both utilities at equilibrium must equal $W/2$.*

The previous remark implies that the value of this constant-sum game is $W/2$. It is well-known that games of this kind admit optimal strategies which are the solution to min-max problems (see, e.g., Osborne and Rubinstein 1994). When the game admits a pure-strategy Nash equilibrium, we refer to a vertex that represents an optimal strategy as a *winning strategy.* We provide an alternative characterization of Nash equilibria that has the advantage of liberating players from having to forecast the action of their opponent.

Definition 1. *We say that a vertex $x \in V$ is a* winning strategy *if $u_1(x, v) \geq W/2$ for all $v \in V$.*

[4]Although having $w(v) = 0$ in some vertices does not change existence results, additional equilibria may exist. In Section 3.3, we consider cases in which some of the vertices may have zero weights, and discuss some of the implications of such relaxation.

[5]Remark 2.1 does not immediately generalize to more players. In fact, the characterization of equilibria with more players is much more involved and establishing conditions that guarantee existence is far from trivial. Moreover, deciding if an equilibrium exists is NP-hard for an arbitrary number of players (Dürr and Thang 2007). We indicate some directions of possible generalizations to more players in Section 7.

The utility obtained by a player when choosing a winning strategy is at least $W/2$ regardless of the selection of the other player. There is a direct connection between the location of winning strategies and that of equilibria. In fact, an equilibrium exists if and only if a winning strategy exists, and any equilibrium must consist of each player locating a facility in a winning strategy. This is formalized in the next result.

Lemma 2.2. *For arbitrary topologies, an equilibrium of a facility location game with two players exists if and only if there exists at least one winning strategy.*

Proof. The result follows from the definition of a winning strategy. If $\bar{x}$ is at equilibrium, $W/2 = u_2(x_1, x_2) \geq u_2(x_1, v)$ for all $v \in V$, which implies that x_1 is a winning strategy because, since this is a constant-sum game, $W/2 \leq u_1(x_1, v)\ \forall v \in V$. To prove the converse, take a winning strategy y and consider $\bar{x} = (y, y)$. By definition $W/2 = u_1(\bar{x}) \leq u_1(y, v)$ for all $v \in V$. The equilibrium condition follows using that this is a constant-sum game again. $\square$

Note that an equilibrium will be unique if and only if there is a single winning strategy. Indeed, if there are m winning strategies, there are $m(m+1)/2$ different equilibria without counting permutations between players, or m^2 equilibria including permutations. The next definition, which deals with the centralized facility location problem, is central to our results.

Definition 2. *For a vertex $x \in V$, the total* consumer cost $C(x) := \sum_{z \in V} w(z) d(x, z)$ *measures the weighted distance to x. We say that a vertex $v \in V$ is a* 1-median, *or simply a* median, *if it minimizes $C(x)$ among $x \in V$.*

Similarly, we compute the consumer cost of an equilibrium. For a given solution $\overline{x}$ and a vertex $v \in V$, $d(v, \overline{x}) = \min\{d(v, x_1), d(v, x_2)\}$ is the distance from v to the nearest facility in $\overline{x}$. The *consumer cost* of an equilibrium $C(\overline{x})$ is defined in a similar way. The location problem where two facilities are chosen by a single agent is known as the 2-*median problem*. In that case, the agent minimizes the consumer cost to those 2 locations, solving $\min\{C(\overline{x}) : \overline{x} \in V^2\}$.

2.1. **Relation to Other Solution Concepts.** We next highlight relations between the equilibria of our game and other equilibrium concepts. First, note that the concept of winning strategy is strongly related to that of *dominant strategies*, which refer to actions that are optimal for an arbitrary action of the opponent. Although the facility location game does not necessarily posses dominant-strategy equilibria, a winning strategy guarantees that the player will be (weakly) better off than the opponent *regardless what strategy the opponent selects*. Nevertheless, a winning strategy needs not be dominant; it is possible that a better strategy exists. To illustrate, consider the path $(v_1, v_2, v_3, v_4, v_5)$ of 5 vertices with unit weights. The unique winning strategy is to choose v_3 (therefore, the only equilibrium is (v_3, v_3)); however, a best response to an opponent that chooses v_5 is to choose v_4, so v_3 is not a dominant strategy.

Second, we relate winning strategies to Stackelberg solutions of the corresponding sequential game. In the *Stackelberg facility location game*, one of the players (the *leader*) plays first and decides where to locate his facility. Once the decision of the first player is fixed, the second player (the *follower*) decides where to locate his facility to maximize his utility. A profile $\bar{x} = (s, s')$ is a *Stackelberg solution* if s maximizes the leader's utility given that after he plays s, the follower chooses s' to maximize his utility. An extensive analysis of Stackelberg solutions on discrete and continuous topologies is given in Mirchandani and Francis (1990).

A Stackelberg solution exists even if a Nash equilibrium does not. But when the game possesses a winning strategy (and hence a Nash equilibrium), the Stackelberg leader also selects it. Indeed, the Stackelberg leader cannot get more than half of the market because the follower can select the same location. Hence, a vertex that is a winning strategy is an optimal decision for the leader. Since in every equilibrium the leader plays a

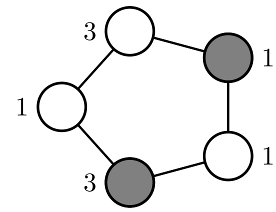


FIGURE 1. A Stackelberg equilibrium that is not a Nash equilibrium.

winning strategy and both players obtain a utility of $W/2$, Nash equilibria are also Stackelberg solutions. We conclude that the leader gets a utility of $W/2$ only if a winning strategy exists (and therefore, by Lemma 2.2, only if an equilibrium exists), and that whenever there is no winning strategy (equivalently, if an equilibrium does not exist) the leader gets strictly less than $W/2$.

Nevertheless, a Stackelberg solution needs not be an equilibrium. For example, consider the instance in Figure 1. Both vertices with demand equal to 3 are winning strategies, hence in a Nash equilibrium each player selects one of them. Without loss of generality, suppose that the Stackelberg leader selects the gray vertex with weight 3. Besides selecting a vertex with weight 3, selecting the other grey vertex (with weight 1) is also an optimal strategy for the follower, achieving a utility of 4.5. However, this solution is not a Nash equilibrium because the leader can move to the other vertex with weight 3 and increase his utility to 5.5.[6]

We conclude with a remark on *symmetric equilibria*, which are equilibria where both players use the same strategy. Following Lemma 2.2, it is immediate that if a winning strategy exists, the solution in which both players select that vertex is a symmetric equilibrium. Hence, a facility location game on an arbitrary graph possesses a symmetric equilibrium if and only if an equilibrium exists. This concrete relation between the existence of symmetric equilibria and the existence of general pure-strategy equilibria implies that the complexity of deciding whether one exists matches the complexity of deciding whether the other exists.

3. Equilibrium Analysis

In this section we study existence conditions for the equilibria of the game described in Section 2. We start by considering simple topologies, such as trees and cycles, and continue towards more complex topologies.

3.1. The Case of Trees.

Analogously to Hotelling's solution for two facilities on a line, a 1-median of a tree—also known as *centroid* in the literature—is a natural candidate for equilibrium location. Kariv and Hakimi (1979) showed that a median of a tree can be defined as a vertex that minimizes the maximum weight of a connected component induced by removing this vertex. They also showed that a vertex v is a median of a tree T of total weight W if and only if the removal of v induces components of weight at most $W/2$.

Eiselt and Laporte (1991) showed that the medians of a tree are the only possible equilibrium locations for a facility location game in the case of a duopoly (see also Wendell and McKelvey 1981). The next result is the starting point of our work: It extends the intuition arising from Hotelling's result by showing that the outcome of duopoly competition coincides with the single-facility centralized optimal solution.

Theorem 3.1 (Eiselt and Laporte (1991)). *In a facility location game with two players on a tree, an equilibrium always exists. Moreover, a solution is an equilibrium if and only if both players select a median of the tree (not necessarily the same one).*

[6]Some models in the literature, specially among those that are played sequentially, assume that there cannot be co-location to prevent symmetric equilibria from happening (see, e.g., Granot et al. 2010). In a simultaneous game like ours, it is not natural to prevent co-location because it is not immediate how to coordinate agents to guarantee they select different locations.

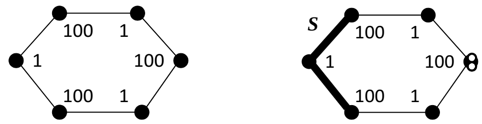


FIGURE 2. (*Left*) No equilibrium with two players in a cycle. (*Right*) The vertex selected by both players is not a winning strategy because $W(S) = 201 > 303/2 = W/2$.

The previous theorem implies that medians of a tree and winning strategies coincide. Because trees must have at least one median, they are guaranteed to admit pure-strategy Nash equilibria. Moreover, a tree with positive weights has either one or two medians (Kariv and Hakimi 1979) so there are at most 4 different Nash equilibria. When multiple medians exist, they all form a connected component (even in the zero-weights case), giving equilibrium locations a sense of proximity, as discussed in the introduction. This characterization provides a $O(|V|)$-time algorithm for finding all the vertices where equilibria can be located in a tree because we can solve the 1-median problem to find all medians of a tree within that time (Goldman 1971). We note that if the number of players is larger than two, an equilibrium may not exist, even if the graph is a tree. We discuss the case of more than two players in Section 7.

We conclude the section with a result that will be useful to characterize equilibria for more general topologies. The key idea of the proof is showing that a tree has two medians if and only if there is a vertex whose removal induces a subtree of total weight $W/2$.

Lemma 3.2. *Consider a tree T with vertices that may have zero weights. For any vertex v that is not a median of T, there is a median such that removing it will leave v in a component T' of weight $W(T') < W/2$.*

3.2. **The Case of Cycles.** Now we direct our attention to the case of cycles. Eaton and Lipsey (1976) studied cycles but for the case when facilities can be also placed along edges. They derived conditions that guarantee the existence of equilibria. Along similar lines, our game on cycles may or may not admit an equilibrium. As an example, consider a 6-cycle with weights that alternate between 1 and 100 (left of Figure 2). If one player selects a vertex of weight 100, the other would select the opposing vertex of weight 1. Conversely, if one player selects a vertex of weight 1, the other would select an adjacent vertex of weight 100, which implies that this instance does not admit winning strategies. However, replacing one weight of 100 by 200 makes that vertex be a winning strategy and hence an equilibrium.

We observe that if two players locate their facilities on different vertices, each of them would be assigned to exactly half of the vertices (up to two vertices may be shared). Based on this observation, we will look at connected half-cycles to characterize the equilibria of the game.

Definition 3. *A half-cycle S is a set of adjacent vertices that span half of the cycle. Suppose that the cycle has cardinality k. When k is even, a half-cycle may be either a set of $k/2$ adjacent vertices, or a set of $k/2+1$ ones where the two extreme vertices get only half of their original weight (we refer to them as* half-vertices*). When k is odd, each half-cycle contains $(k-1)/2$ adjacent vertices and one half-vertex.*

Regardless of whether k is odd or even, there are a total of $2k$ half-cycles. When players are located on different vertices, each of them controls a half-cycle. Therefore, if a player is located on a vertex v, the other player prefers a different vertex only if a half-cycle of total weight exceeding $W/2$ that excludes v exists. The following result characterizes equilibria by finding the winning strategies of a cycle.

Proposition 3.3. *A vertex v is a winning strategy in a cycle if $W(S) \leq W/2$ for every half-cycle S that does not contain v.*

The previous result provides a sufficient condition for a vertex to be a winning strategy that is based on a set of inequalities that must be checked. The condition is reminiscent of that of Eaton and Lipsey (1976) for

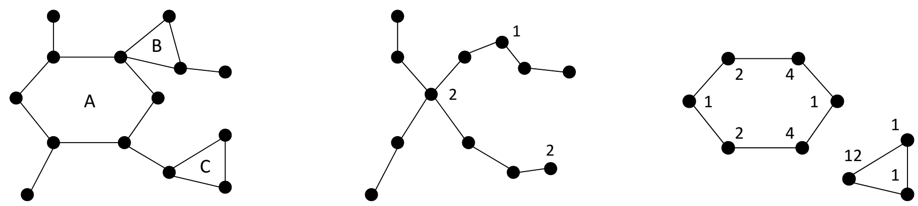


FIGURE 3. (*Left*) A unit-weighted graph G. (*Center*) G^T, the bi-connected components tree representation of G. Vertices without labels represent cutoff vertices and have a unit demand. (*Right*) Projection of G onto two different blocks (projections are introduced formally in Definition 6).

the case of continuous space who determined that an equilibrium exists if "no firm's whole market is smaller than any other firm's half market" (p. 29). Revisiting the instance without equilibria presented earlier, the picture on the right of Figure 2 shows that vertices with weight 100, as well as ones with weight 1, cannot be winning strategies as there is a half-cycle not containing them that has weight larger than $W/2$.

From the winning strategies, one can derive all the possible equilibria for two players in a cycle. In Appendix A, we provide a method to enumerate all winning strategies of a cycle of $|V|$ vertices in $O(|V|)$ time. Notice that when all weights are equal, all the vertices are winning strategies and Lemma 2.2 implies that an equilibrium always exists; any solution is an equilibrium. This coincides with the characterization of equilibria on unit-weight cycles for general number of players by Mavronicolas et al. (2008).

The previous result of this section, which establishes a connection between winning strategies and the 1-median problem, paves the road for more general topologies. We prove that although an equilibrium may not exist, when it does it is always located in a 1-median of a cycle. It is important because it provides a strong necessary condition for the location of equilibria. To put this in perspective, although this is weaker than the result for trees that established that medians and winning strategies coincide, in this case winning strategies may not exist while 1-medians always do.

Theorem 3.4. *If a winning strategy v of a cycle exists, it must solve the 1-median problem.*

3.3. **Bi-connected Components and Transformations Based on Them.** To study equilibria in more general topologies, we transform general graphs into trees that encode the essential information of the instance. The next two subsections show that the location of equilibria can be narrowed down to a subset of vertices, which we refer to as the *central block* of the graph. Our approach will be to represent a graph G as a tree of *maximal bi-connected components* $\{G_i\}$ and *cutoff vertices* $\{c_j\}$. A maximal bi-connected component, usually referred to as *block*, is a maximal subgraph in which each pair of vertices is connected by at least two vertex-disjoint paths. A cutoff vertex (also known as articulation point) is a vertex whose removal disconnects G. This transformation represents a general graph G as a weighted tree G^T that conserves some of the relevant information about the original graph.

Definition 4. *(Harary 1969) A maximal bi-connected components tree (also known as a block-cutoff tree) representation of a weighted graph $G(V,E)$ is a tree $G^T(V^T,E^T)$ on the vertex-set $V^T := \{G_i\} \cup \{c_j\}$. The weight of a vertex $c_j \in V^T$, which corresponds to a cutoff vertex $c_j \in G$, is the same as the original weight. The weight of a vertex $G_i \in V^T$, which represents the block $G_i \subseteq G$, equals the total weight of vertices in G_i that are not cutoff vertices. The tree induced by E^T is formed by connecting vertices corresponding to a block G_i to all the cutoff vertices $c_j \in G_i$, and by connecting all adjacent cutoff vertices that are not in the same block.*

Figure 3 provides an example that illustrates the transformation of a unit-weight graph G (depicted on the left of the figure) into its bi-connected components tree G^T (depicted in the center). Recalling Definition 4, note that blocks in G are represented in G^T by vertices with a weight equal to the sum of weights of vertices belonging to the block, *excluding* cutoff vertices. For instance, block A has 6 unit-weight vertices and 4 of these are cutoff vertices (removing those would break the connectivity of G). Therefore, the weight of the vertex representing A in G^T is $6-4=2$. Similarly, block B has 2 cutoff vertices and 1 interior one, making the weight of the corresponding vertex in G^T equal to $3-2=1$. Block C has 1 cutoff vertex and 2 interior ones; hence the weight of the corresponding vertex in G^T is $3-1=2$. Cutoff vertices in G are represented in G^T by vertices of the same weight. For that reason in Figure 3 all vertices in G^T corresponding to cutoff vertices have unit weight. Finally, arcs in G^T are the subset of arcs in G that are incident to cutoff vertices.

We provide a few additional remarks on this transformation. First, the maximal bi-connected components tree of a given graph is unique because blocks are uniquely defined (Harary 1969). Second, notice that blocks in the tree cannot be adjacent since they must be connected through cutoff vertices; this observation plays a role in the structure of the 1-medians of these trees. Finally, when all the vertices of a block G_i are cutoff vertices, the vertex that represents G_i in V^T will have zero weight. Although the case of zero weights was assumed away in the description of the model, that possibility has minimal consequences in the results presented in this paper. We explicitly comment on this case below. Among the set $\{G_j\}$ of blocks, we pay special attention to those that represent medians of the maximal bi-connected components tree G^T.

Definition 5. *A* central block *of a graph G is a maximal bi-connected component that is represented by a median in the maximal bi-connected components tree G^T.*

We will see that (i) if there is a median of G^T that corresponds to a block, any equilibria of the original graph can only be located in vertices belonging to that block; and (ii) all medians of G^T that corresponds to cutoff vertices define equilibria automatically. In our terms, we show that winning strategies can only be located in cutoff vertices that are medians of G^T or in central blocks.

We note that although a tree with positive weights cannot have more than two medians, the possible presence of zero-weight vertices in the maximal bi-connected components tree may induce more medians. Nevertheless, since between two block vertices there has to be a cutoff vertex with positive weight, there cannot be more than three medians in G^T. For this reason, the tree G^T may have from one to three medians and classifying a median as a block (B) or a cutoff vertex (C), the list of all possibilities for the median solutions is B, C, B-C, C-C, and C-B-C. Other cases are ruled out since in a tree there may be more than two medians only when two positive-weight medians are connected by a path of zero-weights medians. The case C-C is straightforward because one need not consider the rest of the block, equilibria are formed by the combinations of the two cutoff vertices. In the cases B-C and C-B-C (the latter can only occur if the central block has zero weight), the medians that are cutoff vertices belong to the central block. In summary, it is never needed to consider more than one central block.

To connect our approach to the literature, Slater (1975) calls a vertex v a *security center* for the model with unit demands if it maximizes $\min_{w \in V\setminus\{v\}} f(v,w)$, where $f(v,w)$ is the number of vertices that are closer to v than to w minus those that are closer to w than to v. He provides an algorithm that finds a *security center* by sequentially discarding blocks of G until the last one remains. In our context, generalizing $f(v,w)$ to consider arbitrary demands, it can be seen that a winning strategy must be a security center. However, a security center v can only be a winning strategy if $\min_{w \in V\setminus\{v\}} f(v,w) \geq 0$.

Our next definition allows us to focus on the subgraph that matters: we simplify the original graph by projecting it to one of its blocks, typically the central block.

Definition 6. *We consider a weighted graph G and a block $G_j \subseteq G$. If a vertex v is connected to G_j through vertex $x \in G_j$, x is referred to as the projection of v onto G_j. If $v \notin G_j$, then x must be a cutoff vertex. Otherwise $v = x \in G_j$. Using this, we define the weighted graph G'_j as the* projection *of G onto G_j. To* *construct it, we project all vertices in V onto G_j. We set the weight of each vertex $v \in G'_j$ as $w(v)$ plus the weights of all other vertices projected onto v.*

The graph to the right side of Figure 3 depicts two projections of the graph G (shown to the left) on two different blocks: A, and C. One may observe that in each projection each vertex gets its original weight (unit weight in this instance) plus the total weight of the rest of the graph that connects through it.

3.4. **Equilibria are Located in the Central Block.** As previously anticipated, we show that an equilibrium of a facility location game on a general graph G with two players must be located in a central block or in cutoff vertices. The driving idea is to project G onto the central block and prove that a vertex is a winning strategy of this projection if and only if it is a winning strategy of the original graph. We divide the analysis into two separate cases. Recall that the tree G^T may have the following combinations of medians: B, C, B-C, C-C, and C-B-C. If one of the medians of G^T represents a block, then all medians of G^T must be part of that block. In that case, we prove that the winning strategies of G coincide with those of the central block (including possibly its cutoff vertices). In particular, there cannot be a winning strategy outside of the central block. The second case is when G^T will have exactly one or two medians that are cutoff vertices. Then, each median will be a winning strategy in G and there cannot be more winning strategies in G.

First, we prove sufficient conditions for being a winning strategy in G. We look at the cases of cutoff vertices and blocks separately. Each median of G^T that is a cutoff vertex is a winning strategy while if there is a median of G^T that is a block, finding a winning strategy of G reduces to finding a winning strategy in the projection of G onto that block.

Lemma 3.5. *Let $v \in V$ be a vertex that corresponds to a median of G^T that is a cutoff vertex. Then, v is a winning strategy in G.*

When there are two medians of G^T that are cutoff vertices, G has two winning strategies in different blocks. In all other cases, all winning strategies of G belong to just one block. In those cases, we can encode the relevant information of G in the projection onto that central block.

Lemma 3.6. *Let G_j be a block that corresponds to a median of G^T. Then, the winning strategies of G within G_j and the winning strategies of G'_j coincide.*

Notice that winning strategies within G_j include the cutoff vertices that are medians of G^T (if there are any) because those cutoff vertices will be part of the same block. Now that the sufficient conditions for having winning strategies are settled, we are ready to prove that these conditions are also necessary. By combining the two previous lemmas, we provide a full characterization of equilibria in general graphs. This result implies that we can restrict our search for equilibria to the central block of the graph or to two cutoff vertices, depending on the situation. The set of equilibria can be formed by combining all winning strategies arbitrarily, as discussed in Section 2. Concretely, an equilibrium exists if and only if G^T has a median that is a cutoff vertex or if the projection onto the central block of G admits an equilibrium.

Theorem 3.7. *Consider a facility location game with two players on a general graph G. If all medians of G^T are cutoff vertices, the winning strategies are exactly those medians. Otherwise, there is median that represents a block. Referring to that block as G_j, the winning strategies of G coincide with the winning strategies of G'_j.*

The uniqueness of equilibria depends on the number of winning strategies in G. If there are multiple medians in G^T, the equilibrium may or may not be unique. The case of two cutoff vertices that are medians leads to three equilibria in G (one in each cutoff vertex, and one that combines them both). When the only median is a cutoff vertex, there must be a unique equilibrium, while when the only median is a central block, there may be from zero to more than one equilibria. We next specialize this characterization to the case when the central block has a manageable topology. For the class of graphs we are going to define, the combination of results for cycles and trees provides a concrete characterization of equilibria.

Definition 7. *We say that G is a* simple central block *(SCB) graph if every central block of G is either a cycle or a vertex.*

To illustrate, a common family of graphs called *cactus graphs* satisfies Definition 7. Indeed, Geller and Manvel (1969) defined a *cactus* as a graph such that each edge belongs to at most one cycle. Naturally, cacti both generalize cycles and trees because all its blocks are cycles, and so are the corresponding projections.

Corollary 3.8. *Consider a facility location game with two players on a SCB graph G. A vertex v is a winning strategy if and only if v is a cutoff vertex that is a median of G^T or v is a winning strategy of the cycle formed when projecting G onto its unique central block.*

Our previous analysis shows that any combination of winning strategies is an equilibrium. This fully characterizes equilibria for SCB graphs (in particular, for cacti). A general graph can have more complicated central blocks, making it harder to find an equilibrium; yet, our results reduce the problem of finding an equilibrium in a graph to finding one on its central block. Section 4 provides additional results that allows us to study more general central blocks. We complete this section with a short discussion on the complexity of finding equilibria for SCB graphs. Below we present an outline of an algorithm:

(1) List the blocks of the graph in $O(|E|)$ time (Hopcroft and Tarjan 1973, Aho et al. 1974).
(2) Construct the tree G^T in $O(|E|)$ time (Appendix B).
(3) Find the medians of G^T in $O(|V|)$ time (Goldman 1971).
(4) If a median represents a block, project G onto the central block in $O(|V|)$ time.
(5) Find the equilibria in the central block. Since it is a cycle, find winning strategies in $O(|V|)$ time (Section 3.2).

All the steps take linear time in the number of edges or vertices. When G is sparse (e.g., in cactus graphs), we have that $|E| = O(|V|)$ so the whole algorithm goes from $O(|E|)$-time to $O(|V|)$-time. When the graph does not have a simple central block, we have a linear-time reduction of the problem for the full graph to the same problem for the central block, which in some cases may be much smaller than the original graph.

4. Further Connections between Winning Strategies and 1-Medians

As mentioned in the introduction, the first to establish a connection between 1-medians and equilibria in the facility location game were Eiselt and Laporte (1991). Motivated by this result, two natural questions arise. First, we know that not every instance has an equilibrium; however, is it true that when an equilibrium exists every winning strategy must be a solution to the 1-median problem? Second, is it possible to generalize the result obtained in trees? In other words, is there any other class of graphs for which an equilibrium always exists and for which selecting a median is enough to guarantee that players will be at equilibrium?

We now address the first question. It turns out that equilibria naturally correspond to *local* 1-medians. Indeed, whenever a winning strategy exists, we prove that it must be a local minimizer of the 1-median problem, where 'local' is to be understood as a minimum among neighbors in the graph.

Theorem 4.1. *If y is a winning strategy of the facility location game, then it must be a local median. Here, a vertex y is a* local median *of $G = (V, E)$ if $C(y) \leq C(v)$ for all $v \in N(y) := \{z \in V : yz \in E\}$.*

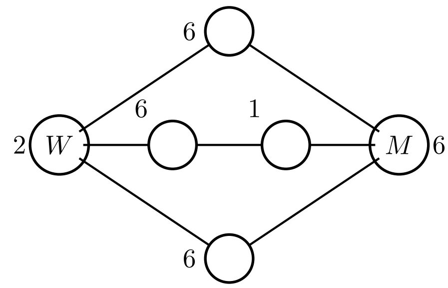


FIGURE 4. This example illustrates that a winning strategy may not be located at the (global) 1-median. Numbers represent the weight of each vertex. The reader can verify that the vertex labeled as M is the only 1-median of the graph, while the vertex labeled as W is the only winning strategy.

Proof. Let y be a winning strategy and let $v \in N(y)$. Let $d_z := d(v,z) - d(y,z)$ for all $z \in V$. Because y and v are neighbors, $d_z \in \{-1, 0, 1\}$ $\forall z$. We consider $\bar{x} = (y, v)$. Since (y, y) is at equilibrium, $W/2 = u_2(y,y) \geq u_2(\bar{x})$, from where $u_1(\bar{x}) - u_2(\bar{x}) \geq 0$ because the game is constant-sum. Let M_y (resp. M_v) be the set of vertices that are strictly closer to y than to v (resp. v to y). The result follows using that $d_z = 1$ for $z \in M_y$ and $d_z = -1$ for $z \in M_v$, because

$$C(v) - C(y) = \sum_{z \in V} w(z)(d(v,z) - d(y,z)) = W(M_v) - W(M_y) = u_1(\bar{x}) - u_2(\bar{x}).$$

□

Even though it is natural to think that a more general property holds with (global) 1-medians, the example in Figure 4 shows this is not necessarily the case (Scoccola 2014). In that instance, there is a unique median and a unique winning strategy, and both are located in different nodes. This provides the motivation to characterize families of graphs for which winning strategies and 1-medians coincide in the same locations.

Although in trees medians must form a connected subset of vertices, in general the set of medians of a graph is arbitrary. Indeed, given a graph G, there exists a graph H for which the subgraph of H induced by the median vertices is isomorphic to G (Slater 1980). In particular, the median-set can induce a disconnected subgraph. However, for certain families of graphs, local 1-medians coincide with the (global) 1-medians. Bandelt and Chepoi (2002) proved that the following conditions are equivalent: (a) The median-set is connected for arbitrary weights w, and, (b) The set of local medians coincide with the median-set for arbitrary rational weights w. Based on this equivalence, we obtain the following corollary that implies that every winning strategy solves the 1-median problem. To describe the result compactly, let us refer to graphs for which equilibria must be located in a 1-median as *MPM graphs*, where MPM means *must play median.* In the previous section, we have proved that cycles and trees are MPM graphs.

Corollary 4.2. *Let G be a graph that belongs to a family for which, for any rational weights w, the solutions to the 1-median problem induce a connected subgraph of G. Then, G is an MPM graph.*

Families of graphs satisfying this property include *median graphs*, *quasi-median graphs*, *pseudo-median graphs*, *Helly graphs* and *strongly chordal graphs.* A complete characterization of graphs with connected median-sets can be found in Bandelt and Chepoi (2002). Among graphs in this family, we put special emphasis on median graphs because they are particularly relevant to us. A median graph is defined as a graph that satisfies that any three vertices a, b, and c have a unique median (which is a vertex that belongs to shortest paths between any two of a, b, and c). This class includes lattices, meshes, and grids, which encode the characteristics of networks found in the real world when dealing with location problems in urban areas. In particular, this sheds some light into why in many urban areas competitors choose to concentrate next to each other, as opposed to covering the network more uniformly.

Now, we prove that arbitrary graphs whose central blocks satisfy the MPM property are MPM graphs. As a corollary this implies that simple central block graphs are MPM graphs. So with much more generality than in the previous sections, we know that we can restrict the search of the location of equilibria to vertices that are 1-medians.

Theorem 4.3. *Consider a facility location game with two players on a graph G whose central block is an MPM graph. Then, G is an MPM graph.*

Proof. Let us start by assuming that (v, z) is an equilibrium of G. We must prove that v (and similarly z) solves the 1-median problem in G. Considering an arbitrary vertex $i \in G$, we will show that $C(i) \geq C(v)$.

Let us first assume that $v \in G_j$ for some central block by Theorem 3.7. Let i' be the projection of i onto the central block G_j. Interpreting i' as the corresponding vertex in G_j, the assumption that the central block is an MPM graph implies that $C(i') \geq C(v)$. If $i = i'$ we are done. Otherwise, removing i' from the graph will leave i in a connected component with a weight of at most $W/2$ since v (which is represented by a median in G^T) is not part of it. Moving the facility from i' to i cannot decrease the total cost. Indeed, at most $W/2$ of the demand decrease its cost by $d(i, i')$ (the demand in the connected component where i lies) and at least $W/2$ of the demand increase its cost $d(i, i')$ (the rest of the demand). Putting it all together, $C(i) \geq C(i') + d(i, i') \cdot W/2 - d(i, i') \cdot W/2 = C(i')$. Second, let us assume that v is a cutoff vertex that is a median of G^T. Then, removing v creates connected components, each with a total weight of at most $W/2$. Reasoning like before, $C(i) \geq C(v)$, completing the proof. $\square$

This results implies that as a general strategy to compute the equilibria of an instance of a facility location game in a duopoly based on an arbitrary MPM graph, we can compute all 1-medians of the central block and check if each of them is a winning strategy or not.

4.1. **Strongly Chordal Graphs.** To address the second question raised at the beginning of Section 4, we now focus on strongly chordal graphs. This family generalizes many well-known classes of graphs such as trees, block graphs and interval graphs. A graph is *chordal* if every cycle with more than three vertices has a *chord* (an edge joining two non-consecutive vertices of the cycle). A *p-sun* is a chordal graph with a vertex set $x_1, \ldots, x_p, y_1, \ldots, y_p$ such that $y_1, \ldots, y_p$ is an independent set, $(x_1, \ldots, x_p, x_1)$ is a cycle, and each vertex y_i has exactly two neighbors $x_i - 1$ and x_i, where $x_0 = x_p$. A graph is *strongly chordal* if it is chordal and contains no p-sun for $p \geq 3$.

Since the set of 1-medians of these graphs induce a connected component, we know from the previous section that they are MPM graphs, establishing a necessary condition for equilibria. Now we prove the converse to get a sufficient condition as well: a vertex that is a 1-median must be a winning strategy. This guarantees that strongly chordal graphs always admit equilibria and that the equilibrium locations and 1-medians coincide. This completely extends the results for trees of Eiselt and Laporte (1991) to this family. As opposed to the class of cacti, each edge may belong to many cycles. Because there are many chords, these graphs are densely populated with triangles.

Theorem 4.4. *Every connected strongly chordal graph has an equilibrium. Furthermore, there is a one-to-one correspondence between winning strategies and the solutions to the 1-median problem.*

The proof follows the methodology of Theorem 1 in Lee and Chang (1994), where it is shown that the median-set of a connected strongly chordal graph is a clique. While we use the same inductive idea, we need to rely on more complex structures. As a corollary, the set of winning strategies of a connected strongly chordal graph is, not only connected as previously discussed, but also a clique. When considering more complex graphs, the ideas presented earlier imply that when the central block of an arbitrary graph is strongly chordal, this graph must have an equilibrium because a median must be a winning strategy.

5. Efficiency of Equilibria

In this section we examine the worst-case inefficiency of equilibria in the facility location game, usually referred to as *price of anarchy*. By inefficiency, we mean that the total consumer cost at an equilibrium does not minimize the total consumer cost among all feasible solutions. The *consumer optimum* refers to the solution of the 2-median problem. This solution is the best possible outcome for the consumers, taken as a group. The problem is that in practice this solution can only be achieved if players coordinate themselves and agree to benefit consumers ignoring their own utilities. We denote the optimal consumer cost by D_{opt}.

The price of anarchy is defined as the worst-case inefficiency of an equilibrium among all instances (Koutsoupias and Papadimitriou 2009). It is a measure of how much it is lost by the lack of central coordination. To compute it, we evaluate the ratio of the consumer cost of an arbitrary equilibrium to that of a consumer optimum, and maximize the ratio over all instances, as given by a graph G and a demand vector associated to the vertices of G. In other words, denoting $\bar{w} = \{w(v)\}_{v\in V}$, we evaluate

$$\text{POA} := \sup_{G,\bar{w}} \frac{D_{\text{eq}}}{D_{\text{opt}}} \, ,$$

Here, $D_{\text{eq}} := \sup_{\overline{x}\in \text{NE}(G,\bar{w})} C(\overline{x})$ is the consumer cost of the worst equilibrium of the instance, where $\text{NE}(G,\bar{w})$ is the set of equilibria. The worst equilibrium can always be achieved at a solution where both players select the same vertex. Otherwise, if one of the players switched to the other location, the solution would still be at equilibrium and the consumer cost cannot go down. We note that the price of anarchy is defined only when an equilibrium exists.[7] The motivation of looking at instances that admit equilibria is that an instance that does not is unstable since players have incentives to switch locations, undermining the notion that this can be the status-quo in practice. In addition, we disregard instances where $D_{\text{opt}} = 0$ because instances with consumer cost equal to zero are extreme but with little interest since inefficiency is trivially unbounded.[8]

Having established a full characterization of equilibria for SCB graphs, this section focuses on this broad class of networks. Since in SCB graphs players at equilibrium select the 1-median, the price of anarchy also quantifies the worst-case gap between solutions to the 1-median and to the 2-median problems with respect to consumer cost, measuring the (centralized) impact of opening an extra facility in the network.

5.1. **Monotonicity.** This section shows that the consumer cost at equilibrium is monotone with respect to edge removals in SCB graphs. Although it may be intuitive that removing edges can only increase the consumer cost because demand may be pushed further away from facilities, it is well-known that removing edges in network routing games may—in some cases—make all consumers better off. In the context of the *transportation assignment problem*, this apparently counterintuitive phenomenon has been called the *Braess paradox* (1968). We show that facility location games are well-behaved in this respect: equilibria *cannot* induce a lower consumer cost when an edge is removed. We begin by showing that the existence of an equilibrium in SCB graphs is preserved under edge removals.

[7]Following the assumptions made in Section 2, we restrict our attention to pure-strategy Nash equilibria and assume that the price of anarchy is unbounded when an equilibrium does not exist. Nevertheless, one may also consider an analogous price of anarchy defined for mixed-strategy Nash equilibrium.

[8]For instance, in a game with two players on two vertices connected by a single edge, a consumer optimum is a solution with facilities in both vertices that achieves a cost of zero.

Proposition 5.1. *Assume that a facility location game with two players on an SCB graph $G(V, E)$ admits an equilibrium. For any $E' \subseteq E$ such that $G' = G(V, E')$ is connected, G' admits an equilibrium.*

This result highlights that an SCB graph that admits an equilibrium must also admit one after removing some of its edges (assuming the graph remains connected). To address the change in consumer cost after the removal, we define the concept of *monotonicity* of a graph. We study this property for the case of two players, noting that whether it holds or not depends on the number of players.

Definition 8. *Let $G(V, E)$ be an arbitrary graph that admits an equilibrium. We say that G is* monotone *under edge removals if for any subset of edges $E' \subseteq E$ that induces a connected graph $G'(V, E')$, we have that $D_{eq}(G') \geq D_{eq}(G)$.*

A graph is monotone if whenever we remove edges from it, the worst-case equilibrium can only get worse. The property is transitive: if $G(V, E)$ is monotone and $E' \subseteq E$ induces a connected graph, $G'(V, E')$ is monotone too. Definition 8 does not require the existence of an equilibrium in the modified graph G', under the understanding that the worst-case cost of an equilibrium that does not exist is infinity. Nevertheless, Proposition 5.1 points out that removing edges from SCB graphs will not remove equilibria.

Because Theorem 4.3 allows us to characterize equilibria as solutions to an optimization (1-median) problem, the Braess paradox cannot happen: removing edges can be mapped to adding constraints, and solutions to more constrained problems cannot be better. The following result formalizes this insight.

Theorem 5.2. *SCB graphs are monotone.*

Proof. Let $G(V, E)$ be an arbitrary SCB graph. Let (v, v) be a worst-case equilibrium (without loss of generality we can assume that both players select the same vertex). Removing edges leads to a new graph $G'(V, E')$ with worst-case equilibrium (v', v'). We have to prove that it cannot have a smaller consumer cost. Considering the 1-median problem on the original graph, we know that placing the facilities in v' instead of in v increases the consumer cost (Theorem 4.3). Then, keeping the facility in v' and removing the edges in $E \setminus E'$ from the graph increases the consumer cost even more, which concludes the proof. $\square$

5.2. **Bounding the Inefficiency of Equilibria.** The implication of the monotonicity results is that when looking for SCB instances that are inefficient (according to the price-of-anarchy yardstick), it is enough to consider trees. We prove that removing edges one by one until we are left with a tree monotonically increases the price of anarchy. The following result does not use the fact that there are only two players in the game; it holds as long as the graph is monotone.

Proposition 5.3. *Consider a monotone graph $G(V, E)$ that is not a tree. There exists an edge that can be removed from it without decreasing the coordination ratio D_{eq}/D_{opt}.*

Proof. Let $\overline{x}_{\text{opt}}$ and $\overline{x}_{\text{eq}}$ be optimal and equilibrium solutions, respectively. Connecting all vertices in $\overline{x}_{\text{opt}}$ to a super-sink t with edges with zero cost, the optimal consumer cost D_{opt} is given by the solution of a shortest path problem from every vertex $v \in V$ to t. Because the solution of the previous shortest path problem is a shortest-path tree, for any cycle in the original graph we can find an edge e that is not in the tree. Increasing the weight of e cannot change the shortest path tree because e was already not part of any shortest path. Therefore, removing e does not change D_{opt}. Regarding D_{eq}, the monotonicity of G implies that an equilibrium of $G(V, E \setminus \{e\})$ cannot have lower consumer cost. This concludes the proof. $\square$

The price of anarchy in trees, however, is not bounded, even for two players and unit weights. Figure 5 shows an example of two players on a tree with unit-weight vertices for which there is a unique equilibrium and a unique consumer optimum for any $k \geq 2$ (where a line connecting two star-shaped clusters is of length

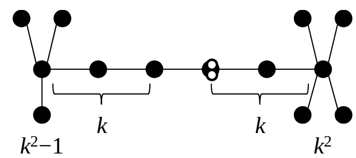


FIGURE 5. A bad tree. Two star-shaped clusters are connected by a line, where the distance between the centers of the stars is $2k+1$. The two white dots represent both players, who choose the unique vertex that is a winning strategy. Under a social optimum, each player is located at the center of a different star.

$2k$). Under the unique equilibrium, both facilities are located in the unique 1-median. There are $O(k^2)$ vertices at a distance of $O(k)$ of the median, totaling a consumer cost of $D_{\text{eq}} = O(k^3)$. On the other hand, it is optimal to place facilities at the extremes, in the center of the clusters, where $O(k^2)$ vertices are at a unit distance from the facilities while $O(k)$ vertices along the segment connecting the two extremes are located at distance of $O(k)$. This results in $D_{\text{opt}} = O(k^2)$, from where we see that the price of anarchy is not bounded.

A similar example can be given for non-uniform weights when keeping the size of the graph constant. The price of anarchy grows when $\delta := \max_{v\in V} w(v)/\min_{v\in V} w(v) \to \infty$. As an example, we can take a path on $|V|$ vertices whose two leaves have a weight of $(W - (|V|-2))/2$ and the interior vertices have unit weight. While D_{opt} is constant, $D_{\text{eq}} \to \infty$ when $W \to \infty$, showing the unboundedness of the price of anarchy.

Given the previous examples, we provide an upper bound on the price of anarchy for trees of diameter[9] at least 2 that is parameterized by the size of the graph, its diameter, and by the spread δ of the consumer demand among the vertices. This bound follows from a lower bound on the consumer cost of the 2-median problem and an upper bound on the consumer cost at an equilibrium. The only possible tree with unit diameter is a line with 2 vertices, which has an unbounded price of anarchy because the consumer cost in its optimum is zero. Without loss of generality we can assume that the minimum demand is one because we can normalize demands without changing any solution.

Proposition 5.4. *For a tree of size $|V|$, diameter $d > 1$, and demands between 1 and δ, the price of anarchy is bounded by*

$$\frac{4\delta(|V|-1)(d+1)}{(d-1)(d+3)}\,.$$

Recalling Proposition 5.3, Proposition 5.4 gives an upper bound for the price of anarchy for any monotone graph. In particular, Theorem 5.2 shows that this bound holds for any SCB graph, e.g., cactus graphs. In the case of vertices with uniform weights, we have that $\delta = 1$, and in the case of lines, we also have that $|V| = d+1$. For instance, for uniform lines of diameter $d \geq 2$, Proposition 5.4 implies a bound of $24/5$. While Appendix C shows that the exact price of anarchy on unit-weight lines is $9/4$, this bound is parametric on the topological parameters d, δ, and $|V|$, and provides intuition about how those parameters effect the inefficiency of equilibria.

The price of anarchy may be bounded for some classes of graphs that do not contain trees because worst-case instances would be excluded. Nevertheless, if one is to expect bounded price of anarchy, the class cannot even include general cycles since it is not hard to construct a cycle with a large gap between an equilibrium and an optimum. We do not include the example here because it is an extension of the instance shown in Figure 5. In any event, cycles are monotone so the bound presented in Proposition 5.4 applies.

Connections to Network Design. A system manager controlling a network may want to introduce changes to its topology, or even design it from scratch in a way that limits the negative effects of mis-coordination

[9]The diameter of a graph is defined as the maximum distance between any pair of vertices.

and improve the efficiency of the facility location game at equilibrium. Proposition 5.4 provides some insights on guidelines that would reduce the inefficiency introduced by competitive behavior. It indicates that size, diameter and variability of demand are drivers that increase the inefficiency. Hence, as a design recommendation, one would try to limit those. For a given size, one can keep the diameter small by having symmetry around the 'center' of the instance, and the variability small by spreading the total demand W as uniformly as possible. This motivates the following network design challenge: If one is given k vertices $\{v_1, \ldots, v_k\}$ with weights $w_1 \geq \ldots \geq w_k$, find a network that connects those vertices in such a way that D_{eq} is minimum. Notice that the objective is different from minimizing the ratio D_{eq}/D_{opt}. The former minimizes the consumer cost experienced at equilibrium while the latter minimizes the inefficiency gap.

To illustrate, we provide a simple example that demonstrates how these guidelines can be used to reduce the inefficiency of equilibria, compared to the bound of Proposition 5.4. We seek the topology that minimizes the consumer cost at equilibrium among all trees because we know that D_{eq} is monotone with respect to edge removals. The solution of this problem is a star in which all vertices are connected to v_1 because both players locate on v_1 at equilibrium, achieving a consumer cost of $W - w_1$. In that case, a socially-optimal solution locates facilities at v_1 and at v_2, making the inefficiency gap equal to $1 + w_2/(W - w_1 - w_2)$. This topology follows all the recommendations given above: the vertex with the highest weight is placed in the 'center', all other vertices are at distance 1, and the diameter is only 2. In the case of unit weights, the ratio is $1 + 1/(k - 2)$, which tends to 1 when k grows. This is small, compared to the case of lines which had an inefficiency gap of 9/4, because the diameter does not grow when k goes to infinity.

Related to this, Ravi and Sinha (2006) look at the centralized facility location problem from the perspective of network design but they take the topology as fixed and instead search for capacities for the edges of the network. Although their results do not apply directly in our case because equilibria when edges have capacities need not coincide with 1-medians, it would interesting to include edge capacities in our game, analyze equilibria, and optimize the design.

6. Robustness of the Results

In this section we weaken some of the assumptions we have made earlier to test the robustness of our theoretical results. To rely on realistic topologies, we apply a simulation procedure to a large set of random instances extracted from an actual city that exhibits a grid-like structure, without being a grid.

6.1. **Splitting Rule for Heterogeneous Facilities.** So far we have assumed that whenever there are ties, demand is split equally between facilities. We now explore the consequences of players' heterogeneity by assuming that when there is a tie, player 1 gets a fraction $\alpha \in [1/2, 1]$ of the demand while player 2 gets the fraction $1 - \alpha$. The parameter α encodes the attractiveness of player 1 compared to player 2, accounting for potential differences in reputation, quality of service, marketing efforts, etc. Note that we still assume that consumers select a facility closest to it.

Even for simple topologies, this simple change may invalidate the existence of pure-strategy Nash equilibria. Indeed, consider two unit vertices connected by an edge and $\alpha > 1/2$: this facility location game is equivalent to the *matching pennies* game (see, e.g., Osborne and Rubinstein 1994) because player 1 prefers to choose the same vertex as player 2 (to get αW instead of $W/2$), while player 2 prefers to choose a different vertex from player 1 (to get $W/2$ instead of $(1 - \alpha)W$).

However, a tree with two medians requires two perfectly balanced subtrees of total weight $W/2$ each, and such symmetry is rarely found in practical networks. In the more common case of a unique median v in a tree, if an equilibrium exists, it will remain in the median for any value of α which is "close enough" to 1/2. More formally, let $\mathcal{C} := \{T_i\}$ be the set of connected components that remain after removing v from T. The

FIGURE 6. Road map of the city of Buenos Aires, Argentina.

solution (v, v) is at equilibrium if and only if $W(T) \leq (1-\alpha)W$ for all $T \in \mathcal{C}$. Indeed, if a player deviates to another vertex he would get $W(T_i)$, which is smaller than or equal to $(1-\alpha)W < \alpha W$. Hence, the existence and characterization of equilibria for $\alpha = 1/2$ are robust with respect to changes in α, as long as $\alpha \leq 1 - \max_{T\in\mathcal{C}} W(T)/W$. When α is larger, the optimal strategy of the constant-sum game is a probability distribution on vertices and the support can contain vertices that are not medians.[10]

In the case of cycles, one may generalize the definition of a winning strategy to consider an arbitrary α. We say that a vertex is an α-winning strategy if $W(S) \leq (1-\alpha)W$ for any half-cycle S that does not contain v. Following the analysis of Section 3.2 one may verify that a vertex can be played at equilibrium if and only if it is an α-winning strategy. As for trees, the condition becomes more stringent if α is further away from $1/2$ so equilibria are less likely to exist. These properties may also be extended to SCB graphs using similar arguments.

6.2. **Numerical Study.** Although we have illustrated that equilibria may not exist if one changes the splitting rule mildly, we next argue that equilibria in networks arising from real-world urban areas exist with high probability for values of α that are not much larger than $1/2$. To that end, we run a set of experiments using random portions of the city of Buenos Aires, Argentina. We chose this city because it resembles a grid for the most part without being one (see Figure 6). Buenos Aires in 2010 had a population of 2.89 million inhabitants, an area of 78.5 square miles, and its road graph had approximately $17,000$ nodes and $30,000$ edges.[11] Each experiment consists of both a graph topology and a random demand, as we explain below.

To create each graph topology, we randomly choose a corner vertex (i.e., an intersection of two streets) to be a root and select a neighborhood of a pre-specified radius that includes vertices at a distance smaller than or equal to that radius. For each root, we take radii equal to 3, 4 and 5. We then draw demands for each vertex independently from a fixed continuous distribution: either uniform with support $[1, 10]$, or a truncated normal with mean 5, variance 5, and support $[1, 10]$.

[10]Consider a simple example of 3 vertices with weights 4, 1, and 4 on a line, and $\alpha = 2/3$. Solving the linear program that characterizes equilibria of a constant-sum game, we see that the player who gets $2/3$ of the demand in case of ties plays the mixed strategy $(3/14, 4/7, 3/14)$ while the player who gets $1/3$ of the demand in case of ties plays the mixed strategy $(3/7, 1/7, 3/7)$. The unique median is the middle vertex with unit demand.

[11]These numbers refer only to the city of Buenos Aires, not to the metropolitan area which is much larger in all respects.

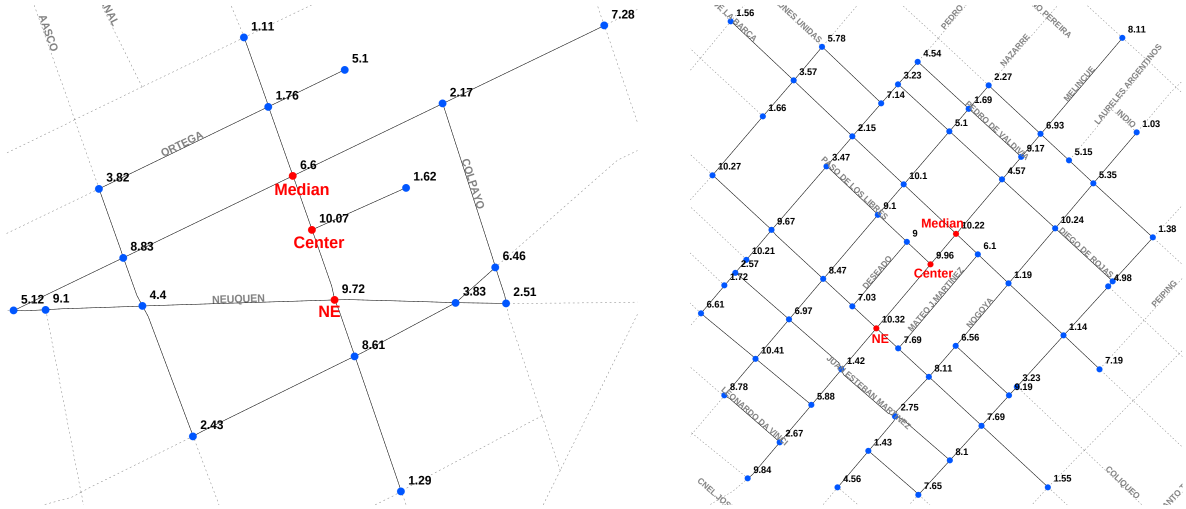


FIGURE 7. Two examples: instances with uniform demand and radii equal to 3 and 5, on the left and right, respectively. Vertex labels represent demands. We also indicate the location of the root ('center'), the equilibrium ('NE') and the median.

The simulations were run on 2,000 instances, varying the radii and the distribution for each (representing a total of $2,000 \times 3 \times 2 = 12,000$ runs for each value of α). We consider $\alpha \in \{0.5, 0.501, 0.51, 0.55, 0.6, 0.7, 0.8, 0.9, 1\}$ to study the impact that this value may have on equilibrium existence and identity. The constructed graphs had an average of 23.48 vertices for a radius of 3, 39.45 vertices for a radius of 4, and 60.16 vertices for a radius of 5. These dimensions are somewhat consistent with those of grids of size 5 by 5, 6 by 6 and 8 by 8 for radii of 3, 4, and 5 respectively. Figure 7 shows two examples that illustrate the instance sizes. These examples were chosen among those that admit equilibria in a location different from the median, in contrast to theoretical results earlier that guarantee that both coincide.

We present a summary of the simulation results in Table 1. Rows corresponding to $\alpha \geq 0.8$ are omitted because they tend to not admit equilibria. As pointed out in Section 4, lattices, meshes, and grids, are useful in representing networks found in the real world when dealing with location problems in urban areas. All those graphs are contained in the class of median graphs, for which we have shown that if a winning strategy exists it must be located in a 1-median. Therefore, we would expect to see that, in most cases, the location of equilibria and medians coincide. The summary presented in Table 1 agrees with that intuition: the percentage of instances for which both locations were different is under 0.4% in all cases (among instances

TABLE 1. Summary of results. Each column refers to a different set of parameters (radius and distribution). Within each, entries under "No Eq" count cases without equilibria while entries under "E≠M" count cases with equilibria but where equilibria and medians are in different locations. Each of those entries is over a total of 2000 trials.

	Uniform						Normal					
radius	3		4		5		3		4		5	
α	No Eq	E≠M	No Eq	E≠M	No Eq	E≠M	No Eq	E≠M	No Eq	E≠M	No Eq	E≠M
0.5	34	3	96	1	165	7	36	2	96	6	171	4
0.501	44	1	118	1	186	6	49	1	116	6	203	4
0.51	180	0	307	0	430	2	187	0	291	2	441	2
0.55	748	0	1043	0	1353	0	707	0	1026	0	1294	0
0.6	1462	0	1779	0	1879	0	1450	0	1788	0	1883	0
0.7	1992	0	1997	0	2000	0	1993	0	1998	0	2000	0

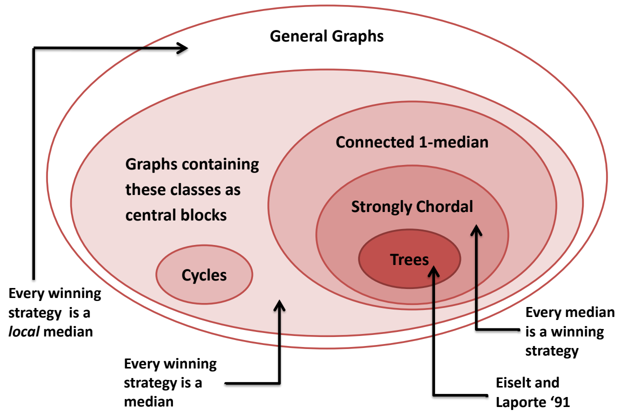


FIGURE 8. Nesting of the families of graphs considered in our results.

that admit equilibria). The instances shown in Figure 7 illustrate these rare situations (since neither instance is a median graph these do not contradict our theoretical results). The simulations provide evidence that instances of that kind do not materialize often in real-world urban networks.

It can also be observed in Table 1 that as α grows, equilibria become less likely. To provide some intuition on this phenomena, let us consider the extreme case in which $\alpha = 1$. Player 1 will always mimic the move of player 2 because if both choose the same vertex all the demand goes to 1. On the other hand, player 2 is better off selecting any other vertex. Therefore, an equilibrium can never exist. Despite that fact, for values of α bounded away from 1, instances inspired from real-world networks tend to admit equilibria, and equilibria agree with medians. This, again, provides evidence that our theoretical results extend when considering topologies inspired by real-world urban networks.

7. Final Remarks

We have provided an exhaustive characterization of equilibria for duopolistic facility location games under different classes of topologies. Figure 8 provides an illustration of some of the classes of graphs that we have considered. For SCB graphs (including trees, cycles, and cacti) and for graphs whose 1-medians form a connected subgraph, we have shown that equilibria must always coincide with 1-medians. For strongly chordal graphs, we have shown that every solution consisting of 1-medians must be an equilibrium. In addition, for SCB graphs we have provided a procedure to decide if an equilibrium exists or not in $O(|V|)$ time and characterized all their locations. Going forward, an interesting challenge may be to characterize all classes for which both are equal whenever an equilibrium exists.

In terms of equilibrium efficiency, we have showed that SCB graphs are monotone, meaning that the consumer cost at equilibrium cannot decrease when edges are removed. We have used this monotonicity property to bound the inefficiency of equilibria for the class of monotone graphs. It would be interesting to identify other classes of monotone graphs, because our results on inefficiency would readily extend to them.

We have also tested the robustness of our result when the demand is not split equally between facilities in case of a tie. Although in theory even a minimal change in the splitting rule can modify the existence of equilibria, we provide empirical evidence that such cases are rare in real-world urban networks. Furthermore, although real-world urban networks may fail to be grid graphs, in the vast majority of instances we tested the winning strategies and the 1-medians were located in the same vertices.

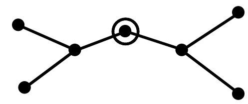


FIGURE 9. For three players on a tree, if the degree of a median is 2, an equilibrium cannot exist.

In the following subsections, we comment on some potential generalizations of our analysis. Besides these, one may consider allowing each player to control multiple facilities. Generalizing our results in that direction is not trivial. Taking the case of trees as a potentially simple example, we see that Theorem 3.1 does not hold for 2 facilities: the solution where both players select the 2-median is not necessarily an equilibrium.

Primitives of the Model. Allowing zero weights on vertices does not effect the existence and characterization results in this work. For trees, it is possible that there are more than two medians, which may happen only when there is a path of zero-weight vertices between two positive-weight vertices that are medians. Theorem 3.1 continues to hold under this relaxation. However, when vertices can have zero weights there may be more than two medians in a tree, and therefore we may have additional equilibria. Considering cycles, the definition of a winning strategy, as well as Theorem 3.4, do not assume strictly positive weights, and therefore hold without changes. Hence, we can extend our results to SCB graphs with zero weights.

Furthermore, some of our results also hold when edge lengths are arbitrary, instead of 1. In that case, we can approximate lengths with rational numbers, which allows us to assume that they are integral. Theorem 3.1 is for general lengths and holds without modification. With respect to cycles, we can subdivide edges with zero-weight vertices at unit distances from each other. These artificial vertices do not constitute feasible locations for facilities, and should not be taken into account when determining if a vertex is a winning strategy. The computational effort in looking for winning strategy vertices does not change, and Theorem 3.4 holds as well. Hence, we can extend our results to SCB graphs with arbitrary lengths.

More Than Two Players. Another direction of further research is to characterize equilibria for an arbitrary number of players in trees and in cycles with arbitrary weights. One difficulty that arises when considering more than two players is that an equilibrium may not split the market in equal parts, as Remark 2.1 indicates it happens for duopolies. In the cases of trees, an equilibrium may not exist and if it does it may not be located on a median. For the case of three players, one can show that there is an equilibrium only if any median of that tree has a degree strictly larger than two (see Figure 9), which generalizes a result of Lerner and Singer (1937) whereby an equilibrium does not exist for three players on a unit-weight line. More generally, Eiselt and Laporte (1993) provided sufficient conditions and a characterization of equilibria on a tree with three players, under various assumptions. While a generalization to an arbitrary number of players on a tree is difficult, such generalization may be successful for simpler structures. For instance, in sharp contrast to the case of a line with three players, Gur (2009) shows that equilibria always exist on a line with k unit-weight vertices with four or more players. He also provides the equilibrium profiles, which depend on the values of n and k.

In the cases of cycles, the characterization in Theorem 3.4 is based on how players split the vertices and the total weight of the cycle; this property does not hold for more than two players. Nevertheless, the intuition gained in the treatment for two players may help when considering necessary and sufficient conditions for equilibria for more than two players.

Acknowledgement

The authors thank Ehud Lehrer, Miquel Oliu Barton, Iván Rapaport and Luis Scoccola for insightful discussions and enriching comments. This work was partially supported by the Chazen Institute of International Business at Columbia Business School, by Conicet Argentina grant Resolución 4541/12 and by ANPCyT Argentina PICT-2012-1324. Preliminary versions including some of the results that are presented in this paper appear in Gur and Stier-Moses (2011) and Saban and Stier-Moses (2012).

References

H.-K. Ahn, S.-W. Cheng, O. Cheong, M. Golin, and R. van Oostrum. Competitive facility location: the Voronoi game. *Theoretical Computer Science*, 310:457–467, 2004.

A. V. Aho, J. E. Hopcroft, and J. D. Ullman. *The Design and Analysis of Computer Algorithms.* Addison-Wesley, Reading, MA, 1974.

H. Bandelt and V. Chepoi. Graphs with connected medians. *SIAM Journal Discrete Mathematics*, 15(2): 268–282, 2002.

B. Ben-Moshe, B. Bhattacharya, Q. Shi, and A. Tamir. Efficient algorithms for center problems in cactus graphs. *Theoretical Computer Science*, 378(3):237-252, 2007.

J. Bhadury and H. A. Eiselt. Stability of Nash equilibria in locational games. *Recherche opérationnelle*, 29 (1):19–33, 1995.

S. Bharathi, D. Kempe, and M. Salek. Competitive influence maximization in social networks. In *Proceedings of the 3rd international conference on Internet and network economics (WINE)*, pages 306–311, Springer, Berlin, 2007.

D. Braess. Über ein Paradoxon aus der Verkehrsplanung. *Unternehmensforschung*, 12:258–268, 1968. An English translation appears in Transportation Science, 39:(4):446–450, 2005.

A. Dasci. Conditional location problems on networks and in the plane. In H. A. Eiselt and V. Marianov, editors, *Foundations of Location Analysis*, International Series in Operations Research & Management Science, pages 179–206, Springer, New York, 2011.

C. D'Aspremont, J. J. Gabszewicz, and J. F. Thisse. Product differences and prices. *Economic Letters*, 11: 19–23, 1983.

A. de Palma, V. Ginsberg, Y. Y. Papageorgiou, and J. F. Thisse. The principle of minimum differential holds under sufficient heterogeneity. *Econometrica*, 53(4):767–781, 1985.

G. A. Dirac. On rigid circuit graphs. Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg 25, Universität Hamburg, 1961.

T. Drezner. Competitive facility location in the plane. In Z. Drezner, editor, *Facility Location: A Survey of Applications and Methods*, pages 285–300, New York, 1995a. Springer.

T. Drezner and Z. Drezner. Competitive facilities: Market share and location with random utility. *Journal of Regional Science*, 36(1):1–15, 1996.

T. Drezner, Z Drezner, and S. Salhi. Solving the multiple competitive facilities location problem. *European Journal of Operational Research*, 142:138–151, 2002.

T. Drezner, Z. Drezner, and P. Kalczynski. A cover-based competitive location model. *Journal of the Operational Research Society*, 62:100–113, 2011.

Z. Drezner. Competitive location strategies for two facilities. *Regional Science and Urban Economics*, 12: 485–493, 1982.

Z. Drezner. *Facility Location: A Survey of Applications and Methods.* Springer, New York, 1995b.

Z. Drezner and T. Drezner. Finding the optimal solution to the Huff competitive location model. *Computational Management Science*, 1(2):193–208, 2004.

Z. Drezner, G. O. Wesolowsky, and T. Drezner. On the logit approach to competitive facility location. *Journal of Regional Science*, 38:313–327, 1998.

C. Dürr and N. K. Thang. Nash equilibria in Voronoi games on graphs. In L. Arge, M. Hoffmann, and E. Welzl, editors, *Proceedings of the 15th European Symposium on Algorithms (ESA)*, volume 4698, pages 17–28. Springer, Heidelberg, 2007.

B. Eaton and R. Lipsey. The principle of minimum differentiation reconsidered: Some new developements in the theory of spatial competition. *Review of Economic Studies*, 42:7–49, 1976.

H. A. Eiselt and G. Laporte. Competitive spatial models. *European Journal of Operational Research*, 39: 231–242, 1989.

H. A. Eiselt and G. Laporte. Locational equilibrium of two players on a tree. *Recherche opérationnelle*, 25 (1):5–18, 1991.

H. A. Eiselt and G. Laporte. The existence of equilibria in the 3-facility Hotelling model in a tree. *Transportation Science*, 27(1):39–43, 1993.

H. A. Eiselt and G. Laporte. Review of sequential location problems. *European Journal of Operational Research*, 96:217–231, 1996.

H. A. Eiselt and V. Marianov. *Foundations of Location Analysis*. International Series in Operations Research & Management Science. Springer, New York, 2011.

H. A. Eiselt, G. Laporte, and J.-F. Thisse. Competitive location models: a framework and bibliography. *Transportation Science*, 27:44–54, 1993.

J. J. Gabszewicz and J. F. Thisse. Spatial competition and location of firms. *Fundamentals of Pure and Applied Economics*, 5:1–71, 1986.

D. Geller and B. Manvel. Reconstruction of cacti. *Canadian Journal of Mathematics*, 21:1354–1360, 1969.

M. X. Goemans, V. S. Mirrokni, and A. Vetta. Sink equilibria and convergence. In *Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science*, pages 142–154, Pittsburgh PA, 2005.

A. J. Goldman. Optimal center locations in simple networks. *Transportation Science*, 5:212–221, 1971.

D. Graitson. Spatial competition à la Hotelling: A selective survey. *Journal of Industrial Economics*, 31: 13–25, 1982.

D. Granot, F. Granot, and T. Raviv. On competitive sequential location in a network with a decreasing demand intensity. *European Journal of Operational Research*, 205:301–312, 2010.

Y. Gur. On Nash equilibria in network location games. Master's thesis, School of Mathematical Science, Tel-Aviv University, 2009.

Y. Gur and N. E. Stier-Moses. A facility location problem under competition. In J. Douceur and A. Ozdaglar, editors, *Proceedings of the Workshop on the Economics of Networks, Systems and Computation (NetEcon)*, 2011.

S. L. Hakimi. On locating new facilities in a competitive environment. *European Journal of Operational Research*, 12:29–35, 1983.

S. L. Hakimi. P-Median theorems for competitive locations. *Annals of Operations Research*, 6:79–98, 1986.

P. Hansen, J.-F. Thisse, and R. Wendell. Equilibrium analysis for voting and competitive location problems. In P. B. Mirchandani and R. L. Francis, editors, *Discrete Location Theory*, pages 439–478, Wiley, New York, 1990.

F. Harary. *Graph Theory*. Addison-Wesley, Reading, MA, 1969.

J. Hatzl. Median problems on wheels and cactus graphs. *Computing*, 80(4):377–393, 2007.

D. A. Hay. Sequential entry and entry-deterring strategies in spatial competition. *Oxford Economic Papers*, 28:240–257, 1976.

J. Hopcroft and R. Tarjan. Efficient algorithms for graph manipulation. *Communications of the ACM*, 16 (6):372–378, 1973.

H. Hotelling. Stability in competition. *The Economic Journal*, 39(153):41–57, 1929.

K. Jain and M. Mahdian. Cost sharing. In N. Nisan, T. Roughgarden, É. Tardos, and V. V. Vazirani, editors, *Algorithmic Game Theory*, pages 385–410, Cambridge University Press, UK, 2007.

O. Kariv and S. L. Hakimi. An algorithmic approach to network location problems. II: The p-medians. *SIAM Journal on Applied Mathematics*, 37(3):539–560, 1979.

J. Kostka, Y. A. Oswald, and R. Wattenhofer. Word of mouth: Rumor dissemination in social networks. In *Proceedings of the 15th international colloquium on Structural Information and Communication Complexity (SIROCCO)*, volume 5058 of *Lecture Notes in Computer Science*, pages 185–196, Springer, Berlin, 2008.

E. Koutsoupias and C. H. Papadimitriou. Worst-case equilibria. *Computer Science Review*, 3(2):65–69, 2009. Conference version in *16th Annual Symposium on Theoretical Aspects of Computer Science*, pages 404–413, Trier, Germany, 1999.

D. Kress and E. Pesch. Sequential competitive location on networks. *European Journal of Operational Research*, 217(3):483–499, 2012.

H. Lee and G. J. Chang. The w-median of a connected strongly chordal graph. *Journal of Graph Theory*, 18(7):673–680, 1994.

G. Leonardi and R. Tadei. Random utility demand models and service location. *Regional Science and Urban Economics*, 14(3):399–431, 1984.

A. P. Lerner and H. W. Singer. Some notes on duopoly and spatial competition. *Journal of Political Economy*, 45:145–186, 1937.

M. Mavronicolas, B. Monien, V. G. Papadopoulou, and F. Schoppmann. Voronoi games on cycle graphs. In *Proceedings of the 33rd International Symposium on Mathematical Foundations of Computer Science (MFCS)*, volume 5162, pages 503–514, Springer, Berlin, 2008.

P. B. Mirchandani and R. L. Francis. *Discrete Location Theory*. Wiley, New York, NY, 1990.

V. S. Mirrokni and A. Vetta. Convergence issues in competitive games. In *International Workshop on Approximation Algorithms for Combinatorial Optimization Problemas (APPROX)*, volume 3122 of *Lecture Notes in Computer Science*, pages 183–194. Springer, Berlin, 2004.

J. F. Nash. Noncooperative games. *Annals of Mathematics*, 54(2):286–295, 1951.

M. J. Osborne and A. Rubinstein. *A Course in Game Theory*. MIT Press, Cambridge, MA, 1994.

C. H. Papadimitriou. Algorithms, games, and the Internet. In *Proceedings of the 33rd Annual ACM Symposium on Theory of Computing (STOC)*, pages 749–753, ACM Press, New York, 2001.

F. Plastria. Static competitive facility location: An overview of optimisation approaches. *European Journal of Operational Research*, 129:461–470, 2001.

E. C. Prescott and M. Visscher. Sequential location among firms with foresight. *Bell Journal of Economics*, 8:378–393, 1977.

R. Ravi and A. Sinha. Approximation algorithms for problems combining facility location and network design. *Operations Research*, 54(1):73–81, 2006.

C. S. ReVelle. The maximum capture or sphere of influence location problem: Hotelling revisited on a network. *Journal of Regional Science*, 26:343–357, 1986.

C. S. ReVelle and H. A. Eiselt. Location analysis: A synthesis and survey. *European Journal of Operational Research*, 165:1–19, 2005.

T. Roughgarden and É. Tardos. How bad is selfish routing? *Journal of the ACM*, 49(2):236–259, 2002.

D. Saban and N. E. Stier-Moses. The competitive facility location problem in a duopoly: Relations to the 1-median problem. In P.W. Goldberg and M. Guo, editors, *8th Workshop on Internet & Network Economics (WINE)*, volume 7695 of *Lecture Notes in Computer Science*, pages 540–546, Liverpool, UK, 2012. Springer, Heidelberg.

S. C. Salop. Monopolistic competition with outside goods. *Bell Journal of Economics*, 10:141–156, 1979.

L. Scoccola. The competitive facility location problem. Master's thesis, Computer Science Department, Universidad de Buenos Aires, 2014.

P. J. Slater. Maximin facility location. *Journal of Research of the National Bureau of Standards*, B79: 107–115, 1975.

P. J. Slater. Medians of arbitrary graphs. *Journal of Graph Theory*, 4:389–392, 1980.

H. K. Smith, G. Laporte, and P. R. Harper. Locational analysis: highlights of growth to maturity. *Journal of the Operational Research Society*, 60(1):140–148, 2009.

M. Taylor. Review article: mathematical political theory. *British Journal of Political Science*, 1:339–381, 1971.

A. Vetta. Nash equilibria in competitive societies with applications to facility location, traffic routing and auctions. In *Proceedings of the 43rd Annual IEEE Symposium on Foundations of Computer Science (FOCS)*, pages 416–428, Vancouver, Canada, 2002. IEEE Computer Society Press, Los Alamitos, CA.

R. E. Wendell and R. D. McKelvey. New prospectives in competition location theory. *European Journal of Operational Research*, 6(2):174–182, 1981.

Appendix A. Finding all Equilibria in a Cycle in $\mathbf{O}(|\mathbf{V}|)$ Complexity

Denote $|V| = k$. To list all the equilibria in a cycle for the case of two players, it is sufficient to find all the winning strategies of that cycle. We provide an algorithm that finds all of them in $O(k)$ time complexity. We assume that k is even; the case of odd k is omitted because it is similar.

We first create a list of the weights $W(S_i)$ of the $2k$ half-cycles $\{S_i\}_{i=1,\dots,2k}$ of the cycle. To compute the weight of S_1, we sum the weights of the $k/2$ successive vertices, which takes $O(k)$ time. For each additional half-cycle, we only need to subtract half the weight of a vertex and to add half the weight of another vertex. Therefore, the rest of the weights of the half-cycles are produced in $O(k)$ as well.

With this list, identifying a winning strategy vertex involves finding a sequence of $k-1$ consecutive half-cycles that have weight of at most $W/2$. To do that, we can go over the list once and output all the winning strategies. This can be also done in $O(k)$ time because basically for each half-cycle that has weight larger than $W/2$ we know that all vertices outside it cannot be winning strategies. All together, we can list all winning strategies in $O(k)$ time and arbitrary pairs of these vertices form all the equilibria of the game.

Appendix B. Building a Maximal Bi-connected Components Tree

The algorithm described by Aho et al. (1974) provides description of the blocks of an arbitrary input graph $G(V, E)$, in $O(|E|)$ time. The algorithm assigns each edge to a class, where each class contains edges that belong to a block (these classes contain at least two edges), and outputs single edges that are not contained in any block (and therefore connects two cutoff vertices). We build a maximal bi-connected components tree G^T going over the classes. When we finish a class that represents a block, we select an arbitrary vertex as its representative in G^T, and set its weight to the sum of the weights of the vertices of the class. For edges that are not part of blocks, we represent both cutoff vertices in G^T and connect them with an edge. A vertex we already observed in a previous class must be a cutoff vertex. This vertex will be represented by a vertex in G^T, and we deduce its weight from any blocks it was contained in so far (if we did not deduced it already). This way, we can output G^T using the list of edges provided by Aho et al. (1974) only once, in $O(|E|)$ time.

Appendix C. Price of Anarchy on a Uniform Line

We consider a line of n vertices with unit weight. Without loss of generality, we assume that at equilibrium both players locate in the same median. The consumer optimum is such that players locate on vertices that are as close as possible to vertices $n/4$ and $3n/4$ of the line. To be precise, we need to consider four different cases, depending on the remainder after dividing by 4. When $n = 4k$ we have that

$$D_{eq} = \sum_{i=1}^{2k-1} i + \sum_{i=1}^{2k} i = 4k^2 \qquad \text{and} \qquad D_{opt} = 4\sum_{i=1}^{k-1} i + 2k = 2k^2 \,.$$

Then, $D_{eq}/D_{opt} = 2$ for any value of k. When $n = 4k+1$ we have that

$$D_{eq} = 2\sum_{i=1}^{2k} i = 4k^2 + 2k \qquad \text{and} \qquad D_{opt} = 4\sum_{i=1}^{k-1} i + 3k = 2k^2 + k \,.$$

Then, $D_{eq}/D_{opt} = 2$ for any value of k. When $n = 4k+2$ we have that

$$D_{eq} = \sum_{i=1}^{2k} i + \sum_{i=1}^{2k+1} i = 4k^2 + 4k + 1 \qquad \text{and} \qquad D_{opt} = 4\sum_{i=1}^{k} i = 2k^2 + 2k \,.$$

Then, $D_{eq}/D_{opt} \leq 9/4$. The upper bound is achieved when $k = 1$ and the ratio decreases with k. Finally, when $n = 4k + 3$ we have that

$$D_{eq} = 2 \sum_{i=1}^{2k+1} i = 4k^2 + 6k + 2 \qquad \text{and} \qquad D_{opt} = 4 \sum_{i=1}^{k} i + (k+1) = 2k^2 + 3k + 1\,.$$

Then, $D_{eq}/D_{opt} = 2$ for any value of k. Altogether, we see that a tight upper bound on D_{eq}/D_{opt} is 9/4, achieved on a line with 6 vertices with unit weight. The analysis above also provides a lower bound for the consumer cost on a line of n vertices with unit weight. Rewriting the expression for D_{opt} as a function of n in each case, we get $n^2/8$, $(n^2 - 1)/8$, $(n^2 - 4)/8$ and $(n^2 - 1)/8$, respectively. Then, $D_{\text{opt}} \geq (n^2 - 4)/8$.

Appendix D. Proofs Deferred to the Appendix

Proof of Lemma 3.2. Let x be the median that is closest to v and $T' \ni v$ be the connected component that remains if x is removed from the tree T. We consider that $W(T') = W/2$ because otherwise the result holds. In this case the neighbor x' of x in T' must be a median too because removing it generates components with weight bounded by $W/2$. This is straightforward for components included in T' and $W(T \setminus T') = W/2$. The claim follows from the contradiction that x is not the closest median to v.

Proof of Lemma 3.3. Suppose player one is located on a vertex v_j that is not a winning strategy; i.e., there is a half-cycle S^* not including v_j for which $W(S^*) > W/2$. We will find a vertex v_i from which player two can get all the demand in S^*. First, suppose that our cycle contains an even number of vertices and let v' be the vertex in S^* that is closest to v_j. If S^* consists of $k/2$ vertices, player two selects the vertex $v_i \in S^*$ that satisfies $d(v_i, v') = d(v_j, v') - 1$ and gets the full demand in both extremes of S^*. If S^* consists of $k/2 - 1$ vertices and 2 half-vertices, player two selects the vertex $v_i \in S^*$ that satisfies $d(v_i, v') = d(v_j, v')$ and gets half of the demand in both extremes of S^*. The only case left is when our cycle contains an odd number of vertices and S^* consists of $(k-1)/2$ vertices and a half-vertex. We let v' be the full-vertex in the extreme of S^*. As before, by selecting $v_i \in S^*$ that satisfies $d(v_i, v') = d(v_j, v') - 1$, player two gets the demand of v' and half of that of the half-vertex. In the three cases, player two gets all S^*, as needed.

Proof of Theorem 3.4. Let us assume that v is a winning strategy of a cycle graph $G = (V, E)$ with total consumer cost $C(v) := \sum_{i \in V} w(i) d(i, v)$. We want to prove that $C(v') \geq C(v)$ for any vertex $v' \in V$. To compare both costs, we compute:

$$C(v') - C(v) = \sum_{i \in V} d_i w(i) \tag{1}$$

where we have defined $d_i := d(i, v') - d(i, v)$. The labels d_i measure the difference in cost corresponding to a unit demand in i, when locating the facility in vertex v' instead of in the original one v. Referring to the distance between both vertices by d, these labels are integer numbers between $-d$ and $+d$.

To show the result, we will regroup the terms in (1) in a way that they are all non-negative, using that v is a winning strategy. We refer to the set of vertices along the shortest path between v and v' (not including v and v') as P (if both vertices are opposite each other in the cycle, we may choose either of the two equal paths). We denote the half-cycle that begins at vertex $i \in P$ and that contains v (resp., v') by S_{iv} (resp., $S_{iv'}$). In addition, we denote the half-cycle that begins at v and that does not contain v' by S_v, and similarly the half-cycle that begins at v' and does not contain v by $S_{v'}$ (Figure 10 shows an example of this construction). In the previous definitions, all the indicated vertices where the half-cycles start are full-vertices; the half-cycles may contain half-vertices at the other end.

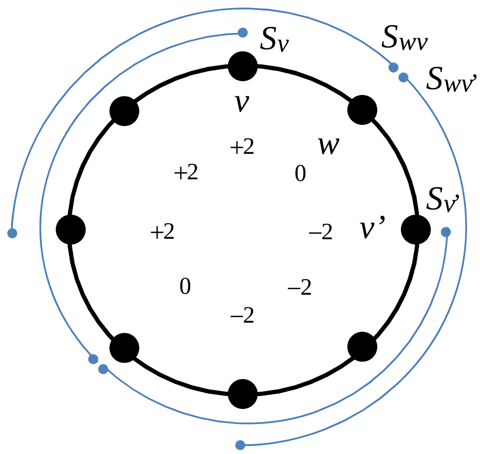


FIGURE 10. Example of a cycle with $|V| = 8$ and $d = 2$. We represent the half-cycles used in the proof of Theorem 3.4 outside the cycle, the vertex labels close to the vertices, and the labels d_i inside it.

We group all these $2d$ half-cycles in two sets, depending on whether they include v or v'. We let

$$\mathcal{S}_+ := S_v \cup \bigcup_{i \in P} S_{iv} \qquad \textit{and} \qquad \mathcal{S}_- := S_{v'} \cup \bigcup_{i \in P} S_{iv'}.$$

The key of this proof is that each label d_i is equal to the number of half-cycles that include i, counting with the correct sign depending on whether the half-cycle includes v or v'. Indeed,

$$d_i = |\{S \in \mathcal{S}_+ | i \in S\}| - |\{S \in \mathcal{S}_- | i \in S\}|$$

for all $i \in V$. We note that for every $S \in \mathcal{S}_-$ the complementary half-cycle S^C belongs to $\mathcal{S}_+$. Because half-vertices can only occur at the extreme of a half-cycle, every time a half-cycle contains a half-vertex, the other half-vertex is contained in the complement of the half-cycle, hence half-vertices do not influence d_i. Since v is a winning strategy and $v \in S$ for all $S \in \mathcal{S}_+$, we have $W(S) \geq W(S^C)$ for all $S \in \mathcal{S}_+$. Therefore, the structure of the constructed half-cycles implies that

$$\sum_{i \in V} d_i w(i) = \sum_{S \in \mathcal{S}_+} W(S) - \sum_{S \in \mathcal{S}_-} W(S) = \sum_{S \in \mathcal{S}_+} \left(W(S) - W(S^C)\right) \geq 0,$$

which implies that $C(v') \geq C(v)$. Since v' was an arbitrary vertex in the cycle, this is true for all $v' \in V$, and therefore v solves the 1-median problem in G.

Proof of Lemma 3.5. Let v be a vertex in G corresponding to a median of G^T that is a cutoff vertex and let x be any other vertex in G. We need to prove that $u_1(v, x) \geq W/2$. This follows because $u_1(v, x) \geq W(V \setminus T_x)$ where T_x denotes the component containing x that remains after removing v from G. Finally, the LHS cannot be smaller than $W/2$ because v is a median of G^T so T_x has total weight at most $W/2$.

Proof of Lemma 3.6. Let G_j be the central block of G. Utilities under G and the projection G'_j coincide for any two vertices in G_j. Let $v \in G_j$ be a winning strategy of G. Then, $v \in G'_j$ and the conditions for being a winning strategy of G'_j are trivially satisfied. For the other direction, if v is a winning strategy of G_j, we must see that v is a winning strategy of G. It remains to be seen that $u_1(v, x) \geq W/2$ for all $x \notin G_j$. To get a contradiction, suppose that there exists $x \notin G_j$ such that $u_1(v, x) < W/2$. Letting x' be the projection of x onto G_j, we have that $u_1(v, x') \leq u_1(v, x) < W/2$. The contradiction follows because $x' \in G_j$.

Proof of Theorem 3.7. First, assume that all medians of G^T are cutoff vertices. Then, vertex v that is not a median of G^T cannot be a winning strategy of G: Lemma 3.2 implies that there is a median $v' \in G^T$ such that removing v' from G^T leaves the vertex in G^T corresponding to v in a component of weight strictly less than $W/2$. Since that median also belongs to G, $u_1(v, v') < W/2$, and therefore v is not a winning strategy.

Second, we show that if one median of G^T represents a block, a vertex v that is not part of that block cannot be a winning strategy of G. We extend the argument of the previous paragraph to handle both cutoff vertices and blocks. Denoting that block by G_j, if $v \notin G_j$, Lemma 3.2 implies that there is a median $v' \in G^T$ such that removing v' from G^T leaves the vertex in G^T that corresponds to v in a component of weight strictly less than $W/2$. If that median is a cutoff vertex, then it also belongs to G and hence $u_1(v, v') < W/2$. If that median represents a block of G, we let v'' be the cutoff vertex adjacent to v' in the path from v' to v. Since v is not in the central block, it cannot be v''. Thus, $u_1(v, v'') < W/2$. Since the opponent can get more than half of the demand in both cases, v cannot be a winning strategy.

Proof of Proposition 5.1. Let us first assume that v is a median of G^T that is a cutoff vertex. We show that v is a winning strategy in G' by showing that it is a is a median of $(G')^T$ that is a cutoff vertex. That v is a cutoff vertex follows because G' contains less edges than G. To see that v is also a median notice that even though $(G')^T$ may be different from G^T, the connected components that remain after removing v from each of those graphs have exactly the same weight. Hence, all of them have weight bounded by $W/2$ implying that v is a median of $(G')^T$, as required.

We now consider the case when G^T has one median that is a block. We refer to the central block as cycle C (G has a simple central block). Removing edges outside C does not change components and at most one edge can be removed from C without disconnecting the graph. Hence, we can write without loss of generality that $E' = E \setminus \{e\}$ for some $e \in C$. The smaller graph G' consists of a line (the broken cycle C) and all the components that are connected to C in G. Similarly to what we did for cycles in Section 3.3, we project the whole graph G' onto the line $C \setminus \{e\}$ with weights that represent the whole component connected to each of the vertices. Let us call this projection L. The median v of L, which must exist because the graph is a line, must be a median of G'. This is because removing v from G' creates components that have weight bounded by $W/2$. The two components in L satisfy the bound because v is a median of L. The component connected to G' through v satisfies that bound because C was a central block of G^T. Finally, since that median is a cutoff vertex of G', it must be a winning strategy.

Proof of Proposition 5.4. First, let us prove that the consumer cost of a solution to a 2-median problem is lower bounded by $(d^2 + 2d - 3)/8$. Indeed, the consumer cost can be subdivided in that incurred by the first unit of demand for each vertex and the consumer cost of the rest of the demand. Bounding the former provides the result because the latter is non-negative. Hence, we can assume that all vertices have unit weight. Any tree of diameter d must have a line of size $d+1$ as a subgraph, so the consumer cost must be at least the one incurred along this line. The case-by-case analysis of lines provided in Appendix C shows that the minimal consumer cost is equal to $(d-1)(d+3)/8$, which is attained when the diameter has the form of $d = 4k + 1$ for some integer k. The consumer cost for the complete tree must be even higher.

Second, let us see that the consumer cost at equilibrium is upper bounded by $\delta(|V|-1)(d+1)/2$. We let v be the vertex achieving the graph radius $\min_{v_i \in V}\{\max_{v_j \in V} d(v_i, v_j)\}$ (called the *central vertex* of the graph). In particular, v lies in a line of $d+1$ vertices and the distance between v and any other vertex in the tree is at most $(d+1)/2$. Considering the other $|V|-1$ vertices in the tree that have a maximum weight of δ, the consumer cost when both facilities are located at v is at most $\delta(|V|-1)(d+1)/2$. Finally, since equilibrium locations in trees must be 1-median solutions, the consumer cost at equilibrium can only be lower.

The result follows from the combination of the bounds in the previous two paragraphs.

Appendix E. Proof of Result for Strongly Chordal Graphs

In this section, we prove Theorem 4.4, which establishes that every connected strongly chordal graph has an equilibrium, and that there is a one-to-one correspondence between winning strategies and the solutions to the 1-median problem. The proof and the presentation follow the methodology of Lee and Chang (1994).

E.1. **Strongly Chordal Graphs and Related Definitions.** Recall that a graph is *chordal* if every cycle with more than three vertices has a *chord* (an edge joining two nonconsecutive vertices of the cycle). A *p-sun* is a chordal graph with a vertex set $x_1, \ldots, x_p, y_1, \ldots, y_p$ such that $y_1, \ldots, y_p$ is an independent set, $(x_1, \ldots, x_p, x_1)$ is a cycle, and each vertex y_i has exactly two neighbors x_{i-1} and x_i with the understanding that $x_0 = x_p$. A graph G is *strongly chordal* if it is chordal and contains no p-sun for $p \geq 3$. An important property is that any induced subgraph of a strongly chordal graph is also strongly chordal.

The (open) neighborhood $N_G(y)$ of a vertex y in G is the set $\{z \in V : yz \in E\}$. The closed neighborhood $N_G[y]$ of vertex y is $N_G(y) \cup \{y\}$. A *clique* is a set of pairwise adjacent vertices. A vertex u is *simplicial* if $N_G(u)$ is a clique. Suppose u is a simplicial vertex of a chordal graph G. For any two vertices $x, y \in G \setminus \{v\}$, note that a shortest x-y path in G cannot contain u. Indeed, $x, \ldots, x', u, y', \ldots, y$ cannot be a shortest because $x, \ldots, x', y', \ldots, y$ is a feasible path that is shorter. Consequently, $d_{G\setminus\{v\}}(x, y) = d(x, y)$ for any two vertices $x, y \in G \setminus \{v\}$, where $d_{G\setminus\{v\}}$ is the distance induced by that subgraph. Therefore, the graph $G \setminus \{v\}$ is called a *distance invariant subgraph* of G.

A vertex u is *simple* if for any two $x, y \in N_G[u]$ either $N_G[x] \subseteq N_G[y]$ or $N_G[y] \subseteq N_G[x]$. Note that a simple vertex is a simplicial vertex. A *maximal* neighbor of a simple vertex v is a vertex $m \in N_G[v]$ such that $N_G[x] \subseteq N_G[m]$ for all $x \in N_G[u]$. Below we will use that every strongly chordal graph that is not complete has two nonadjacent simple (hence, also simplicial) vertices (Dirac 1961).

E.2. **Proof Sketch.** As in Lee and Chang (1994), we introduce the concept of *cost w-median*, which is more general than the median, to be able to remove vertices without changing the solution in resulting graph. To do that, besides weights w, we also associate nonnegative costs c to each vertex in V. The *cost w-distance sum* of a vertex $y \in V$ is $D_{G,w,c}(y) := \sum_{v \in V} d(y, v)w(v) - c(y)$, where we subtract the cost from the weighted distance to a vertex. The *cost w-median* $M_{G,w,c}$ of G is the set of vertices that minimize that objective function; i.e., $\{y \in V : D_{G,w,c}(y) \leq D_{G,w,c}(z)\ \forall z \in V\}$. Clearly, $M_{G,w,c}$ reduces to the median-set of G when $c(y) = 0$ for all $y \in V$.

The proof starts with a connected strongly chordal graph G with positive weights w and costs $c(y) = 0$ for all $y \in V$. We apply an inductive step that removes a simple vertex to obtain the smaller graph G'. In this step, we modify weights and costs to w' and c', guaranteeing that $M_{G,w,c} = M_{G',w',c'}$. Furthermore, we show that the set of winning strategies in G' coincide with those of G. We repeat this induction until the resulting graph is complete, in which case the problem can be easily solved for an arbitrary cost function. The main difference from the from proof by Lee and Chang is that we need to keep track of more details in each iteration to make the accounting work and keep winning strategies invariant.

E.3. **Proof Setup.** In iteration i, we select a simple vertex v^i of the current strongly chordal graph G^i and remove it. We also select a maximal neighbor m^i of v^i that will "absorb" v^i in that iteration. When iteration i removes vertex v^i, we increase the weight of its maximal neighbor m^i by $w^i(v^i)$, and we increase the cost of all the other neighbors by $w^i(v^i)$ to compensate and maintain the same medians (details later). We denote the vector of changes in costs by Δ^i, and maintain a current vector of weights and costs in w^i and c^i, respectively. To maintain the same winning strategies throughout the iterations, we consider that players are given subsidies: in addition to her corresponding market-share, when a player selects vertex x_1 in iteration i, she receives a subsidy $\sigma^i(x_1, x_2)$. Notice that the subsidy also depends on the vertex x_2 chosen by the other player. The reason we consider vertex-dependent subsidies is that removing vertices changes the relative distances between vertices so we need to correct for that.

For $y, x \in V^i$, we let the logical expression $Q^i(y,x)$ denote whether the inequality

$$\sigma^i(y,x) + \sum_{z \in C_{G^i}[y,x]} w^i(z) \geq \sigma^i(x,y) + \sum_{z \in C_{G^i}[x,y]} w^i(z) \tag{2}$$

holds, where we let $C_{G^i}[x,y]$ denote the set of vertices in V^i such that their distance to x is strictly smaller than their distance to y; i.e., $C_{G^i}[x,y] := \{z \in V^i : d_{G^i}(x,z) < d_{G^i}(y,z)\}$. Extending the notion introduced earlier, we say that y is a winning strategy in iteration i if $Q^i(y,x)$ holds for all $x \in V^i$. Accordingly, we denote the set of winning strategies by WS^i. Clearly, when $\sigma^i = 0$ for all vertices, this definition matches the regular one, so winning strategies coincide.

To help us choose the correct vertices in each iteration, we also construct a sequence $S^i(x) = \{S^i_0(x) = x, S^i_1(x), \dots, S^i_{n^i(x)}(x)\} \subseteq N_G[x]$ for each vertex $x \in V$, where $n^i(x)$ is implicitly defined as the length of the sequence at iteration i. Throughout the proof we maintain the following two invariants on these sequences:

- **(C1):** $N_G[S^i_0(x)] \subseteq N_G[S^i_1(x)] \subseteq \ldots \subseteq N_G[S^i_{n^i(x)}(x)]$.
- **(C2):** If $c^i(x) > 0$, then $n^i(x) \geq 1$, $c^i(x) < \sum_{j=1}^{n^i(x)} w^i(S^i_j(x))$, and $c^i(x) < \sum_{j=1}^{k} w^i(S^i_j(x)) + c^i(S^i_k(x))$ for $1 \leq k < n^i(x)$.

Finally, we define a partially ordered set (poset) P on the vertex-set V. Two vertices $y <_P z$ are ordered with respect to P at iteration i if $y = S^i_j(x)$ and $z = S^i_k(x)$ for some $x \in V$ with $0 \leq j < k \leq n^i(x)$. Note that P will not necessarily be poset if sequences $S(x)$ are not constructed properly.

E.4. **Proof Details.** We are now ready to present the full proof.

Theorem 4.4. *Every connected strongly chordal graph with positive weights has an equilibrium. Furthermore, there is a one-to-one correspondence between winning strategies and the solutions to the 1-median problem.*

Proof. We prove that the median-set at each iteration M_{G,w^i,c^i} is nonempty and constant. To complete the proof we also show that at each iteration the set of winning strategies WS^i is constant and coincides with the median-set referred to earlier. We prove this by induction as follows.

Setup: We initialize our procedure with weight vector $w^0 = w$, zero costs $c^0 = 0$, zero subsidies $\sigma^0 = 0$, and the sequences of vertices $S^0(x) = \{x\}$ for all $x \in V$. This choice of S^0 makes Conditions (Cl) and (C2) hold trivially.

Inductive step i: Let i be the current inductive step. Here, we transform the current graph G^i, which is not complete, into G^{i+1}. We choose a pair of nonadjacent simple vertices v^i and u^i of G^i such that $w^i(v^i) + c^i(v^i) \leq w^i(u^i) + c^i(u^i)$, which are guaranteed to exist. Let m^i be a maximal neighbor of v^i in G^i. As discussed in Lee and Chang (1994), v^i and u^i can be chosen so they are minimal with respect to the poset P and m^i can be chosen so it is maximal with respect to P. We let G^{i+1} be $G^i \setminus \{v^i\}$, which is a connected, distance-invariant, strongly chordal graph. For the new graph, we define the updated weights and costs for $x \in V^{i+1}$ as follows:

- $w^{i+1}(x) = \begin{cases} w^i(x) + w^i(v^i) & \text{if } x = m^i, \\ w^i(x) & \text{otherwise.} \end{cases}$
- $\Delta^{i+1}(x) = \begin{cases} w^i(v^i) & \text{if } x \in N_{G^i}(v^i) \setminus \{m^i\}, \\ 0 & \text{otherwise.} \end{cases}$
- $c^{i+1}(x) = c^i(x) + \Delta^{i+1}(x)$.

- $S^{i+1}(x) = \begin{cases} S^i(x) \cup \{m^i\} \setminus \{v^i\} & \text{if } x \in N_{G^i}(v^i) \text{ and } m^i \notin S^i(x), \\ S^i(x) \setminus \{v^i\} & \text{otherwise.} \end{cases}$
- Last, for $\{x_1, x_2\} \subseteq V^{i+1}$, we set $\sigma^{i+1}(x_1, x_2) = \begin{cases} \sigma^i(x_1, x_2) + \Delta^{i+1}(x_1) & \text{if } x_2 \in N_{G^{i+1}}[m^i], \\ \sigma^i(x_1, x_2) & \text{otherwise.} \end{cases}$

The vector w^{i+1} is positive and c^{i+1} is non-negative, as required. Note that since the weight of the vertex that is deleted is transferred to another vertex, the total weight of the graph is constant throughout; that is, $\sum_{x \in V^i} w^i(x) = \sum_{x \in V^0} w^0(x)$. Also, the definition maintains the invariant $c^{i+1}(x) = \sum_{j=1}^{i+1} \Delta^j(x)$ and $\sigma^{i+1}(x_1, x_2) = \sum_{j \in \{1,\ldots,i+1\}: x_2 \in N_{G^j}[m^{j-1}]} \Delta^j(x_1)$ at all times. Hence, $\sigma^{i+1}(x_1, x_2) \leq c^{i+1}(x_1)$ for arbitrary x_1 and x_2 because the LHS only sums a subset of all terms. Note that if we effectively modify the subsidy of a vertex, then it has to be adjacent to v^i but not m^i and the second vertex cannot be its neighbor. (Because m^i is a maximal vertex of v^i, if x_1 and x_2 were neighbors, m^i and x_2 would neighbors as well.) This update strategy for the subsidies allows us to compensate relative changes to distances.

Since we constructed the updates of weights and costs in a way that corresponds to those used by Lemma 1 of Lee and Chang (1994), we already have that $M_{G^{i+1}, w^{i+1}, c^{i+1}} = M_{G^i, w^i, c^i}$. First, we start with a technical result that is used to show that an iteration of the procedure does not remove a winning strategy from the graph by verifying that $\exists x \in V^i : \neg Q^i(v^i, x)$. Next we prove that inequalities (2) are invariant throughout, and finally we prove that winning strategies do not change.

Claim 1: If $x \in N_{G^\ell}(v^\ell) \setminus \{m^\ell\}$, then it cannot happen that at a later iteration $j > \ell$ we set $w^{j+1}(x) = w^j(x) + w^j(m^\ell)$. In other words, if $v^j = m^\ell$ for some $j > \ell$, then $m^j \notin N_{G^\ell}(v^\ell)$.

To see this, consider that for the operation $w^{j+1}(x) = w^j(x) + w^j(m^\ell)$ to be performed in some step $j \geq \ell$, we need m^ℓ to be simple and x to be the maximal neighbor of m^ℓ in G^j. Since m^ℓ was the maximal neighbor of v^ℓ in G^ℓ and $x \in N_{G^\ell}[v^\ell]$, it follows that $N_{G^\ell}[x] \subseteq N_{G^\ell}[m^\ell]$ and thus $N_{G^j}[x] \subseteq N_{G^j}[m^\ell]$. Furthermore, $N_{G^j}[m^\ell]$ must be a clique, and $N_{G^j}[y] \subseteq N_{G^j}[x]$ for $y \in N_{G^j}[m^\ell]$ by maximality of x. Putting the two inclusions together, $N_{G^j}[y] \subseteq N_{G^j}[m^\ell]$, which implies that G^j is complete. This is a contradiction because the iterations would have stopped before the update in step j.

Claim 2: $v^i \notin WS^i$. We have to prove that there is an $x \in V^i$ such that

$$\alpha(x) := \sum_{z \in C_{G^i}[x, v^i]} w^i(z) + \sigma^i(x, v^i) - \sum_{z \in C_{G^i}[v^i, x]} w^i(z) - \sigma^i(v^i, x) > 0.$$

Using the topology of the network, that $\sigma^i(v^i, m^i) \leq c^i(v^i)$, and that $w^i(v^i) + c^i(v^i) \leq w^i(u^i) + c^i(u^i)$,

$$\begin{aligned} \alpha(m^i) &\geq \sum_{z \in V^i \setminus N_{G^i}[v^i]} w^i(z) + w^i(m^i) + \sigma^i(m^i, v^i) - w^i(v^i) - c^i(v^i) \\ &\geq \sum_{z \in V^i \setminus N_{G^i}[v^i]} w^i(z) + w^i(m^i) - w^i(u^i) - c^i(u^i). \end{aligned}$$

When $c^i(u^i) = 0$, $\alpha(m^i) \geq w^i(m^i) > 0$. When $c^i(u^i) > 0$ but $S^i(u^i) \cap N_{G^i}(v^i) = \emptyset$, then $\alpha(m^i) \geq \sum_{j=1}^{n^i(u^i)} w^i(S^i_j(u^i)) - c^i(u^i) + w^i(m^i)$. By (C2), $\sum_{j=1}^{n^i(u^i)} w^i(S^i_j(u^i)) - c^i(u^i) > 0$, hence $\alpha(m^i) > 0$. For the last case, when $S^i(u^i) \cap N_{G^i}(v^i) \neq \emptyset$, we use Claim 1. We choose the smallest index k such that $S^i_k(u^i) \in N_{G^i}(v^i)$ and refer to that vertex by r. As before, $\sigma^i(v^i, r) \leq c^i(v^i)$ and $\alpha(r) \geq \sum_{z \in V^i \setminus N_{G^i}[v^i] \setminus \{u^i\}} w^i(z) + w^i(r) + \sigma^i(r, v^i) - c^i(u^i)$. To prove that the RHS is positive, we see that if iteration $\ell \leq i$ adds $\Delta^{\ell+1}(u^i)$ to $c^{\ell+1}(u^i)$, then that value is part of another term that is summed. When $\Delta^{\ell+1}(u^i) = 0$, there is nothing to prove; then, assume that iteration ℓ selects vertex v^ℓ and that $u^i \in N_{G^\ell}(v^\ell) \setminus \{m^\ell\}$. If $r = m^\ell$, iteration ℓ sets $w^{\ell+1}(r) = w^\ell(r) + w^\ell(v^\ell)$ so the term $w^i(r)$ compensates $\Delta^{\ell+1}(u^i)$. Otherwise, we see that $\Delta^{\ell+1}(u^i)$ is part of the subsidy $\sigma^i(r, v^i)$ by verifying that $\Delta^{\ell+1}(r) = \Delta^{\ell+1}(u^i)$ and $v^i \in N_{G^{\ell+1}}(m^\ell)$. The former holds because (C1) implies that $N[u^i] \subseteq N[r]$, both in the original graph and in iteration ℓ. Hence, $v^\ell \in N_{G^\ell}[r]$. The latter holds because r is both a neighbor of v^ℓ and v^i; thus, the maximality of m^ℓ implies that m^ℓ and v^i are neighbors too.

Now that we know that winning strategies are not removed after an update, we need to show that the condition that defines them is invariant.

Claim 3: $Q^i(y,x) = Q^{i+1}(y,x)\ \forall y, x \in V^{i+1}$. To conclude this, we show that

$$(3)\quad \sigma^i(y,x) + \sum_{z \in C_{G^i}[y,x]} w^i(z) - \sigma^i(x,y) - \sum_{z \in C_{G^i}[x,y]} w^i(z) = \sigma^{i+1}(y,x) + \sum_{z \in C_{G^{i+1}}[y,x]} w^{i+1}(z) - \sigma^{i+1}(x,y) - \sum_{z \in C_{G^{i+1}}[x,y]} w^{i+1}(z)$$

for every pair of vertices $x, y \in V^{i+1}$. Note that all terms of the subsidies cancel at both sides of the inequality, except the last term of the subsidies after the update. The possible cases are:

- Case $y = m^i$ and $x \in N_{G^i}(v^i)$: The update keeps the value constant because $w^{i+1}(m^i)$ and $\sigma^{i+1}(x, m^i)$ increase by $w^i(v^i)$.
- Case $y = m^i$ and $x \notin N_{G^i}(v^i)$: The update removes v^i from $C_{G^i}[y,x]$ but its weight goes to y which belongs to $C_{G^{i+1}}[y,x]$, keeping all the terms with w's the same. The new terms of the subsidy after the update are zero because of the choice of x and y.
- Case $\{y,x\} \subseteq N_{G^i}(v^i) \setminus \{m^i\}$: After the update both subsidies increase by $w^i(v^i)$ because both vertices are neighbors of m^i.
- Case $y \in N_{G^i}(v^i) \setminus \{m^i\}$, $x \notin N_{G^i}(v^i)$: We first consider that $d_{G^i}(v^i, x) = 2$. By the maximality of m^i, $x \in N_{G^i}(m^i)$. After the update, the terms with w decrease by $w^i(v^i)$ because the weight of v^i goes to m^i which is equidistant between x and y. At the same time, the subsidy $\sigma^{i+1}(y,x)$ increases by $w^i(v^i)$ and compensates. Instead, when $d_{G^i}(v^i, x) > 2$, all terms are equal after the update.
- Case $x, y \notin N_{G^i}(v^i)$: By maximality of m^i, both shortest paths from x and y to v^i go through m^i. Hence, all terms remain equal after the update.

With these intermediate steps, we are ready to prove that winning strategies do not change.

Claim 4: $WS^i = WS^{i+1}$. So far we proved that $Q^i(y,x) = Q^{i+1}(y,x)$ for $y, x \in V^{i+1}$ and that $v^i \notin WS^i$. First, we show that $WS^i \subseteq WS^{i+1}$. Consider $y \in WS^i$. Since v^i cannot be a winning strategy, $y \in V^{i+1}$. Hence, $Q^{i+1}(y,x)$ for all $x \in V^{i+1} \subset V^i$, proving that $y \in WS^{i+1}$. To prove $WS^{i+1} \subseteq WS^i$, we establish that $Q^{i+1}(y,x)\ \forall x \in V^{i+1}$ implies $Q^i(y, v^i)$, which completes the result. We consider different cases for y separately. In all cases we assume that $Q^i(y,r)$ holds for a specific vertex $r \in V^{i+1}$, meaning

$$\sigma^i(y,r) + \sum_{z \in C_{G^i}[y,r]} w^i(z) \geq \sigma^i(r,y) + \sum_{z \in C_{G^i}[r,y]} w^i(z),$$

and use it to prove $Q^i(y, v^i)$, or equivalently

$$\sigma^i(y,v^i) + \sum_{z \in C_{G^i}[y,v^i]} w^i(z) \geq \sigma^i(v^i,y) + \sum_{z \in C_{G^i}[v^i,y]} w^i(z).$$

- Case $y \notin N_{G^{i+1}}(m^i)$: Setting $r = m^i$, we have $\sigma^i(y,v^i) + \sum_{z \in C_{G^i}[y,v^i]} w^i(z) = \sum_{z \in C_{G^i}[y,v^i]} w^i(z) \geq \sigma^{i+1}(y,m^i) + \sum_{z \in C_{G^{i+1}}[y,m^i]} w^{i+1}(z)$. The first equality is true because the subsidies are zero since to receive a subsidy in step j, (y, v^{j-1}) and (v^i, m^{j-1}), respectively, should have been neighbors. Hence, (y, m^{j-1}) are neighbors by maximality. If m^{j-1} is still in the graph y is at distance two of

v^i, and if m^{j-1} is not in the graph any longer, that would imply that v^i and y are neighbors. Both possibilities contradict this case. To see that the inequality is true, we prove that $\sum_{z\in C_{G^i}[y,v^i]} w^i(z) - \sum_{z\in C_{G^{i+1}}[y,m^i]} w^{i+1}(z) = \sum_{z\in C_{G^i}[y,v^i]\setminus C_{G^{i+1}}[y,m^i]} w^i(z) \geq \sigma^{i+1}(y,m^i)$. For the inequality to be true, every time we add to the subsidy $\sigma^{i+1}(y,m^i)$, we must show that we also add to one of the weights in $C_{G^i}[y,v^i] \setminus C_{G^{i+1}}[y,m^i]$. If $d_{G^i}(y,v^i) > 3$, then we never add to the subsidy because $d^i_G(y,m^i) > 2$. Otherwise, $d_{G^i}(y,v^i) = 3$. If we add to that subsidy in step j, then G^{j-1} contains the triangle $\{y, v^{j-1}, m^{j-1}\}$ and (m^i, m^{j-1}) are neighbors. Notice that $m^j \in G^i$ because y and m^i are not neighbors in it. Hence, $m^{j-1} \in C_{G^i}[y,v^i] \setminus C_{G^{i+1}}[y,m^i]$ and the weight $w^{j-1}(v^{j-1}) = \Delta^j(y)$ that was added to the subsidy was also added to $w^j(m^{j-1})$. Continuing with the initial inequality chain using that $Q(y,m^i)$ holds, $\sigma^{i+1}(y,m^i) + \sum_{z\in C_{G^{i+1}}[y,m^i]} w^{i+1}(z) \geq \sigma^{i+1}(m^i,y) + \sum_{z\in C_{G^{i+1}}[m^i,y]} w^{i+1}(z) \geq \sum_{z\in C_{G^{i+1}}[m^i,y]} w^{i+1}(z) = \sigma^i(v^i,y) + \sum_{z\in C_{G^i}[v^i,y]} w^i(z)$. In the last equality, we have used the update rule for the weight and that the subsidy $\sigma^i(v^i,y) = 0$ for the same reason as before.

- Case $y = m^i$: To get to a contradiction, we assume that $\neg Q^i(m^i,v^i)$, which means that $\sigma^i(m^i,v^i) + \sum_{z\in C_{G^i}[m^i,v^i]} w^i(z) < c^i(v^i) + w^i(v^i)$. This is because $c^i(v^i) = \sigma^i(v^i,m^i)$. First, let us consider the case of $S^i(u^i) \cap N_{G^i}(v^i) \subseteq \{m^i\}$. Hence, the LHS of the previous inequality is greater than $\sum_{j=0}^{n^i(u^i)} w^i(S^i_j(u^i))$, which is greater than $w^i(u^i) + c^i(u^i)$ by (C2). That is a contradiction to the choice of v^i and u^i. Next, we consider that $S^i(u^i) \cap N_{G^i}(v^i)$ contains an element that is not m^i. We refer to the element $S^i_k(u^i)$ in that set with minimum index k by r. If $m^i \in S^i(u^i)$, it has to be the last element because it was chosen to be a maximal element of the poset. Considering our assumption earlier, we are going to prove that $\neg Q^i(m^i,r)$ which would be a contradiction since $m^i \in WS^{i+1}$. We start with $\sigma^i(m^i,r) + \sum_{z\in C_{G^i}[m^i,r]} w^i(z) = \sigma^i(m^i,v^i) + \sum_{z\in C_{G^i}[m^i,r]} w^i(z)$ where both subsidies coincide because both r and v^i are adjacent to m^i and to m^{j-1} if iteration j added something to those subsidies. This is, in turn, less than or equal to $\sigma^i(m^i,v^i) + \sum_{z\in C_{G^i}[m^i,v^i]} w^i(z) - \sum_{j=0}^{k-1} w^i(S^i_j(u^i))$ because v^i is further away from the rest of the graph than r and the elements in $S^i(u^i)$ were not summed before so we can remove them now. By $\neg Q^i(m^i,v^i)$, this is strictly less than $w^i(v^i) + c^i(v^i) - \sum_{j=0}^{k-1} w^i(S^i_j(u^i))$, which by the choice of u^i and v^i is less than $w^i(u^i) + c^i(u^i) - \sum_{j=0}^{k-1} w^i(S^i_j(u^i))$. Now, (C2) gives us the bound of $c^i(r) + w^i(r) = \sigma^i(r,m^i) + \sum_{z\in C_{G^i}[r,m^i]} w^i(z)$, proving the desired inequality.
- Case $y \in N_{G^{i+1}}(v^i)\setminus\{m^i\}$: To get to a contradiction, we assume that $\neg Q^i(y,v^i)$. First, let us consider the case of $S^i(u^i)\cap N_{G^i}(v^i) = \emptyset$. We set $r = m^i$ and prove $\neg Q^i(y,m^i)$, which would be a contradiction since $y \in WS^{i+1}$. We start with $\sigma^i(y,m^i) + \sum_{z\in C_{G^i}[y,m^i]} w^i(z) = \sigma^i(y,v^i) + \sum_{z\in C_{G^i}[y,v^i]} w^i(z) - \sum_{z\in X} w^i(z)$, where $X = \{z \in V^i : d_{G^i}(z,y) = d_{G^i}(z,m^i)\} \setminus N_{G^i}(v^i)$. Indeed, both subsidies are equal because both m^i and v^i are adjacent to y and to m^{j-1} if iteration j added to those subsidies. At both sides of the equality we sum the weights of the same vertices. By $\neg Q^i(y,v^i)$, this is strictly less than $\sigma^i(v^i,y) + \sum_{z\in C_{G^i}[v^i,y]} w^i(z) - \sum_{z\in X} w^i(z) = c^i(v^i) + w^i(v^i) - \sum_{z\in X} w^i(z) \leq c^i(u^i) + w^i(u^i) - \sum_{z\in X} w^i(z)$. The equality follows from the definition of the subsidies and the inequality from the choice of u^i and v^i. Applying (C2), we get $w^i(u^i) + \sum_{j=1}^{n^i(u^i)} w^i(S^i_j(u^i)) - \sum_{z\in X} w^i(z)$. To prove the bound of $\sigma^i(m^i,y) + \sum_{z\in C_{G^i}[m^i,y]} w^i(z)$, note that every weight summed in the previous expression is also summed in this one. For $j \in \{0,\ldots,n^i(u^i)\}$, if $S^i_j(u^i) \in X$, it will not be summed in either expression; otherwise, it will be summed in both.

  Next, we consider that $S^i(u^i) \cap N_{G^i}(v^i)$ is nonempty. We refer to the element $S^i_k(u^i)$ in that set with minimum index k by r. We assume that $r \neq y$ (this includes that the element may be m^i). Considering our assumption earlier, we are going to prove that $\neg Q^i(y,r)$ which would be a contradiction since $y \in WS^{i+1}$. We start with $\sigma^i(y,r) + \sum_{z\in C_{G^i}[y,r]} w^i(z) = \sigma^i(y,v^i) + \sum_{z\in C_{G^i}[y,v^i]} w^i(z) - \sum_{z\in X} w^i(z)$, where $X = \{z \in V^i : d_{G^i}(z,y) = d_{G^i}(z,r) = d_{G^i}(z,m^i)\} \setminus$

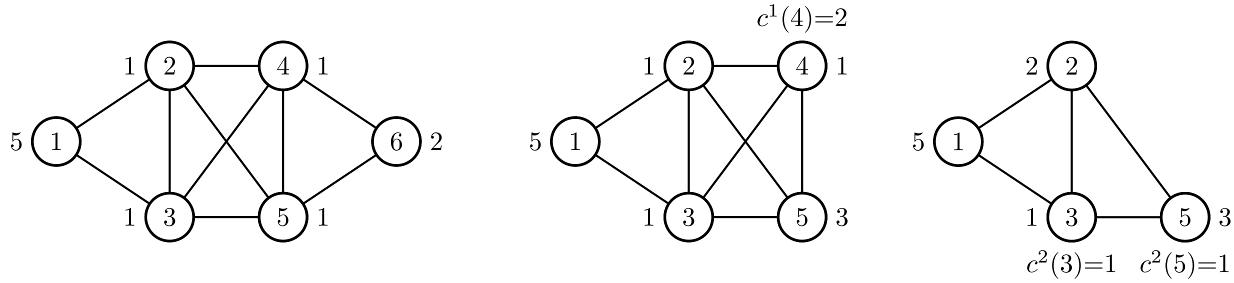
$N_{G^i}(v^i)$. Indeed, both subsidies are equal because both r and v^i are adjacent to y and to m^{j-1} if iteration j added to those subsidies. At both sides of the equality we sum the weights of the same vertices. By $\neg Q^i(y, v^i)$, this is strictly less than $\sigma^i(v^i, y) + \sum_{z \in C_{G^i}[v^i,y]} w^i(z) - \sum_{z \in X} w^i(z) = c^i(v^i) + w^i(v^i) - \sum_{z \in X} w^i(z) \leq c^i(u^i) + w^i(u^i) - \sum_{z \in X} w^i(z)$. The equality follows from the definition of the subsidies and the inequality from the choice of u^i and v^i. Applying (C2), we get a bound of $w^i(u^i) + \sum_{j=1}^{k} w^i(S^i_j(u^i)) + c^i(r) - \sum_{z \in X} w^i(z)$. To prove the bound of $\sigma^i(r, y) + \sum_{z \in C_{G^i}[r,y]} w^i(z)$, note that $c^i(r) = \sigma^i(r, y)$ by the definition of subsidies and that every weight summed in the previous expression is also summed in this one. Knowing that u^i is a neighbor of r and m^i, if it is a neighbor of y too, then it is added and subtracted in the first formula and does not appear in the second. If u^i is not a neighbor of y, it is summed in both. For each $S^i_j(u^i)$ we proceed in a similar way: because it is a neighbor of u^i and u^i is simplicial, it is a neighbor of r and m^i. Then, the same argument works, completing the case.

Finally, we consider that $r = y$ and arrive to a contradiction assuming that $\neg Q(y, v^i)$. Indeed, $\sigma^i(y, v^i) + \sum_{z \in C_{G^i}[y,v^i]} w^i(z) < c^i(v^i) + w^i(v^i) \leq c^i(u^i) + w^i(u^i) \leq w^i(u^i) + \sum_{j=1}^{k} w^i(S^i_j(u^i)) + c^i(y) \leq c^i(u^i) + w^i(u^i) \leq \sum_{z \in C_{G^i}[y,v^i]} w^i(z) + c^i(y)$, where we have used the assumption, the choice of u^i and v^i, and (C2), respectively. This is a contradiction because $\sigma^i(y, v^i) = c^i(y)$ and the chain of inequalities is strict.

- Case $y \in N_{G^{i+1}}(m^i) \setminus N_{G^{i+1}}(v^i)$: First, we consider that $\sigma^i(v^i, y) = 0$. Letting $X := (N_{G^i}(m^i) \cap N_{G^i}(y)) \setminus N_{G^i}(v^i)$, we bound $\sigma^i(y, v^i) + \sum_{z \in C_{G^i}[y,v^i]} w^i(z) \geq \sigma^i(y, v^i) + \sum_{z \in X} w^i(z) + \sum_{z \in C_{G^i}[y,m^i]} w^i(z)$. This holds because $X \cap C_{G^i}[y, m^i] = \emptyset$ and $X \cup C_{G^i}[y, m^i] \subseteq C_{G^i}[y, v^i]$. Notice that $\sigma^i(y, v^i) + \sum_{z \in X} w^i(z) \geq \sigma^i(y, m^i)$ because every time a weight is added to $\sigma^i(y, m^i)$, it is either added to $\sigma^i(y, v^i)$ or to the weight of a neighbor of y (depending on whether that neighbor and v^i are adjacent or not). If that neighbor belongs to V^i then it also must belong to X. Otherwise, at a later iteration the weight of the neighbor must have been added to $\sigma^i(y, v^i)$ or to the weight of another neighbor of y, and so on. Putting the two inequalities together, we have the lower bound $\sigma^i(y, m^i) + \sum_{z \in C_{G^i}[y,m^i]} w^i(z)$, which is bigger than or equal to $\sigma^i(m^i, y) + \sum_{z \in C_{G^i}[m^i,y]} w^i(z)$ by $Q^i(y, m^i)$. The last is bounded by $\sigma^i(v^i, y) + \sum_{z \in C_{G^i}[v^i,y]} w^i(z)$ because $\sigma^i(v^i, y) = 0$, and v^i is further away from y than m^i, proving $Q^i(y, v^i)$.

  If $\sigma^i(v^i, y) > 0$, we refer to the element $S^i_k(v^i) \in N_{G^i}(y)$ with minimum index k by r. This element must exist because some iteration added a subsidy to $\sigma^i(v^i, y)$ and at that stage a neighbor of y was added to $S^i(v^i)$. That vertex could not have subsequently been removed from the graph because v^i and y are not adjacent. As in the previous case, we let $X := (N_{G^i}(r) \cap N_{G^i}(y)) \setminus N_{G^i}(v^i)$, and bound $\sigma^i(y, v^i) + \sum_{z \in C_{G^i}[y,v^i]} w^i(z) \geq \sigma^i(y, v^i) + \sum_{z \in X} w^i(z) + \sum_{z \in C_{G^i}[y,r]} w^i(z)$. This holds because $X \cap C_{G^i}[y, r] = \emptyset$ and $X \cup C_{G^i}[y, r] \subseteq C_{G^i}[y, v^i]$. Reasoning as in the previous case, we have the lower bound $\sigma^i(y, r) + \sum_{z \in C_{G^i}[y,r]} w^i(z)$, which is bigger than or equal to $\sigma^i(r, y) + \sum_{z \in C_{G^i}[r,y]} w^i(z)$, by $Q^i(y, r)$. The last is, in turn, bounded by $\sigma^i(v^i, y) + \sum_{z \in C_{G^i}[v^i,y]} w^i(z)$ because $C_{G^i}[v^i, y] = N_{G^i}[v^i] \setminus N_{G^i}(y) \subseteq C_{G^i}[r, y] \setminus \{r\}$, and, to compare the subsidies, note that if iteration j adds $\Delta^j(v^i)$ to $\sigma^j(v^i, y)$, v^i must have been adjacent to v^j. In that case, r must also be adjacent to v^j since $N_G(v^i) \subseteq N_G(r)$ by (C1). Then the iteration must have added $\Delta^j(v^i) = w^j(v^j)$ to $\sigma^j(r, y)$ if $r \neq m^j$ or must have added that quantity to $w^j(r)$ if $r = m^j$. This proves that $\sigma^i(v^i, y) \leq \sigma^i(r, y) + w^i(r)$, completing the case.

Last step: When G^i is a complete graph, $D_{G^i,w^i,c^i}(y) = \sum_{v \in V^i} w^i(v) - w^i(y) - c^i(y)$, so M_{G^i,w^i,c^i} is the set of vertices that maximizes $w^i(y) + c^i(y)$. Also, $C_{G^i}[y, x] = \{y\}$. At this step, $c^i(y) = \sigma^i(y, x)$ for all $x, y \in V^i$. If not, there would exist an iteration j in which $\Delta^{j+1}(y) > 0$ and x was not a neighbor of m^j.



(A) Initial graph G^0 (B) Graph after one step G^1 (C) Graph after two steps G^2

FIGURE 11. Illustration of the proof. Labels are displayed inside vertices and weights outside.

By maximality of m^j, this would also imply that x was not a neighbor of y at iteration j. This would be a contradiction to the completeness of G^i because the induction never adds edges to the graph. Hence, y is a winning strategy when $w^i(y) + c^i(y) \geq w^i(x) + c^i(x)$ for all $x \in V$. This implies that $M_{G^i,w^i,c^i} = WS^i$, from where we conclude that we can compute the winning strategies of the original graph G^0 by examining all vertices in V^i. □

To conclude, Figure 11 provides an illustration of the proof. We display labels inside vertices and weights w outside. The set of generalized winning strategies (and generalized medians) in all iterations is $\{1, 2, 3\}$. At the beginning, costs are zero. Since vertices 1 and 6 are simple and $w(6) = 2 < w(1) = 5$, in the first step, we set $u^0 = 1$ and $v^0 = 6$, and remove v^0, arriving to graph G^1. We choose $m^0 = 5$ so $w^1(5) = 3$. To update costs, we compute $\Delta^1(4) = 2$ and 0 otherwise, which gives $c^1 = \Delta^1$, in this case. The subsidies $\sigma^1(x, y) = 2$ for $x = 4$ and $y \in \{2, \ldots, 5\}$ and 0 otherwise. This reflects that when a player selects vertex 4 and the other selects a vertex in $\{2, \ldots, 5\}$, the computation of market share is misleading compared to the original graph because of the removal of vertex 6. In those cases, we assign an extra demand of 2 to the player that chose vertex 4 to compensate the additional weight in vertex 5. For example, if $x_1 = 3$ and $x_2 = 4$, then evaluating (2) for G^0 and for G^1 gives $6 > 3$ in both cases.

At the next iteration, $u^1 = 1$ and $v^1 = 4$ because $S^1(4) = \{4, 5\}$, which makes the simple vertex 5 not minimal, as required. A maximal neighbor of v^1 is $m^1 = 2$. We transform the graph and update the weights and costs to get G^2. In this case $\sigma^2(x, y) = 1$ if $x \in \{3, 5\}$ and 0 otherwise. In the next step we select $u^2 = 1$, $v^2 = 5$ and $m^2 = 2$ (because $S^2(3) = \{3, 2\}$). We remove v^2 to get G^3, which is a complete graph with weights 5, 5 and 1, respectively. Also, $\Delta^3(3) = 3$ and $c^3(3) = 4$ and both are zero for all other vertices. In this case $\sigma^3(x, y) = 4$ if $x = 3$ and 0 otherwise. We finish, declaring that $\{1, 2, 3\}$ are winning strategies because $w(y) + c(y)$ is constant across vertices.

Corollary E.1. *The set of winning strategies of a connected strongly chordal graph is a clique.*

Proof. Follows from Theorem 1 in Lee and Chang (1994). □