Absolute Stability Regions for LMMs and RKMs

Adams-Bashforth Methods

These are k-step explicit methods of order k, for any integer $k \geqslant 1$. The k-step method has characteristic polynomial $\rho(z) = z^k - z^{k-1} = (z-1)z^{k-1}$, and hence is zero-stable and thus convergent for any $k \geqslant 1$.

Low order methods:

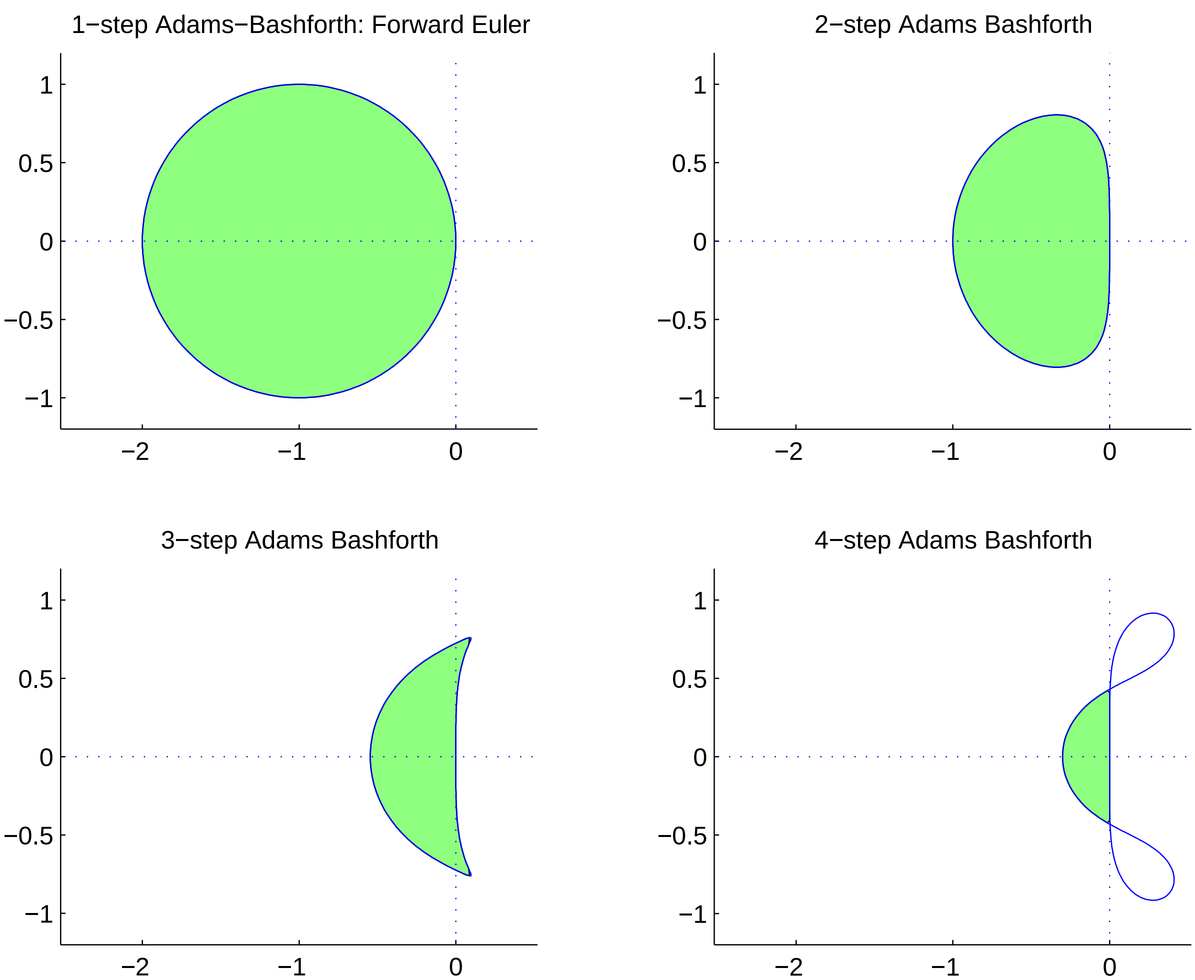
$$k = 1: \qquad y_{n+1} = y_n + hf_n \qquad \text{(Forward Euler)}$$

$$k = 2: \qquad y_{n+2} = y_{n+1} + h\left(\frac{3}{2}f_{n+1} - \frac{1}{2}f_n\right)$$

$$k = 3: \qquad y_{n+3} = y_{n+2} + h\left(\frac{23}{12}f_{n+2} - \frac{16}{12}f_{n+1} + \frac{5}{12}f_n\right)$$

$$k = 4: \qquad y_{n+4} = y_{n+3} + h\left(\frac{55}{24}f_{n+3} - \frac{59}{24}f_{n+2} + \frac{37}{24}f_{n+1} - \frac{9}{24}f_n\right)$$

The regions of absolute stability for the k-step methods for $k = 1, 2, 3, 4$ are as shown below.



The region of absolute stability $\mathcal{S}$ is given by the shaded region.

Note that $k = 1$ is the forward Euler method with root locus

$$z = \frac{\rho(\xi)}{\sigma(\xi)} = \frac{\xi - 1}{1} = e^{i\theta} - 1,$$

which defines the unit circle centred at -1.

For $k = 2, 3, 4$ we compute z for a range of values of $\theta \in [0, 2\pi]$ and plot the graphs numerically. The case $k = 4$ is interesting in that the boundary locus curve intersects itself and forms three loops. The region of absolute stability is given by the central loop. In the upper and lower loops we find that a second root ξ_i crosses the unit circle, so that in these regions the method has two roots with modulus greater than 1.

Note that these graphs were produced in matlab. The root locus curve can be found relatively easily. To determine the region of absolute stability $\mathcal{S}$ is a little more tricky; I did so by taking a grid of points z in the complex plane and using the $roots(\tau)$ command in matlab to compute the roots xi_i of tau then $norm(roots(\tau), inf)$ to find the root of largest modulus, then plotting a filled contour (contourf) shading the region on which all roots are less than one in modulus!

Adams-Moulton Methods

These are k-step implicit ($\beta_k \neq 0$) methods of order $k + 1$, for any integer $k \geqslant 1$. The k-step method has characteristic polynomial $\rho(z) = z^k - z^{k-1}$, and hence is zero-stable and thus convergent for any $k \geqslant 1$.

Low order methods:

$$k = 0: \qquad y_{n+1} = y_n + hf_{n+1} \qquad \text{(Backward Euler, see note below)}$$

$$k = 1: \qquad y_{n+1} = y_n + h\left(\frac{1}{2}f_n + \frac{1}{2}f_{n+1}\right) \qquad \text{(Trapezoidal Rule)}$$

$$k = 2: \qquad y_{n+2} = y_{n+1} + h\left(\frac{5}{12}f_{n+2} + \frac{8}{12}f_{n+1} - \frac{1}{12}f_n\right)$$

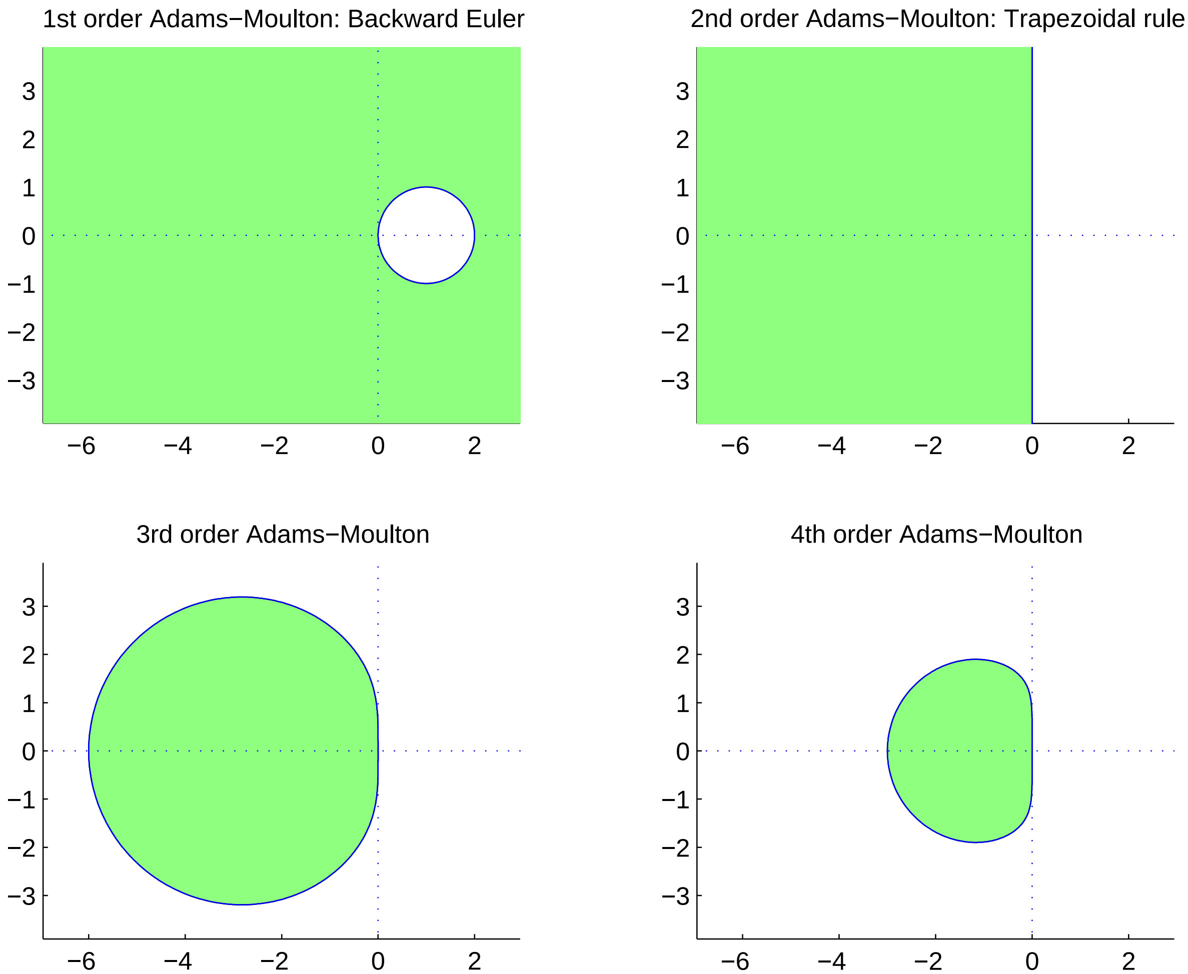
$$k = 3: \qquad y_{n+3} = y_{n+2} + h\left(\frac{9}{24}f_{n+3} + \frac{19}{24}f_{n+2} - \frac{5}{24}f_{n+1} + \frac{1}{24}f_n\right)$$

The regions of absolute stability for the p-th order methods for $p = 1, 2, 3, 4$ are as shown below. Note that both the order $p = 1$ and $p = 2$ methods are one-step methods whilst for $k \geqslant 2$ the k-step method has order $p = k + 1$.

The region of absolute stability $\mathcal{S}$ is the shaded region and hence for $p = 1$ and $p = 2$ it is unbounded.

Comparing with the graphs for the Adams-Bashforth methods (note that the scales are not the same) we see that the implicit methods have larger regions of absolute stability than the explicit methods. This is true in general for numerical methods.

We also note that for both classes of methods the regions of absolute stability become smaller as the order increases; this is true in general for LMMs (but not for Runge-Kutta methods).



Backward Differentiation Formulae (BDF) methods

The Adams methods are characterized by having the characteristic polynomial $\rho(z) = z^k - z^{k-1}$. We now consider a class of methods which is defined not through ρ but instead is defined by its characteristic polynomial $\sigma(z)$.

The BDF methods are implicit methods, with the k-step method being of order k and characterized by the characteristic polynomial

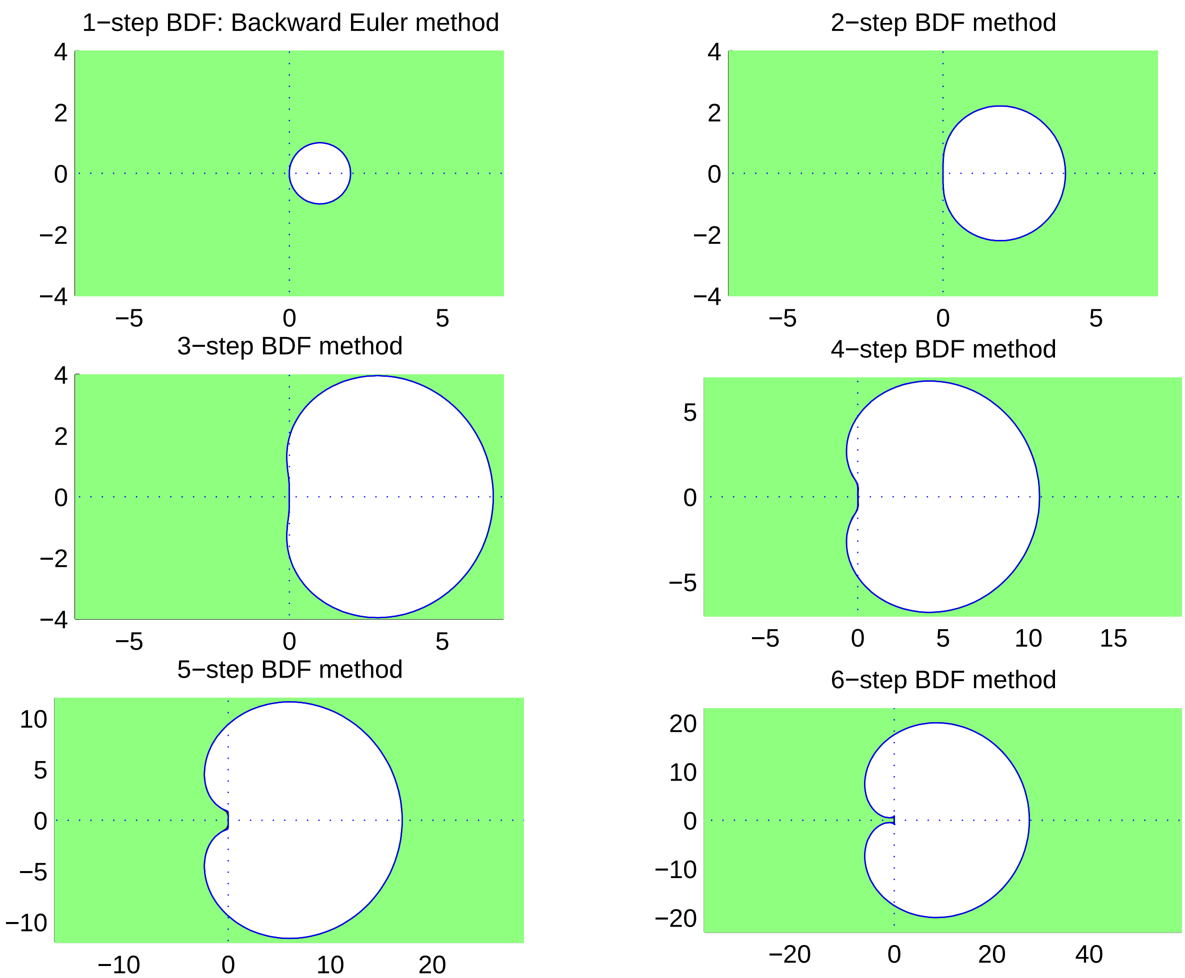
$$\sigma(z) = z^k.$$

Using $\sigma(z)$ as given, and the order conditions we can compute these methods for arbitrary k. We obtain

$$
\begin{aligned}
k = 1: &\qquad y_{n+1} = y_n + hf_{n+1} \qquad \text{(Backward Euler)} \\
k = 2: &\qquad \frac{3}{2}y_{n+2} - 2y_{n+1} + \frac{1}{2}y_n = hf_{n+2} \\
k = 3: &\qquad \frac{11}{6}y_{n+3} - 3y_{n+2} + \frac{3}{2}y_{n+1} - \frac{1}{3}y_n = hf_{n+3} \\
k = 4: &\qquad \frac{25}{12}y_{n+4} - 4y_{n+3} + 3y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{4}y_n = hf_{n+4},
\end{aligned}
$$

and so on. Unlike the Adams methods the BDF methods do not have zero-stability built in, and we need to compute the roots of the polynomials $\rho(z)$ appearing in the formulae above to verify whether these methods are convergent.

It is known that the methods are zero-stable and hence convergent if and only if $k \leqslant 6$. Thus the BDF methods are only convergent for $k = 1, 2, 3, 4, 5, 6$. Nevertheless these low order methods have large regions of absolute stability, as shown, and hence are excellent for solving problems with stable fixed points.



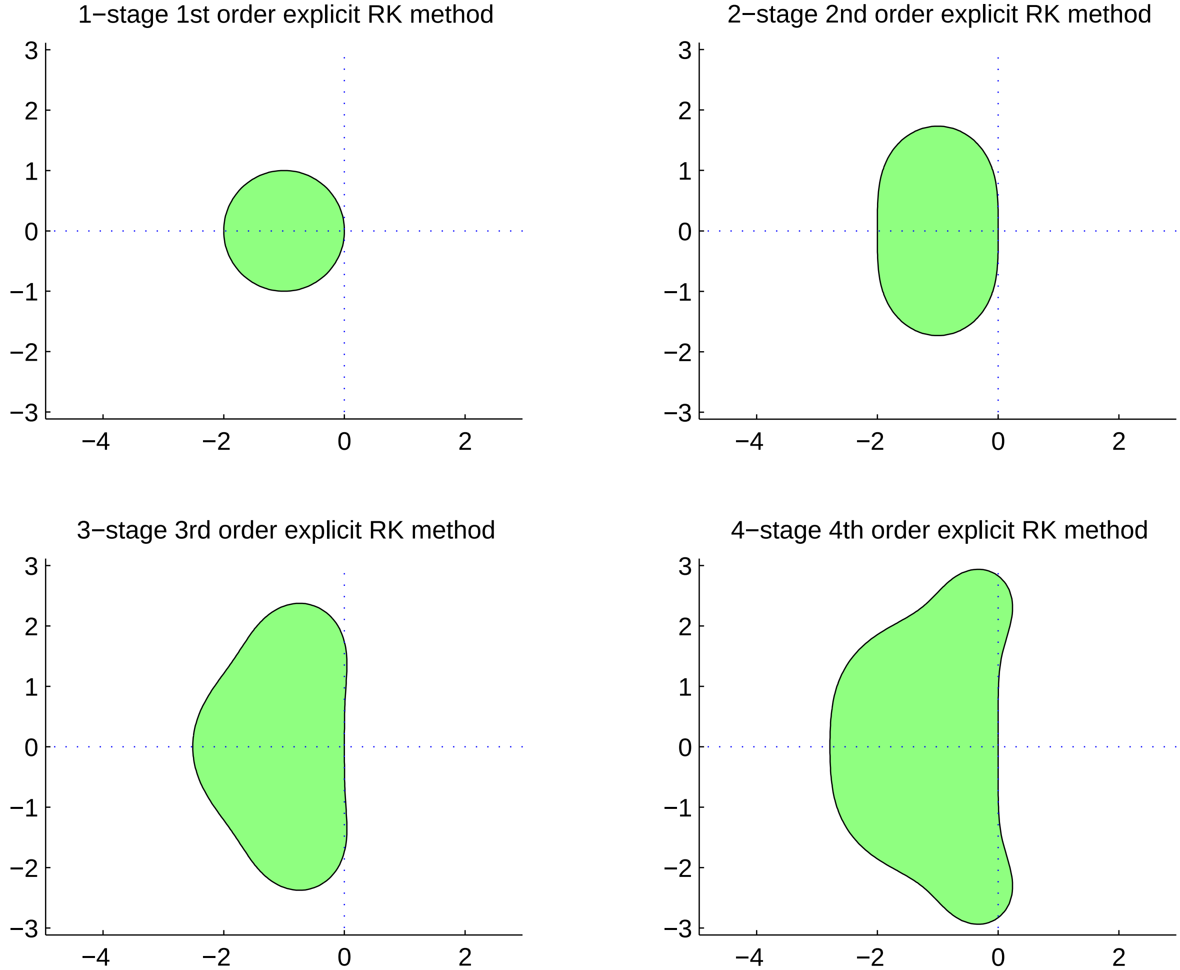
Note that in each case the region of absolute stability $\mathcal{S}$ is the shaded region and hence is unbounded. As k is increased the unshaded region not in $\mathcal{S}$ grows, until for $k = 7$ the point $z = 0$ is in the interior of this region and the method is not zero-stable and hence not convergent.

Runge-Kutta methods

The stability function of every s-stage order s explicit Runge-Kutta method is

$$R(z) = 1 + z + \frac{1}{2!}z^2 + \ldots + \frac{1}{s!}z^s.$$

The corresponding regions of absolute stability are shown below for $s = 1, 2, 3, 4$.



Note that the unique one-stage first order method is the forward Euler method, and that as s increases so does the size of the stability region. This is in direct contrast to Adams Bashforth methods for which $k = 1$ corresponded to the forward Euler method and the size of the stability region decreased as we increased k. Recall that for $s \geqslant 5$ there do *not* exist explicit s-stage Runge-Kutta methods of order s.