Supplement to "Commitment without Reputation: Renegotiation-Proof Contracts under Asymmetric Information"

(not for publication)

Emanuele Gerratana
SIPA, Columbia University

Levent Koçkesen
Koç University

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In what follows we provide the omitted proofs of the statements made in our paper "Commitment without Reputation: Renegotiation-Proof Contracts under Asymmetric Information." In order to distinguish statements made in that paper from the ones made in this document we will add a note "(of the main paper)" after those from the main paper.

It is well-known that if b_2 is increasing, then, under increasing differences, incentive compatibility reduces to local incentive compatibility. We state it as a claim for future reference.

Claim 1. *If u_2 has increasing differences in $(\succsim_\theta, \succsim_2)$ and $b_2 \in A_2^{A_1 \times \Theta}$ is increasing in $(\succsim_\theta, \succsim_2)$, then for any $f \in \mathscr{C}$*

$$u_2(a_1, b_2(a_1,\theta^i),\theta^i) - f(a_1, b_2(a_1,\theta^i)) \geq u_2(a_1, b_2(a_1,\theta^j),\theta^i) - f(a_1, b_2(a_1,\theta^j)), \text{ for all } i, j = 1,2,\ldots,n$$

holds if and only if

$$u_2(a_1, b_2(a_1,\theta^i),\theta^i) - f(a_1, b_2(a_1,\theta^i)) \geq u_2(a_1, b_2(a_1,\theta^{i-1}),\theta^i) - f(a_1, b_2(a_1,\theta^{i-1})), \text{ for all } i = 2,\ldots,n,$$

and

$$u_2(a_1, b_2(a_1,\theta^i),\theta^i) - f(a_1, b_2(a_1,\theta^i)) \geq u_2(a_1, b_2(a_1,\theta^{i+1}),\theta^i) - f(a_1, b_2(a_1,\theta^{i+1})), \text{ for all } i = 1,2,\ldots,n-1.$$

Proof of Proposition 2 (of the main paper). **(Only if)** Suppose that b_2 is incentive compatible, i.e., there exists a contract f such that (f, b_2) is incentive compatible. Fix orders $(\succsim_\theta, \succsim_2)$ in which u_2 has strictly increasing differences. Take any $a_1 \in A_1$ and $\theta, \theta' \in \Theta$ and assume without loss of generality, that $\theta \succ_\theta \theta'$. Suppose, for contradiction, that $b_2(a_1,\theta') \succ_2 b_2(a_1,\theta)$. Sequential rationality of player 2 implies

$$u_2(a_1, b_2(a_1,\theta),\theta) - f(a_1, b_2(a_1,\theta)) \geq u_2(a_1, b_2(a_1,\theta'),\theta) - f(a_1, b_2(a_1,\theta'))$$
$$u_2(a_1, b_2(a_1,\theta'),\theta') - f(a_1, b_2(a_1,\theta')) \geq u_2(a_1, b_2(a_1,\theta),\theta') - f(a_1, b_2(a_1,\theta))$$

and hence

$$u_2(a_1, b_2(a_1,\theta'),\theta) - u_2(a_1, b_2(a_1,\theta),\theta) \leq u_2(a_1, b_2(a_1,\theta'),\theta') - u_2(a_1, b_2(a_1,\theta),\theta'),$$

contradicting that u_2 has strictly increasing differences in $(\succsim_\theta, \succsim_2)$. Therefore, b_2 must be increasing in $(\succsim_\theta, \succsim_2)$.

[If] Suppose u_2 has strictly increasing differences and b_2 is increasing. We need to prove the existence of a contract $f \in \mathscr{C}$ such that

$$u_2(a_1, b_2(a_1, \theta^i), \theta^i) - f(a_1, b_2(a_1, \theta^i)) \geq u_2(a_1, b_2(a_1, \theta^j), \theta^i) - f(a_1, b_2(a_1, \theta^j)), \text{ for all } i, j = 1, 2, ..., n. \tag{1}$$

By Claim 1, (1) holds if and only if $Df(a_1, b_2) \leq \vec{U}_2(a_1, b_2)$. Therefore, we need to show that for any $a_1 \in A_1$ there exists $f(a_1, b_2) \in \mathbb{R}^n$ such that $Df(a_1, b_2) \leq \vec{U}_2(a_1, b_2^*)$. By Gale's theorem for linear inequalities (Mangasarian (1994), p. 33), there exists such an $f(a_1, b_2) \in \mathbb{R}^n$ if and only if for any $y \in \mathbb{R}_+^{2(n-1)}$, $D'y = 0$ implies $y'\vec{U}_2(a_1, b_2^*) \geq 0$. It is easy to show that $D'y = 0$ if and only if $y_1 = y_2, y_3 = y_4, \cdots, y_{2(n-1)-1} = y_{2(n-1)}$. Let $\vec{U}_2(a_1, b_2)_i$ denote the i^{th} row of $\vec{U}_2(a_1, b_2)$ and note that since b_2 is increasing and u_2 has strictly increasing differences, $\vec{U}_2(a_1, b_2)_{2i-1} + \vec{U}_2(a_1, b_2)_{2i} \geq 0$, for any $i = 1, 2, \ldots, n-1$. Therefore,

$$y'\vec{U}_2(a_1, b_2^*) = \sum_{i=1}^{n-1} (\vec{U}_2(a_1, b_2)_{2i-1} + \vec{U}_2(a_1, b_2)_{2i}) y_{2i-1} \geq 0$$

This proves the existence of a $f(a_1, b_2) \in \mathbb{R}^n$ such that (1) is satisfied for all $a_1 \in A_1$. We can complete the proof by defining $\tilde{f} \in \mathscr{C}$ as

$$\tilde{f}(a_1, a_2) = \begin{cases} f(a_1, a_2), & \exists \theta : a_2 = b_2(a_1, \theta) \\ \infty, & \text{otherwise} \end{cases}$$

□

Proof of Lemma 1 (of the main paper). By definition $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1 \times \Theta}$ is not renegotiation-proof if and only if there exist $a_1 \in A_1$, $i = 1, 2, \ldots, n$ and an incentive compatible $(g, b_2) \in \mathscr{C} \times A_2^{A_1 \times \Theta}$ such that $u_2(a_1, b_2(a_1, \theta^i), \theta^i) - g(a_1, b_2(a_1, \theta^i)) > u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) - f(a_1, b_2^*(a_1, \theta^i))$ and $g(a_1, b_2(a_1, \theta^j)) > f(a_1, b_2^*(a_1, \theta^j))$ for all $j = 1, 2, \ldots, n$. For any $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1 \times \Theta}$, let $f(a_1, b_2^*) \in \mathbb{R}^n$ be a vector whose j-th component, $j = 1, 2, \ldots, n$, is given by $f(a_1, b_2^*(a_1, \theta^j))$. Note that incentive compatibility of $(g, b_2) \in \mathscr{C} \times A_2^{A_1 \times \Theta}$ is equivalent to $Dg(a_1, b_2) \leq \vec{U}_2(a_1, b_2)$ for all $a_1 \in A_1$. Therefore, $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1 \times \Theta}$ is not renegotiation-proof if and only if there exist $a_1 \in A_1$, $i = 1, 2, \ldots, n$ and $(g(a_1, b_2), b_2) \in \mathbb{R}^n \times A_2^{A_1 \times \Theta}$ such that $Dg(a_1, b_2) \leq \vec{U}_2(a_1, b_2)$, $u_2(a_1, b_2(a_1, \theta^i), \theta^i) - g(a_1, b_2(a_1, \theta^i)) > u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) - f(a_1, b_2^*(a_1, \theta^i))$, and $g(a_1, b_2) \gg f(a_1, b_2^*)$. Also note that $g(a_1, b_2) \gg f(a_1, b_2^*)$ if and only if there exists an $\varepsilon \gg 0$ such that $g(a_1, b_2) = f(a_1, b_2^*) + \varepsilon$. Therefore, we have the following

Lemma 1. *$(f, b_2^*) \in \mathscr{C} \times A_2^{A_1 \times \Theta}$ is not renegotiation-proof if and only if there exist $a_1 \in A_1$, $i = 1, 2, \ldots, n$, $b_2 \in A_2^{A_1 \times \Theta}$, and $\varepsilon \in \mathbb{R}^n$ such that $D(f(a_1, b_2^*) + \varepsilon) \leq \vec{U}_2(a_1, b_2)$, $\varepsilon_i < u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1^i, \theta^i), \theta^i)$, and $\varepsilon \gg 0$.*

We first state a theorem of the alternative, which we will use in the sequel.

Lemma 2 (Motzkin's Theorem)**.** *Let A and C be given matrices, with A being non-vacuous. Then either*

1. *$Ax \gg 0$ and $Cx \geq 0$ has a solution x*

   or

2. *$A'y_1 + C'y_2 = 0$, $y_1 > 0$, $y_2 \geq 0$ has a solution y_1, y_2*

but not both.

Proof of Lemma 2. See Mangasarian (1994), p. 28. □

For any $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1 \times \Theta}$, $a_1 \in A_1$, $b_2 \in A_2^{A_1, \times \Theta}$, and $i = 1, 2, \ldots, n$, define $V = \vec{U}_2(a_1, b_2) - Df(a_1, b_2^*)$, $C = \begin{pmatrix} V & -D \end{pmatrix}$, and

$$A = \begin{pmatrix} I_{n+1} \\ l_i \end{pmatrix}$$

where $l_i = (u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i))e_1 - e_{i+1}$. Note that C and A depend on and are uniquely defined by (f, b_2^*), a_1 and (i, b_2) but we suppress this dependency for notational convenience. The following lemma uses Motzkin's Theorem to express renegotiation-proofness as an alternative.

Lemma 3. *$(f, b_2^*) \in \mathscr{C} \times A_2^{A_1 \times \Theta}$ is renegotiation-proof if and only if for any $a_1 \in A_1$, $i = 1, 2, \ldots, n$ and $b_2 \in A_2^{A_1 \times \Theta}$ there exist $y \in \mathbb{R}^{n+2}$ and $z \in \mathbb{R}^{2(n-1)}$ such that $A'y + C'z = 0$, $y > 0$, $z \geq 0$.*

Proof of Lemma 3. By Lemma 1, (f, b_2^*) is not renegotiation-proof if and only if there exist $a_1 \in A_1$, $i = 1, 2, \ldots, n$, $b_2 \in A_2^{A_1 \times \Theta}$, and $\varepsilon \in \mathbb{R}^n$ such that $D(f(a_1, b_2^*) + \varepsilon) \leq \vec{U}_2(a_1, b_2)$, $\varepsilon_i < u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i)$, and $\varepsilon \gg 0$. This is true if and only if for some a_1, i and b_2 there exists an $x \in \mathbb{R}^{n+1}$ such that $Ax \gg 0$ and $Cx \geq 0$. To see this let $\xi > 0$ and define

$$x = \begin{pmatrix} \xi \\ \xi\varepsilon \end{pmatrix}$$

Then $D(f(a_1, b_2^*) + \varepsilon) \leq \vec{U}_2(a_1, b_2)$ if and only if $Cx \geq 0$. Also, $\varepsilon \gg 0$ and $\varepsilon_i < u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i)$ if and only if $Ax \gg 0$. The lemma then follows from Motzkin's Theorem. □

For any $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1 \times \Theta}$, $b_2 \in A_2^{A_1 \times \Theta}$, $a_1 \in A_1$, and $i = 1, 2, \ldots, n$, let $\vec{U}_2(a_1, b_2)_j$ denote the j-th component of vector $\vec{U}_2(a_1, b_2)$ and define $\alpha_1 = 1$, $\alpha_{i+1} = u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i)$, and

$$\begin{aligned}
\alpha_{k+1} &= \sum_{j=k}^{i-1} \vec{U}_2(a_1, b_2)_{2j-1} + \alpha_{i+1} - f(a_1, b_2^*(a_1, \theta^k)) + f(a_1, b_2^*(a_1, \theta^i)), && \text{for } k = 1, 2, \ldots, i-1, \\
\alpha_{l+1} &= \sum_{j=i+1}^{l} \vec{U}_2(a_1, b_2)_{2(j-1)} + \alpha_{i+1} - f(a_1, b_2^*(a_1, \theta^l)) + f(a_1, b_2^*(a_1, \theta^i)), && \text{for } l = i+1, i+2, \ldots, n, \\
\beta_j &= \vec{U}_2(a_1, b_2)_{2j} + \vec{U}_2(a_1, b_2)_{2j-1}, && \text{for } j = 1, 2, \ldots, n-1.
\end{aligned}$$

Again, note that α_j and β_j depend on and are uniquely defined by (f, b_2^*), a_1 and (i, b_2) but we suppress this dependency in the notation. We have the following lemma.

Lemma 4. *For any $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1 \times \Theta}$, $b_2 \in A_2^{A_1 \times \Theta}$, $a_1 \in A_1$ and $i = 1, 2, \ldots, n$, there exist $y \in \mathbb{R}^{n+2}$ and $z \in \mathbb{R}^{2(n-1)}$ such that $A'y + C'z = 0$, $y > 0$, and $z \geq 0$ if and only if there exist $\hat{y} \in \mathbb{R}^{n+1}$ and $\hat{z} \in \mathbb{R}^{(n-1)}$ such that $\hat{y} > 0$, $\hat{z} \geq 0$, and*

$$\sum_{j=1}^{n+1} \alpha_j \hat{y}_j + \sum_{j=1}^{n-1} \beta_j \hat{z}_j = 0 \tag{2}$$

Proof of Lemma 4. Fix $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1\times\Theta}$, $b_2 \in A_2^{A_1\times\Theta}$, $a_1 \in A_1$ and $i = 1,2,\ldots,n$. First note that for any y and z, $A'y + C'z = 0$ if and only if

$$y_1 + (u_2(a_1, b_2(a_1,\theta^i),\theta^i) - u_2(a_1, b_2^*(a_1,\theta^i),\theta^i))\, y_{n+2} + V'z = 0 \tag{3}$$

$$D'z = \left[A'y\right]_{-1} \tag{4}$$

where $\left[A'y\right]_{-1}$ is the n-dimensional vector obtained from $A'y$ by eliminating the first row. Recursively adding row 1 to row 2, row 2 to row 3, and so on, we can reduce $\begin{pmatrix} D' & \left[A'y\right]_{-1}\end{pmatrix}$ to a row echelon form and show that (4) holds if and only if

$$z_{2j-1} = z_{2j} + \sum_{k=1}^{j} y_{k+1}, \quad j = 1,2,\ldots,i-1 \tag{5}$$

$$z_{2j} = z_{2j-1} + \sum_{k=j+1}^{n} y_{k+1}, \quad j = i, i+1,\ldots,n-1 \tag{6}$$

$$y_{n+2} = \sum_{k=1}^{n} y_{k+1} \tag{7}$$

Substituting (5)-(7) into (3) we get

$$y_1 + \alpha_{i+1}\sum_{k=1}^{n} y_{k+1} + \sum_{j=1}^{i-1} \vec{U}_2(a_1,b_2)_{2j-1}\sum_{k=1}^{j} y_{k+1} + \sum_{j=i}^{n-1}\vec{U}_2(a_1,b_2)_{2j}\sum_{k=j+1}^{n} y_{k+1} + \sum_{j=1}^{i-1}(\vec{U}_2(a_1,b_2)_{2j-1} + \vec{U}_2(a_1,b_2)_{2j})z_{2j}$$
$$+ \sum_{j=i}^{n-1}(\vec{U}_2(a_1,b_2)_{2j-1} + \vec{U}_2(a_1,b_2)_{2j})z_{2j-1} - \sum_{k=1}^{n}(f(a_1, b_2^*(a_1,\theta^k)) - f(a_1, b_2^*(a_1,\theta^i)))\, y_{k+1} = 0 \tag{8}$$

Therefore, $A'y + C'z = 0$ if and only if equations (5) through (8) hold. Now suppose that there exist $y \in \mathbb{R}^{n+2}$ and $z \in \mathbb{R}^{2(n-1)}$ such that $y > 0$, $z \geq 0$, and (5) through (8) hold. Define $\hat{y}_j = y_j$, for $j = 1,\ldots,n+1$ and

$$\hat{z}_j = \begin{cases} z_{2j}, & j = 1,\ldots,i-1 \\ z_{2j-1}, & j = i,\ldots,n-1 \end{cases}$$

It is easy to verify that $\hat{y} > 0$, $\hat{z} \geq 0$, and $\sum_{j=1}^{n+1}\alpha_j\hat{y}_j + \sum_{j=1}^{n-1}\beta_j\hat{z}_j = 0$.

Conversely, suppose that there exist $\hat{y} \in \mathbb{R}^{n+1}$ and $\hat{z} \in \mathbb{R}^{(n-1)}$ such that $\hat{y} > 0$, $\hat{z} \geq 0$, and (2) holds. Define $y_j = \hat{y}_j$ for $j = 1,\ldots,n+1$ and $y_{n+2} = \sum_{i=1}^{n+1}\hat{y}_j$. For any $j = 1,\ldots,i-1$, let $z_{2j-1} = \hat{z}_j + \sum_{k=1}^{j}\hat{y}_{k+1}$ and $z_{2j} = \hat{z}_j$, and for any $j = i,\ldots,n-1$, let $z_{2j-1} = \hat{z}_j$ and $z_{2j} = \hat{z}_j + \sum_{k=j+1}^{n}\hat{y}_{k+1}$. It is straightforward to show that $y > 0$, $z \geq 0$, and (5) through (8) hold. This completes the proof of Lemma 4. □

Lemma 3 and 4 imply that $(f, b_2^*) \in \mathscr{C} \times A_2^{A_1\times\Theta}$ is renegotiation-proof if and only if for any $a_1 \in A_1$, $i \in \{1,2,\ldots,n\}$ and $b_2 \in A_2^{A_1\times\Theta}$, there exist $\hat{y} \in \mathbb{R}^{n+1}$ and $\hat{z} \in \mathbb{R}^{(n-1)}$ such that $\hat{y} > 0$, $\hat{z} \geq 0$, and equation (2) holds. We can now complete the proof of Lemma 1 (of the main paper).

[Only if] Suppose, for contradiction, that there exist $a_1 \in A_1$, $i = 1,2,\ldots,n$ and an increasing $b_2 \in A_2^{A_1\times\Theta}$ such that $u_2(a_1, b_2(a_1,\theta^i),\theta^i) > u_2(a_1, b_2^*(a_1,\theta^i),\theta^i)$, but there is no $k = 1,2,\ldots,i-1$ such that (8) holds and no $l = i+1,\ldots,n$ such that (9) holds. This implies that $\alpha_j > 0$ for all $j = 1,\ldots,n+1$. Since u_2 has increasing differences, $\beta_j \geq 0$ for all $j = 1,\ldots,n-1$. Therefore, $\hat{y} > 0$ and $\hat{z} \geq 0$ imply that $\sum_{j=1}^{n+1}\alpha_j\hat{y}_j + \sum_{j=1}^{n-1}\beta_j\hat{z}_j > 0$, which, by Lemma 4, contradicts that (f, b_2^*) is renegotiation-proof.

[If] Fix arbitrary $a_1 \in A_1$, $i = 1,2,\ldots,n$ and increasing $b_2 \in A_2^{A_1\times\Theta}$ such that $u_2(a_1, b_2(a_1,\theta^i),\theta^i) >$

$u_2(a_1, b_2^*(a_1, \theta^i), \theta^i)$. Suppose first that there exists a $k \in \{1, \ldots, i-1\}$ such that (8) holds. This implies that $\alpha_{i+1} > 0$ and $\alpha_{k+1} \leq 0$. Let $\hat{y}_{k+1} = 1$, $\hat{y}_{i+1} = \frac{-\alpha_{k+1}}{\alpha_{i+1}} \geq 0$, and all the other $\hat{y}_j = 0$ and $\hat{z}_j = 0$. This implies that equation (2) holds and, by Lemma 3 and 4, that (f, b_2^*) is renegotiation-proof. Suppose now that there exists an $l \in \{i+1, \ldots, n\}$ such that (9) holds. Then, $\alpha_{i+1} > 0$ and $\alpha_{l+1} \leq 0$. Let $\hat{y}_{l+1} = 1$, $\hat{y}_{i+1} = \frac{-\alpha_{l+1}}{\alpha_{i+1}} \geq 0$ and all the other $\hat{y}_j = 0$ and $\hat{z}_j = 0$. This, again, implies that (2) holds and that (f, b_2^*) is renegotiation-proof. □

Proof of Lemma 2 (of the main paper). Suppose that b_2^* is renegotiation-proof and fix a_1, $i = 1, \ldots, n$ and a $b_2(a_1, \theta^i) \in \mathfrak{B}(a_1, i, b_2^*)$. For any $j = 1, \ldots, n$, let $c_j = e_i - e_j$, where e_j is the j^{th} standard basis row vector for $\mathbb{R}^n$, and define

$$E_j = \begin{pmatrix} D \\ c_j \end{pmatrix}$$

Also let

$$w_k = u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) + \sum_{j=k}^{i-1} \vec{U}_2(a_1, b_2)_{2j-1}$$

$$w_l = u_2(a_1, b_2(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) + \sum_{j=i+1}^{l} \vec{U}_2(a_1, b_2)_{2(j-1)}$$

for any $k \in \{1, \ldots, i-1\}$ and $l \in \{i+1, \ldots, n\}$ and define

$$V_j = \begin{pmatrix} \vec{U}_2(a_1, b_2^*) \\ -w_j \end{pmatrix}$$

Incentive compatibility of (f, b_2^*) implies that $Df(a_1, b_2^*) \leq \vec{U}_2(a_1, b_2^*)$. Renegotiation proofness, by Lemma 1 (of the main paper), implies that $c_k f(a_1, b_2^*) \leq -w_k$ for some $k \in \{1, \ldots, i-1\}$ or $c_l f(a_1, b_2^*) \leq -w_l$ for some $l \in \{i+1, \ldots, n\}$. Suppose first that there exists a $k \in \{1, \ldots, i-1\}$ such that $c_k f(a_1, b_2^*) \leq -w_k$. Then we must have $E_k f(a_1, b_2^*) \leq V_k$. By Gale's theorem of linear inequalities, this implies that $x \geq 0$ and $E'_k x = 0$ implies $x' V_k \geq 0$. Denote the first $2(n-1)$ elements of x by y and the last element by z. It is easy to show that $E'_k x = 0$ implies that $y_{2j-1} = y_{2j} + z$ for $j \in \{k, k+1, \ldots, i-1\}$ and $y_{2j-1} = y_{2j}$ for $j \notin \{k, k+1, \ldots, i-1\}$. Therefore,

$$\begin{aligned} x' V_k &= \sum_{j=1}^{n-1} \vec{U}_2(a_1, b_2^*)_{2j} y_{2j} + \sum_{j=1}^{n-1} \vec{U}_2(a_1, b_2^*)_{2j-1} y_{2j-1} - z w_k \\ &= \sum_{j=1}^{n-1} (\vec{U}_2(a_1, b_2^*)_{2j} + \vec{U}_2(a_1, b_2^*)_{2j-1}) y_{2j} + z(-w_k + \sum_{j=k}^{i-1} \vec{U}_2(a_1, b_2^*)_{2j-1}) \\ &\geq 0 \end{aligned}$$

This implies that $-w_k + \sum_{j=k}^{i-1} \vec{U}_2(a_1, b_2^*)_{2j-1} \geq 0$ and hence k is a blocking type.

Similarly, we can show that, if there exists an $l \in \{i+1, \ldots, n\}$ such that $c_l f(a_1, b_2^*) \leq -w_l$, then l is a blocking type, and this completes the proof. □

Proof of Lemma 3 (of the main paper). Let $b_2^* \in A_2^{A_1 \times \Theta}$ be an increasing strategy satisfying the conditions of the lemma. We will show that there exist an $f \in \mathscr{C}$ such that (f, b_2^*) is incentive-compatible and renegotiation-proof. Fix an $a_1 \in A_1$ and for each $i = 1, \cdots, n$ and $b_2^i \in \mathscr{B}(a_1, i, b_2^*)$ pick a blocking type $m(b_2^i) = 1, \cdots, n$ that satisfies the conditions given in the proposition. For each $i = 1$ and $b_2^i \in \mathscr{B}(a_1, i, b_2^*)$ define the n-dimensional row vector $c_{b_2^i} = e_i - e_{m(b_2^i)}$, where e_j is the j^{th} standard basis row vector for $\mathbb{R}^n$, and the scalar $w_{b_2^i}$ as

$$w_{b_2^i} = u_2(a_1, b_2^i(a_1, \theta^i), \theta^i) - u_2(a_1, b_2^*(a_1, \theta^i), \theta^i) + \mathbf{1}_{\{m(b_2^i) \le i-1\}} \sum_{j=m(b_2^i)}^{i-1} \vec{U}_2(a_1, b_2^i)_{2j-1} + \mathbf{1}_{\{i \le m(b_2^i-1\}} \sum_{j=i+1}^{m(b_2^i)} \vec{U}_2(a_1, b_2^i)_{2(j-1)}. \quad (9)$$

Note that for a given $a_1 \in A_1$ and $i = 1, \cdots, n$, $\mathscr{B}(a_1, i, b_2^*)$ is finite and let $\sum_{i=1}^n |\mathscr{B}(a_1, i, b_2^*)| = p$. Denote with $C(a_1)$, the $p \times n$ matrix composed of all the rows $c_{b_2^i}$ and with $W(a_1)$the p dimensional vector with component $w_{b_2^i}$ corresponding to each b_2^i. Let $E(a_1)$ be the matrix

$$E(a_1) = \begin{pmatrix} D \\ C(a_1) \end{pmatrix}$$

and $V(a_1)$ the column vector

$$V(a_1) = \begin{pmatrix} \vec{U}_2(a_1, b_2^*) \\ -W(a_1) \end{pmatrix}$$

Now, if for each $a_1 \in A_1$, we can find an $f(a_1, b_2^*)$ such that $E(a_1)f(a_1, b_2^*) \le V(a_1)$ the proof would be completed. In fact, if $E(a_1)f(a_1, b_2^*) \le V(a_1)$, then $Df(a_1, b_2^*) \le \vec{U}_2(a_1, b_2^*)$, which implies that (f, b_2^*) incentive compatible. Furthermore, $E(a_1)f(a_1, b_2^*) \le V(a_1)$ implies $W(a_1) \le -C(a_1)f(a_1, b_2^*)$ and, by Lemma 1 (of the main paper), that (f, b_2^*) is renegotiation-proof. Gale's theorem of linear inequalities implies that there exist $f(a_1, b_2^*) \in \mathbb{R}^n$ such that $E(a_1)f(a_1, b_2^*) \le V(a_1)$ if and only if $x \in \mathbb{R}^{p+2(n-1)}$, $x \ge 0$ and $E(a_1)'x = 0$ implies $x'V(a_1) \ge 0$. Decompose x into two vectors so that the first $2(n-1)$ elements constitute y and the remaining p components constitute z. Notice that for any $i = 1, \ldots, n$ and $b_2^i \in \mathfrak{B}(a_1, i, b_2^*)$ there is a corresponding element of z, which we will denote $z_{b_2^i}$.

Recursively adding row 1 to row 2, row 2 to row 3, and so on, we can reduce $E(a_1)'$ to a row echelon form and show that $E(a_1)'x = 0$ if and only if

$$y_{2j-1} = y_{2j} + \sum_{b_2^i} z_{b_2^i} [\mathbf{1}_{\{m(b_2^i) \le j \le i-1\}} - \mathbf{1}_{\{i \le j \le m(b_2^i)-1\}}] \quad (10)$$

for $j = 1, \ldots, n-1$.

Let $J_- = \{j \in \{1, \ldots, n-1\} : \exists b_2^i$ such that $i \le j \le m(b_2^i)-1\}$ and $J_+ = \{j \in \{1, \ldots, n-1\} : \exists b_2^i$ such that $m(b_2^i) \le j \le i-1\}$ and note that $J_- \cap J_+ = \emptyset$. To see this, suppose, for contradiction, that there exists a $j \in J_- \cap J_+$. Therefore, there exists a b_2^i such that $i \le j \le m(b_2^i) - 1$ and $b_2^{i'}$ such that $m(b_2^{i'}) \le j \le i' - 1$. This implies that $i < i'$, $m(b_2^i) > i$, $m(b_2^{i'}) < i'$, but $m(b_2^i) > m(b_2^{i'})$, contradicting the conditions of the lemma. We can therefore write (10) as

$$y_{2j} = y_{2j-1} + \sum_{b_2^i} z_{b_2^i} \mathbf{1}_{\{i \le j \le m(b_2^i)-1\}} \quad (11)$$

for $j \in J_-$ and

$$y_{2j-1} = y_{2j} + \sum_{b_2^i} z_{b_2^i} \mathbf{1}_{\{m(b_2^i) \le j \le i-1\}} \quad (12)$$

for $j \in J_+$.

Finally note that

$$x'V(a_1) = \sum_{j=1}^{n-1} \vec{U}_2(a_1,b_2^*)_{2j} y_{2j} + \sum_{j=1}^{n-1} \vec{U}_2(a_1,b_2^*)_{2j-1} y_{2j-1} - \sum_{b_2^i} z_{b_2^i} w_{b_2^i}$$

Substituting from (11) and (12) we obtain

$$\begin{aligned} x'V(a_1) = &\sum_{j\in J_-} \left[\vec{U}_2(a_1,b_2^*)_{2j} + \vec{U}_2(a_1,b_2^*)_{2j-1}\right] y_{2j-1} + \sum_{j\in J_+} \left[\vec{U}_2(a_1,b_2^*)_{2j} + \vec{U}_2(a_1,b_2^*)_{2j-1}\right] y_{2j} \\ &+ \sum_{b_2^i} z_{b_2^i} \left[-w_{b_2^i} + \mathbf{1}_{\{m(b_2^i)\le i-1\}} \sum_{j=m(b_2^i)}^{i-1} \vec{U}_2(a_1,b_2^*)_{2j-1} + \mathbf{1}_{\{i\le m(b_2^i)-1\}} \sum_{j=i}^{m(b_2^i)-1} \vec{U}_2(a_1,b_2^*)_{2j} \right] \end{aligned}$$

Increasing differences, the definition of $m(b_2^i)$, and $y,z \ge 0$ imply that $x'V \ge 0$, and the proof is completed. □

Proof of Proposition 3 (of the main paper). Suppose, for contradiction, that there exists an $a_1' \in A_1$ such that (a_1',θ^n) has right deviation at b_2, i.e., there exists an $a_2' \in A_2$ such that $a_2' \succsim_2 b_2(a_1',\theta^n)$ and $u_2(a_1',a_2',\theta^n) > u_2(a_1',b_2(a_1',\theta^n),\theta^n)$. Define

$$b_2'(a_1',\theta) = \begin{cases} a_2', & \theta = \theta^n \\ b_2(a_1',\theta), & \theta \prec_\theta \theta^n \end{cases}$$

Note that b_2' is increasing and therefore $b_2' \in \mathfrak{B}(a_1',n,b_2)$. It is easy to show that for (a_1',n,b_2') there is no blocking type and therefore, by Lemma 2 (of the main paper), b_2 is not renegotiation proof. □

Proof of Proposition 4 (of the main paper). Fix $a_1 \in A_1$, $i \in \{1,\cdots,n\}$, and $b_2^i \in \mathfrak{B}(a_1,i,b_2^*)$. Since A_2 is linearly ordered, we have $b_2^i(a_1,\theta^i) \succsim_2 b_2^*(a_1,\theta^i)$ or $b_2^*(a_1,\theta^i) \succsim_2 b_2^i(a_1,\theta^i)$. First, assume that $b_2^i(a_1,\theta^i) \succsim_2 b_2^*(a_1,\theta^i)$, i.e., (a_1,i) has right deviation at b_2^*, and note that $R(a_1,i) \neq \emptyset$ by assumption. Let $J = \{j \in \mathbb{N} : i+1 \le j \le \min R(a_1,i)-1 \text{ and } b_2^*(a_1,\theta^j) \succ_2 b_2^i(a_1,\theta^j)\}$. If $J=\emptyset$, let $m(b_2^i) = \min R(a_1,i)$ and if $J\neq\emptyset$, let $m(b_2^i) = \min J$. It is simple to show that

$$\begin{aligned} &\sum_{j=i+1}^{m(b_2^i)} \Big(u_2(a_1,b_2^i(a_1,\theta^{j-1}),\theta^j) - u_2(a_1,b_2^*(a_1,\theta^{j-1}),\theta^j) - [u_2(a_1,b_2^i(a_1,\theta^{j-1}),\theta^{j-1}) - u_2(a_1,b_2^*(a_1,\theta^{j-1}),\theta^{j-1})] \Big) \\ &\qquad + u_2(a_1,b_2^*(a_1,\theta^{m(b_2^i)}),\theta^{m(b_2^i)}) - u_2(a_1,b_2^i(a_1,\theta^{m(b_2^i)}),\theta^{m(b_2^i)}) \ge 0 \quad (13) \end{aligned}$$

Inequality (13) implies that $m(b_2^i)$ is a blocking type.

Now assume that $b_2^*(a_1,\theta^i) \succsim_2 b_2^i(a_1,\theta^i)$, i.e., (a_1,i) has left deviation at b_2^*, and note that $L(a_1,i) \neq \emptyset$. Let $J = \{j \in \mathbb{N} : \max L(i)+1 \le j \le i-1 \text{ and } b_2^i(a_1,\theta^j) \succ_2 b_2^*(a_1,\theta^j)\}$. If $J = \emptyset$, let $m(b_2^i) = \max L(i)$ and if $J \neq \emptyset$, let $m(b_2^i) = \max J$ and note that

$$\begin{aligned} &\sum_{j=m(b_2^i)}^{i-1} \Big(u_2(a_1,b_2^*(a_1,\theta^{j+1}),\theta^{j+1}) - u_2(a_1,b_2^i(a_1,\theta^{j+1}),\theta^{j+1}) - [u_2(a_1,b_2^*(a_1,\theta^{j+1}),\theta^j) - u_2(a_1,b_2^i(a_1,\theta^{j+1}),\theta^j)] \Big) \\ &\qquad + u_2(a_1,b_2^*(a_1,\theta^{m(b_2^i)}),\theta^{m(b_2^i)}) - u_2(a_1,b_2^i(a_1,\theta^{m(b_2^i)}),\theta^{m(b_2^i)}) \ge 0 \quad (14) \end{aligned}$$

Inequality (14) implies that $m(b_2^i)$ is a blocking type.

Finally assume that there exist (a_1, i_1) and (a_1, i_2) with $i_1 < i_2$ such that $m(b_2^{i_1}) > i_1$ and $m(b_2^{i_2}) < i_2$. This implies that (a_1, i_1) has right deviation and (a_1, i_2) has left deviation at b_2^*, which imply that $R(a_1, i_1) \neq \emptyset$, $L(a_1, i_2) \neq \emptyset$ and $R(a_1, i_1) \cap L(a_1, i_2) \neq \emptyset$. But this implies that $m(b_2^{i_1}) \leq m(b_2^{i_2})$ and the proof is completed by applying Lemma 3 (of the main paper). □

References

[1] Mangasarian, O. L. (1994) *Nonlinear Programming*, New York: McGraw-Hill.