ERRATA TO "MEAN FIELD GAMES WITH COMMON NOISE"

RENÉ CARMONA, FRANÇOIS DELARUE, AND DANIEL LACKER

Abstract. This note corrects Lemma 3.7 in our paper [1]. The main results of the paper remain correct as stated.

This note corrects an error in [1, Lemma 3.7]. The lemma is not correct as stated, and the first conclusion must instead be stated as a hypothesis. This erratum corrects the statement of the lemma and then shows that the additional hypothesis is satisfied in each of the three applications of the lemma later in the paper. The main results of the paper remain unchanged.

The error in [1, Lemma 3.7] is at the end of the "first step" of the proof. Specifically, the last equation before the "second step" (lines 5-6 of page 3769) is not accurate, because the preceding equation was proven only for all $\mathcal{F}^{\mu}_t$-measurable functions $\phi_1(\mu)$, not for all $\mathcal{F}^{\mu}_T$-measurable functions. We rewrite the lemma as follows, stating equivalence between its two claims as well as a third and often more convenient form:

Lemma 3.7*. *Let $P \in \mathcal{P}^p(\Omega)$ such that (B, W) is a Wiener process with respect to the filtration $(\mathcal{F}^{\xi,B,W,\mu,\Lambda,X}_t)_{t\in[0,T]}$ under P, and define $\rho := P \circ (\xi, B, W, \mu)^{-1}$. Suppose that (1) and (3) of Definition 3.4 are satisfied, that $P(X_0 = \xi) = 1$, and that the state equation (3.3) holds under P. The following are equivalent:*

(A) For $P \circ \mu^{-1}$-almost every $\nu \in \mathcal{P}^p(\mathcal{X})$, it holds that $(W_t)_{t\in[0,T]}$ is an $(\mathcal{F}^{W,\Lambda,X}_t)_{t\in[0,T]}$-Wiener process under ν.
(B) Under P, $\mathcal{F}^{B,\mu}_T \vee \mathcal{F}^{\xi,W,\Lambda}_t$ is independent of $\sigma\{W_s - W_t : s \in [t, T]\}$ for every $t \in [0, T)$.
(C) P is an MFG pre-solution

Proof.
($A \Rightarrow C$): Let $Q = P \circ (\xi, B, W, \mu, \Lambda)^{-1}$. Assuming (A) holds, the second and third steps of the original proof [1, Lemma 3.7] are correct and show that $Q \in \mathcal{A}(\rho)$. As all of the other defining properties of an MFG pre-solution hold by assumption, we deduce (C).

($C \Rightarrow B$): Note that (C) entails that $\mathcal{F}^{\Lambda}_t$ is conditionally independent of $\mathcal{F}^{\xi,B,W,\mu}_T$ given $\mathcal{F}^{\xi,B,W,\mu}_t$ under P, for every $t \in [0, T)$. Fix $t \in [0, T)$, and fix bounded functions ϕ_t, ψ_T, ψ_t, and h_{t+} such that $\phi_t(\Lambda)$ is $\mathcal{F}^{\Lambda}_t$-measurable, $\psi_T(B, \mu)$ is $\mathcal{F}^{B,\mu}_T$-measurable, $\psi_t(\xi, W)$ is $\mathcal{F}^{\xi,W}_t$-measurable, and $h^+_t(W)$ is $\sigma\{W_s - W_t : s \in [t, T]\}$-measurable. The conditional independence yields

$$\mathbb{E}\left[\phi_t(\Lambda)|\,\mathcal{F}^{\xi,B,W,\mu}_T\right] = \mathbb{E}\left[\phi_t(\Lambda)|\,\mathcal{F}^{\xi,B,W,\mu}_t\right],\ a.s.$$

The independence of ξ, (B, μ), and W easily implies that $\mathcal{F}^{B,\mu}_T \vee \mathcal{F}^{\xi,W}_t$ is independent of $\sigma\{W_s - W_t : s \in [t, T]\}$, and we deduce

$$\begin{aligned}
&\mathbb{E}\left[\phi_t(\Lambda)\psi_T(B,\mu)\psi_t(\xi,W)h_{t+}(W)\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\phi_t(\Lambda)|\,\mathcal{F}^{\xi,B,W,\mu}_t\right]\psi_T(B,\mu)\psi_t(\xi,W)h_{t+}(W)\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\phi_t(\Lambda)|\,\mathcal{F}^{\xi,B,W,\mu}_t\right]\psi_T(B,\mu)\psi_t(\xi,W)\right]\mathbb{E}\left[h_{t+}(W)\right]\\
&= \mathbb{E}\left[\phi_t(\Lambda)\psi_T(B,\mu)\psi_t(\xi,W)\right]\mathbb{E}\left[h_{t+}(W)\right].
\end{aligned}$$

$(B \Rightarrow A)$: Because the state equation (3.3) holds, and because strong uniqueness holds for this SDE, we have $\mathcal{F}^X_t \subset \mathcal{F}^{\xi,B,W,\mu,\Lambda}_t$ a.s., which implies $\mathcal{F}^{B,\mu}_T \vee \mathcal{F}^{W,\Lambda,X}_t = \mathcal{F}^{B,\mu}_T \vee \mathcal{F}^{\xi,W,\Lambda}_t$ a.s. under P, for each t. Hence, (B) implies $\mathcal{F}^{B,\mu}_T \vee \mathcal{F}^{W,\Lambda,X}_t$ is independent of $\sigma\{W_s - W_t : s \in [t,T]\}$ under P, for each t. Now fix $t \in [0,T)$, and fix bounded functions ψ_T, ϕ_t, and h_{t+}, such that $\psi_T(\mu)$ is $\mathcal{F}^\mu_T$-measurable, $\phi_t(W,\Lambda,X)$ is $\mathcal{F}^{W,\Lambda,X}_t$-measurable, and $h_{t+}(W)$ is $\sigma\{W_s - W_t : s \in [t,T]\}$-measurable. The fixed point condition (3) of Definition 3.4 and independence imply

$$\begin{aligned}
&\mathbb{E}^P\left[\psi_T(\mu)\int_{\mathcal{X}} \phi_t(w,q,x)h_{t+}(w)\,\mu(dw,dq,dx)\right] \\
&\quad = \mathbb{E}^P[\psi_T(\mu)\phi_t(W,\Lambda,X)h_{t+}(W)] \\
&\quad = \mathbb{E}^P[\psi_T(\mu)\phi_t(W,\Lambda,X)]\mathbb{E}^P[h_{t+}(W)] \\
&\quad = \mathbb{E}^P\left[\psi_T(\mu)\int_{\mathcal{X}} \phi_t(w,q,x)\,\mu(dw,dq,dx)\right]\mathbb{E}^P[h_{t+}(W)].
\end{aligned}$$

As this holds for every $\mathcal{F}^\mu_T$-measurable ψ_T, we deduce

$$\int_{\mathcal{X}} \phi_t(w,q,x)h_{t+}(w)\,\mu(dw,dq,dx) = \mathbb{E}^P[h_{t+}(W)]\int_{\mathcal{X}} \phi_t(w,q,x)\,\mu(dw,dq,dx), \ \ a.s.$$

By working with a countable convergence-determining class, we conclude that $\mathcal{F}^{W,\Lambda,X}_t$ is independent of $\sigma\{W_s - W_t : s \in [t,T]\}$, under P-a.e. realization of μ. Finally, use conditions (1) and (3) of Definition 3.4 to deduce that

$$\mu \circ W^{-1} = P(W \in \cdot \mid B,\mu) = P \circ W^{-1}, \ \ a.s.$$

Thus $\nu \circ W^{-1}$ equals Wiener measure for $P \circ \mu^{-1}$-a.e. $\nu \in \mathcal{P}^p(\mathcal{X})$, and (A) follows. □

We next show how to apply Lemma 3.7* in each of the places the original [1, Lemma 3.7] was applied, in the proofs of Lemma 3.6, Lemma 3.16, and Theorem 4.1.

Addition to proof of Lemma 3.6. To check the condition (B) of Lemma 3.7*, it suffices to check that $\mathcal{F}^{B,\mu}_T \vee \mathcal{F}^{\xi,W,\Lambda}_t$ is independent of $\sigma\{W_s - W_t : s \in [t,T]\}$ for every $t \in [0,T)$, under $\overline{P}_n$, for each n. Indeed, this independence readily passes to the $n \to \infty$ limit. Under $\overline{P}_n$ we have $\mathcal{F}^\mu_t \subset \mathcal{F}^B_t$ a.s., for each t. Hence, it suffices to check that $\mathcal{F}^B_T \vee \mathcal{F}^{\xi,W,\Lambda}_t$ is independent of $\sigma\{W_s - W_t : s \in [t,T]\}$ for every $t \in [0,T)$, under $\overline{P}_n$. But this follows from the independence of $\mathcal{F}^{\xi,B,W,\Lambda}_t$, $\sigma\{W_s - W_t : s \in [t,T]\}$, and $\sigma\{B_s - B_t : s \in [t,T]\}$.

Addition to proof of Lemma 3.16. We again check that P satisfies the condition (B) of Lemma 3.7*. To see this, note that each P_{n_k} is an MFG pre-solution and thus satisfies (B), by Lemma 3.7*. Passing to the limit $P_{n_k} \to P$, we deduce that P also satisfies (B).

Addition to proof of Theorem 4.1. We will check that $\overline{P} = P' \circ (\xi, B, W, \bar{\mu}, \Lambda, X)^{-1}$ satisfies condition (B) of Lemma 3.7*. First, because $P' \in \mathcal{RA}(\rho)$, it is immediate that P' satisfies properties (1) and (2) of Definition 3.4. Using the implication $(C \Rightarrow B)$ of Lemma 3.7*, it holds under P' that $\mathcal{F}^{B,\mu}_T \vee \mathcal{F}^{\xi,W,\Lambda}_t$ is independent of $\sigma\{W_s - W_t : s \in [t,T]\}$, for each t. Because $\overline{\mu}$ is (B,μ)-measurable, this implies that $\mathcal{F}^{B,\bar{\mu}}_T \vee \mathcal{F}^{\xi,W,\Lambda}_t$ is independent of $\sigma\{W_s - W_t : s \in [t,T]\}$, for each t, under P'. Lemma 3.7* then implies that $\overline{P}$ is an MFG pre-solution.

References

1. R. Carmona, F. Delarue, and D. Lacker, *Mean field games with common noise*, The Annals of Probability **44** (2016), no. 6, 3740–3803.