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# Games with Externalities and Delegation to a Common Agent*

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**Abstract**

I present a model in which the players of a game have the option to delegate parts of their strategies to a third party who has an interest in the outcome of the game. I analyze whether the game with delegation to a common agent improves over the equilibrium of the original game. This paper contributes to the literature on private common agency and to the failure of the revelation principle with multiple principals. One contribution of this paper is the characterization of the complete set of equilibrium outcomes for the game with delegation, including the asymmetric outcomes. I also provide an answer to the question whether the results of the existing models of private common agency are robust to mixed strategy deviations and shed light on the persistence of the failure of revelation principle.

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# 1 Introduction

Suppose that two firms, which compete in an oligopolistic market, are both borrowing from the same bank. It is conceivable that the firms might try to exploit this common link by coordinating their actions in order to achieve a more collusive outcome. At the same time it is likely that the bank might have an interest in the performance of the firms and this could make the coordination role of the bank less (or more) difficult to achieve. A similar situation might occur if instead of the bank we consider a common retailer of the firms or a common supplier of two downstream manufacturers.

As another example, consider two countries that are involved in a dispute over the control of some resources. These countries might decide to use a neighboring country or an international organization to which they both belong as a mediator. Also in this case, it is not far fetched to imagine that this third party has preferences regarding the outcome of the dispute between the two countries, and that both countries take those preferences into account when deciding how much power to delegate to the mediator.

These examples share the following features: the actions of a group of players that we can embed in a game impose externalities on some other agent (the bank, the international organization). Moreover, the players of the original game may decide to use this third party to improve the outcome they can achieve.

I call a *game with externality*, $G$, any two-person strategic form game where each player (from now on *principal*) chooses his own strategy and where the third party (from now on *agent*) has preferences over the strategy profiles, even if she has no action to take. For any given game with externality, I define a *delegation game* (or *game with menus*), $G_M$, as follows. There are two stages: in the first stage, the principals can offer menus (a subset of their strategies); in the second stage the agent (now a player of the game) selects a strategy for each principal within the menu offered to her. I also allow each principal to offer a monetary transfer depending on the strategy that the agent chooses so that the principals can try to affect the preferences of the agent and therefore her choice. The question is whether the game with delegation has equilibrium outcomes that are not achievable in the original game and whether and how the answer depends on the preferences of the agent.

This paper contributes to several strands of literature. First, it is related to the literature on common agency. In a typical common agency model there are several principals and an agent who may sign binding contracts with the principals.[1] Common agency models may be considered as a natural extension of standard principal-agent models with the extra dimension introduced by the strategic interaction between the principals during the contracting stage (see Martimort 2006). This paper focuses on the strategic dimension, neglecting the issues

[1]In the original version of the common agency model (Bernheim and Whinston 1986a), the agent has an action to take and the principals offer monetary transfers to influence the choice of the agent; the focus is on efficiency that can be achieved with a particular class of contracts (truthful mechanisms).

typical of incentive problems (i.e., asymmetric information and unobservable effort).

The common agency model has been applied to a variety of settings and generalized in different ways.[2] This paper focuses on one such generalization, namely private common agency (also known as bidding games), which describes a situation where the monetary transfers between each principal and the agent can depend only on a subset of the choice set of the agent, even if the payoffs of each principal depend on the whole choice set. Martimort and Stole (2003) and Segal and Whinston (2003, section 7) study models of private common agency showing that efficiency cannot be achieved as a consequence of the limited contracting ability of the principals. In fact, in their models, menus enlarge the set of equilibrium outcomes, but without increasing the efficiency of the outcome. However, they assume that preferences of the agent are convex and restrict the analysis to symmetric equilibria. In this paper, I allow more general agent preferences and characterize both symmetric and asymmetric equilibria. One contribution of this paper therefore is to assess whether we can recover efficiency in a more general structure.[3] The conclusions are that efficiency via private common agency can be recovered either by restricting the preferences of the agent in a significant manner (constant differences) or accepting asymmetric equilibrium outcomes (e.g. one principal produces the monopoly outcome while the other does not produce at all).[4]

The literature on common agency, including the above mentioned papers by Martimort and Stole (2003) and Segal and Whinston (2003) has restricted the principals' strategies to pure contracts and therefore begs the question as to whether these results are robust to the introduction of mixed contracts. A second contribution of this paper is to provide a precise answer to this question. As it will be made evident in the course of this paper, mixed contracts have a peculiar role in common agency models: they result in enlarging the set of feasible deviations thereby reducing the set of equilibrium outcomes. A mixed contract is equivalent, in this paper, to a mixed strategy in the original game, and therefore considering their role is more than just a theoretical curiosity. As soon as we allow the principals to play the game themselves, rather than delegating to the agent, mixed strategies become as natural as they are in any game. It turns out that the introduction of mixed strategy deviations has a first order effect if the class of agent preferences are supermodular, while only a second order effect if the agent payoffs are submodular. In the former case, it is shown that only asymmetric outcomes can be supported as equilibria of the delegation game. In the latter case, allowing the

[2]A partial list includes Bernheim and Whinston (1986b), Biglaiser and Mezzetti (1993), Dixit et al.(1997), Laussel and Le Breton (2001), Martimort (1999), Olsen and Osmundsen (2001), Parlour and Rajan (2001), Prat and Rustichini (2003), Segal (1999).

[3]Both Martimort and Stole (2003) and Segal and Winston (2003) are more general than the model in this paper in other dimensions. Martimort and Stole (2003) study both complete and incomplete information while Segal and Winston (2003) study different ways in which the bargaining power is divided between the principals and the agent.

[4]By an asymmetric outcome, I mean an outcome $(y_i, y_j)$ such that for one principal the optimal action is $y_j$ but the reverse is not true.

principals to deviate using a mixed strategy does not reduce the set of equilibrium outcomes in the delegation game as long as a condition relating payoffs of principals and payoffs of the agent is satisfied.[5]

This paper is also related to the literature on the failure of the revelation principle with multiple principals. In single agency models, the revelation principle shows that for every incentive compatible mechanism the principal might design, there is an incentive compatible direct mechanism that gives the principal the same payoff (see, for example, Myerson 1979). As a result, attention can be restricted to direct mechanisms with no loss of generality. This is not true with several principals because when considering a principal-agent pair, the agent does not have only private information about her preferences, but also information on the contracts offered by the other principals.[6]

One of the proposed remedies to the failure of the revelation principle is based on the result that all "interesting" equilibrium outcomes can be supported as equilibrium of a game in which principals offer menus (Martimort and Stole 2002, Peters 2001).[7] In other words, the consequences of the failure of the revelation principle in common agency are captured by the gap between the equilibrium allocations supported by direct mechanisms and the ones supported by menus. Peters (2003a) uses this result to show that in environments in which a "no-externality" condition is satisfied, this gap disappears, restoring the validity of the revelation principle.[8] In the simple environment of this paper, a direct mechanism corresponds to a strategy of the original game $G$, while a menu corresponds to a strategy of the delegation game $G_M$. Moreover, since in the original game the agent has no effort, the no-externality condition is arguably too strong as it requires the agent to rank the actions of one principal independently of the action chosen by the other principal. Therefore, a contribution of this paper is to investigate the consequence of the failure of the revelation principle for a large class of preferences of the agent and to investigate whether there are reasonable assumptions on the class of preferences of the agent for which the revelation principle holds.[9] The results suggest that in order to recover the validity of the revelation principle in an environment with

[5]The existing models (Martimort and Stole 2003 and Segal and Winston 2003) on private common agency assume that the agent's payoff are submodular and therefore are likely to be robust to mixed strategy deviations.

[6]See Epstein and Peters (1999), Martimort and Stole (2002) and Peck (1997) for examples and discussion. Epstein and Peters (1999) provides a full fledged revelation principle valid with multiple principals. The limit of the contribution rests in the difficulty of translate this result in workable models.

[7]The failure of revelation principle with multiple principals has received considerable attention and several remedies have been proposed both for the case of multiple principals-one agent (common agency) and for the case of multiple principals-multiple agents. For the latter see Attar et al. (2006b), Epstein and Peters (1999), Han (2006, 2007a), Peters (2003b); for the former see Martimort and Stole (2002), Peters (2001, 2003a) Pavan and Calzolari (2007), Page and Monteiro (2003).

[8]See also Attar et al. (2006a) for a weaker version of the no-externality condition proposed by Peters (2003a).

[9]This task is made easier because I explicitly consider mixed strategy deviations: therefore the set of equilibrium outcomes that are supportable by menus but not by Nash equilibrium of the original game shrinks.

externality, we must either focus on a model in which monetary transfers are not allowed or be interested exclusively in symmetric equilibria.

Finally, my paper is related to the existing literature on strategic delegation. Following Schelling (1960), several authors (see Fershtman and Judd 1987, Katz 1991, Kockesen and Ok 2004) study whether delegation to an agent can take place as a means of credible precommitment. This literature focuses on exclusive principal-agent pairs, rather than common agency and therefore the comparison with my model is not immediate. Nonetheless, their emphasis on the comparison between the equilibrium outcomes of the original game with those of the delegation game is similar to mine. In particular, Fershtman, Judd and Kalai (1991) study the extent to which the set of equilibria of a game changes when both players can use agents to play the game instead of themselves and show that the cooperative outcome can be achieved. The results of my paper contrast those in Fershtman, Judd and Kalai (1991) as in this model the cooperative outcome is achievable only if the agent has a payoff function separable in the strategies of the principals.

The paper is organized as follows: Section 2 begins with a motivating example based on the prisoners' dilemma game. The results obtained for the prisoners' dilemma are stronger than the ones I obtain in the general model, but they deliver the flavor of this paper's themes. Section 3 presents a simplified model in which the principals cannot offer monetary transfers to the agent. Here too the results are stronger than the results obtained in the model with monetary transfers. In particular, I recover the revelation principle for a large class of preferences of the agent. However, since the absence of monetary transfer is an unappealing feature, the interest in this section stems mainly from the fact that those results are used in the analysis of the complete model with monetary transfers in section 4. Section 5 contains a variant of a well known example based on vertical contracting and illustrates how the results of the paper apply in a more specific setting. The conclusion is set forth in section 6. The proofs of the propositions stated in the paper are included within the appendix.

## 2 Prisoners' dilemma with externalities

In order to illustrate the basic idea of this paper, first a simple example based on the prisoners' dilemma with an interested mediator will be analyzed.

In the original game $G$, each principal selects among the actions $\{C, D\}$ that in the prisoners' dilemma have the usual interpretation of cooperate and defect. The agent has no action to take. The following table contains the payoffs of the principals and the agent for any outcome of the game; in each cell, the first two entries represent the payoff of the first principal (the row player) and the second principal (the column player) respectively while the third entry indicates the payoff of the agent.

| | $C$ | $D$ |
|---|---|---|
| $C$ | $4, 4, u_{CC}$ | $0, 6, u_{CD}$ |
| $D$ | $6, 0, u_{DC}$ | $1, 1, u_{DD}$ |

The only Nash equilibrium of this game is $(D, D)$ with payoffs of 1 for both principals and $u_{DD}$ for the agent.

The delegation game $G_M$ is defined as follows: in the first stage each principal simultaneously and secretly communicates to the agent whether he delegates the choice of the action to the agent or not. If principal $i$ decides to delegate, then the agent will choose between the actions $C$ and $D$ in the second stage. The principal can also indicate a monetary transfer from him to the agent depending on which action the agent selects.[10] If principal $i$ does not delegate, he must communicate to the agent which strategy he intends to play: this can be either $\{C\}$, $\{D\}$ or any mixed strategy (any probability distribution over the action space $\{C, D\}$). In the second stage of the delegation game the agent chooses the actions for the principals who chose to delegate and the game ends. For example, assume that the first principal's strategy is $\{(C, t_C^1), (D, t_D^1)\}$, that is to delegate the choice between $C$ and $D$ to the agent, where $t_C^1$ (respectively $t_D^1$) represents the monetary transfer from the principal to the agent if she chooses outcome $C$ (respectively $D$); and assume that the strategy of the second principal is not to delegate by choosing action $D$. In this case, if the agent chooses $C$ from the first principal, the outcome will be $(C, D)$ and the payoff will be $-t_C$ for the first principal, 6 for the second principal, and $u_{CD} + t_C$ for the agent. If the agent chooses $D$ from the first principal, the outcome will be $(D, D)$ and the payoff will be $1 - t_D$ for the first principal, 1 for the second principal, and $u_{DD} + t_D$ for the agent.

The question is whether the cooperative outcome $(C, C)$ is achievable in the game with delegation and the answer turns out to depend on the payoff function of the agent. The space of the agent's payoff functions can be partitioned into three regions. For payoff functions satisfying $u_{CC} + u_{DD} < u_{CD} + u_{DC}$ the cooperative outcome $(C, C)$ cannot be achieved as an equilibrium of the delegation game because giving to the agent the incentive to choose the outcome $(C, C)$ is incompatible with giving the agent the incentive to punish the deviation to the action $D$.[11]

For payoff functions satisfying $u_{CC} + u_{DD} > u_{CD} + u_{DC}$, the cooperative outcome can be achieved as an equilibrium of $G_M$ only if we restrict the principals to pure strategies. As soon as we allow principals to use mixed strategies $(C, C)$ ceases to be an equilibrium outcome of the delegation game $G_M$. This is one of the main themes of the paper and deserves some clarification. Assume that $u_{CC} = 6$, $u_{CD} = u_{DC} = 2$, and $u_{DD} = 1$; if each principal chooses

[10] Each principal is restricted to choose at least one non-negative transfer, that is either $t_C^i \geq 0$ or $t_D^i \geq 0$ (or both). See the section 4 for a discussion.

[11] At the end of section 4 I will get back to this example to show that for such payoff functions of the agent there might be equilibria in which the agent randomizes between the outcomes $(C, D)$ and $(D, C)$.

to offer the agent the following menu $M_i = \{(C, t_i^c = 0); (D, t_1^d = 2)\}$, the payoffs for the principals and the agent become

| | $(C, 0)$ | $(D, 2)$ |
|---|---|---|
| $(C, 0)$ | $4, 4, 6$ | $0, 4, 4$ |
| $(D, 2)$ | $4, 0, 4$ | $-1, -1, 5$ |

Faced with these menus, the agent chooses the cooperative strategy (with no monetary transfers) from each principal and the payoffs received are $(4, 4, 6)$. Now if any principal, say principal 2, deviates to a non-delegating strategy choosing $D$, the choice of the agent is between $\{(C, 0), D\}$ and $\{(D, 2), D\}$, with the agent's payoffs as 2 and 3 respectively; the agent will therefore punish the deviation by choosing $(D, 2)$ from the menu offered by principal 1, making the deviation of principal 2 unprofitable.

This is not the end of the story, though. In fact, if we allow principals to deviate to a non-delegating strategy and to choose a mixed strategy in the original game, a principal has a profitable deviation. Consider the following mixed deviation for principal 2: $C$ with probability $\frac{1}{2}$ and $D$ with probability $\frac{1}{2}$; the choice of the agent between the two elements of menu $(C, 0)$ and $(D, 2)$ offered by the non-deviating principal 1 now becomes a choice between two lotteries: if the agent changes the selection from the menu offered by principal 1 to $(D, 2)$, she receives an expected payoff of 3.5 while if she selects $(C, 0)$ from the menu offered by principal 1, her expected payoff is equal to 4. Hence, the cooperative outcome is an equilibrium if principals can deviate by using solely pure strategy actions, but it is not robust to mixed strategy deviations.

If the agent's payoff function satisfies the knife-edge condition $u_{CC} + u_{DD} = u_{CD} + u_{DC}$, then the cooperative outcome $(C, C)$ can be supported as an equilibrium of $G_M$. Consider for example the following payoff function: $u_{CC} = 3$, $u_{CD} = u_{DC} = 2$ and $u_{DD} = 1$; if each principal chooses to offer to the agent the following menu $M_i = \{(C, t_i^c = 0); (D, t_1^d = 1)\}$, the payoffs for the principals and the agent become

| | $(C, 0)$ | $(D, 1)$ |
|---|---|---|
| $(C, 0)$ | $4, 4, 3$ | $0, 5, 3$ |
| $(D, 1)$ | $5, 0, 3$ | $0, 0, 3$ |

Since the agent is now indifferent among all the menu outcomes offered by the principals, there exists an equilibrium strategy for the agent that selects $(C, C)$, and that punishes all deviations (pure and mixed) by selecting $(D, 1)$ from the non-deviating principal.

The results obtained for the prisoner's dilemma with an interested mediator are stronger than those obtained for the general model studied in this paper. Nevertheless, some general ideas carry through. In particular there is a large class of the agent's preferences (those that satisfy increasing differences), under which mixed strategy deviations considerably reduce the set of equilibrium outcomes of the delegation game. The efficient outcome can be achieved for a small class of preferences (those that satisfy constant differences) and this equilibrium is robust also to mixed strategy deviations.

# 3 Delegation without monetary transfers

Before analyzing the games with monetary transfer, this section covers a simplified version of the delegation game, in which the principals cannot offer monetary transfers to the agent. The interest in this simplified model goes beyond the theoretical curiosity of what is achievable with delegation when principals do not have access to monetary transfers. In fact, the model studied in this section turns out to be the only case in which it is possible to recover the revelation principle for a large class of games. Moreover, the results of the model with monetary transfers rely heavily on those obtained in this simplified model.

A *game with externalities* $G$ is a two-person game in strategic form, with an extra player who has no proper action.[12] That is $G = \{(1,2), (Y_1, Y_2), (v_1, v_2, u)\}$ where $Y_i$ indicates the set of finite strategies for player $i$ (from now on principal $i$), and $v_i$ represents his payoff function. The payoff function of the "added" player (from now on agent) is designated as $u$.

The *delegation game without transfers* $G_M$ (or game with menus) is a two stage game where the agent becomes a proper player. In the first stage, the principals (the proper players of $G$) simultaneously decide whether to delegate the choice over the strategies (or part of them) to the agent. In the second stage, the agent makes a choice among the alternatives he has been delegated to decide upon.

More precisely, in the game with delegation $G_M$, the strategy space for each principal $j$ is given by $\mathcal{M}_j = (2^{Y_j} \backslash \emptyset) \cup \Delta(Y_j)$. Denote an element of this strategy space as $M_j \in \mathcal{M}_j$. In the second stage the agent chooses an element $(m_1, m_2) \in M_1 \times M_2$.

In other words, if principal $j$ decides to delegate, he chooses a subset of the pure strategies $Y_j$. If principal $j$ decides not to delegate, then he chooses a single strategy either pure or mixed (that is denoted as $\delta_j$). In the second stage, the agent makes a choice among the alternatives to which she has been delegated. Note that if no principal chooses to delegate to the agent, then the second stage of the game is superfluous.

A crucial feature of the proposed model is the treatment of mixed strategies. In most of the common agency models, authors do not allow principals to offer stochastic mechanisms. This modeling choice follows a tradition of standard principal agent models and it is motivated by realism, empirical observations as well as simplicity.[13] In a common agency setting, though,

[12]Some results can be extended trivially to the N-person case, but for others the restriction to two players is important. The search for the right argument to extend the results of this paper to an arbitrary but finite number of principals, is an open question.

[13]See, however, Arnott and Stiglitz (1988) and Strausz (2004) for the problems of restricting the analysis to deterministic mechanisms in principal-agent models.

the role of stochastic contracts is different and arguably more important. As it is shown in the previous section, it is possible to have outcomes supported by equilibria in a model where principals are restricted to offer deterministic contracts while if we allow principals to offer also stochastic mechanisms they fail to be supported.[14] This is a distinctive feature of this class of models. In fact, allowing for mixed strategies in this paper serves to simplify the analysis rather than further complicating it.

Agent's strategies and equilibrium concepts

The strategy for the agent, denoted with $\sigma_A$, is a mapping from $\mathcal{M}_1 \times \mathcal{M}_2$ to $\Delta(Y) \cup \Delta(Y_1) \cup \Delta(Y_2)$ such that if both principals choose to delegate then $\sigma_A(M_1 \times M_2) \in \Delta(Y)$, and only if principal $i$ chooses to delegate $\sigma_A(M_1 \times M_2) \in \Delta(Y_i)$.[15]

A strategy profile $(M_1, M_2, \sigma_A)$ induces a finite probability distribution $\mu$ over $Y$ in the following way: $\mu(y) = \sigma_A(M)(y)$ if both principals choose to delegate; $\mu(y) = \sigma_A(M)(y_i)\delta_j(y_j)$ if only principal $i$ chooses to delegate and principal $j$ does not delegate but chooses the (mixed) strategy $\delta_j$; and finally $\mu(y) = \delta_i(y_i)\delta_j(y_j)$ if neither principal delegates. The expected payoffs of principals and the agent are given respectively by $V_j(M_1, M_2, \sigma_A) = \sum_{y \in Y} v_j(y)\mu(y)$ and $U(M_1, M_2, \sigma_A) = \sum_{y \in Y} u(y)\mu(y)$. At times, if there is no ambiguity on the notation, we simply write $V_j(M_1, M_2, \sigma_A) = V_j(\mu)$ and $U(M_1, M_2, \sigma_A) = U(\mu)$

I use standard equilibrium concepts; in particular, I want to compare the set of Nash equilibria $NE(G)$ of the game $G$ and the set of subgame perfect Nash equilibria $SPNE(G_M)$ of the game $G_M$. I will also compare the set of equilibrium payoffs of the principals corresponding to the two games, which I denote respectively with $\Pi_{NE}(G)$ and $\Pi_{SPNE}(G_M)$.

Preliminary results

The first proposition of the paper serves to clarify the role of delegation to a common agent. It states that the game with menus $G_M$ cannot eliminate equilibria of the original game, but only add new equilibria, and this is true for any payoff function of the agent.

**Proposition 1** *For any game $G$, if $\{\delta_1^\star, \delta_2^\star\}$ is a Nash equilibrium of $G$, then there exists a subgame perfect Nash equilibrium of $G_M$ that induces the same distribution over $Y$ (that is $\mu(y) = \delta_1^\star(y_1)\delta_2^\star(y_2)$)*

Proposition 1 is closely related to Theorem 1 in Peters (2003a) that states, in a more general setting, that any pure-strategy equilibrium of the original game $G$ is robust to deviations of a principal to a more complicated strategy (i.e., menus of strategies) than the ones

[14] See Piaser (2005) for an example with a continuum of strategies for principals and a discussion of this phenomenon.

[15] If neither principal delegates it does not matter how we define $\sigma_A(M_1 \times M_2)$.

allowed in the game $G$.[16] The intuition of the result is straightforward: if neither principal is delegating, (in the revelation principle parlance, if they are both offering direct mechanisms to the agent), the agent does not have any way to affect the outcome. If principal $j$ deviates choosing to delegate (that is he offers a menu) the agent must only choose one element of the action space of principal $j$. Given the strategy of the non-deviating principal, there cannot be an action $y_j$ that gives higher payoff to principal $j$ than his Nash equilibrium strategy.

Incidentally, I note that proposition 1 together with the assumption of finiteness of $Y_1$ and $Y_2$, guarantees the existence of a subgame perfect Nash equilibrium for $G_M$.[17] We can express proposition 1 in terms of payoffs, with the following corollary.

**Corollary 1** *For any game G (and in particular for any payoffs of the agent u) we have* $\Pi_{NE}(G) \subset \Pi_{SPNE}(G_M)$.

Corollary 1 tells us that no matter what the payoff function of the agent in the original game $G$ is, any equilibrium payoff for the game $G$ is still an equilibrium payoff for the modified game with menus $G_M$. Therefore, in this setting, the sole purpose of menus is to enlarge the set of equilibrium payoffs.

The main question is whether there exists some class of preferences for the agent that sustains equilibria that Pareto dominate $\Pi_{NE}(G)$. The next proposition provides a positive answer and can be considered a benchmark as it sets an upper bound for the set of equilibrium payoffs sustainable with delegation. Before presenting proposition 2, let me recall the following definitions:[18]

**Definition 1** *Given a game G, a **correlated strategy** is a probability distribution* $\mu \in \Delta(Y)$.

**Definition 2** *Given a game G, the **minimax value** $\underline{v}_i$ for principal i, is given by*
$\underline{v}_i = \min_{\mu_j \in \Delta(Y_j)}(\max_{y_i \in Y_i} \sum_{y_j \in Y_j} v_i(y_i, y_j)\mu_j(y_j))$

**Definition 3** *Given a game G, a correlated strategy $\mu \in \Delta(Y)$ is said to be **individually rational** for player i if* $V_i(\mu) = \sum_{y \in Y} v_i(y)\mu(y) > \underline{v}_i$

**Proposition 2** *If the agent's payoff u is constant, then any individually rational correlated strategy $\mu$ for the game G can be induced by a subgame perfect Nash equilibrium of $G_M$.*

Proposition 2 implies that if the agent is indifferent over the set of outcomes, then $\Pi_{NE}(G)$ is strictly contained in $\Pi_{SP}(G_M)$, which, in turn, coincides with the set of payoffs sustained by

[16]Peters (2003b) offers counterexamples to this result for the case with multiple agents. Also Segal and Whinston (2003), considering equilibria that are robust to different assumptions on the division of bargaining power between principals and agent, obtain a different result.

[17]For work on the existence of an equilibrium in a more general common agency setting, see Page and Monteiro (2003) and Carmona and Fajardo (2006).

[18]See Myerson (1991) pages 244-249.

individually rational correlated strategies for the game $G$. Therefore, this result characterizes environments where the role of delegation is "maximal."[19]

Mixed deviations

Proposition 2 implies the existence of an agent payoff function such that $G_M$ has a set of equilibrium payoffs much larger than does $G$. But it depends on the very particular case of complete indifference of the agent over the set of outcomes $Y$. As soon as we perturb the payoff function of the agent, the results change considerably. The reason is that the requirement for the agent to have an incentive to punish all mixed-strategy deviation restricts the class of agent payoff functions quite severely. This is formalized in the following lemma.

**Lemma 1** *Let $y^*$ be supported by a subgame perfect Nash equilibrium of $G_M$. If there exists an action $y_i$ such that $v_i(y_i, y_j^*) > v_i(y_i^*, y_j^*)$, then there exists an action $y_j \in M_j \subset Y_j$ such that $y_j \neq y_j^*$ and $u(y_i^*, y_j) = u(y_i^*, y_j^*)$.*

Lemma 1 provides necessary conditions for an outcome $y^* \in Y$ to be supported by a subgame perfect Nash equilibrium. It plays a very important role in this paper as it is used to prove most of the remaining results. The lemma does not hold true if the principals are not allowed to have mixed deviations.

In order to understand the role of mixed strategies, consider a candidate for an equilibrium outcome $y^*$. What test does the equilibrium candidate have to pass in order to survive? First of all the agent must find it optimal to chose $y^*$ rather than any of the other outcomes available to her. This condition implies that the agent's payoff function must satisfy a set of (weak) inequalities over the menus offered by the principals (that is $u(y^*) \geq u(y)$ for all $y \in M_1 \times M_2$). Moreover, for any principal who has a pure-strategy deviation in the original game (that is for any $y_i$ such that $v_i(y_i, y_j^*) > v_i(y_i^*, y_j^*)$), the agent must find it profitable to punish the deviation $y_i$. This means that there exists an action $y_j \in M_j$ that acts as a punishment and that the agent finds it profitable to choose over the equilibrium action, that is $u(y_i, y_j) \geq u(y_i^*, y_j)$. These requirements based on inequalities are sufficient to punish pure strategy deviations, but are not sufficient if we allow principals to deviate by choosing a mixed strategy. In order for the agent to have an incentive to punish all the mixed-strategy deviations, some of the inequalities above must in fact be equalities. This observation alone (formalized in lemma 1) severely restricts the class of agent preferences for which menus add to equilibrium payoffs.

The first application of lemma 1 provides a sufficient condition, namely agent genericity, under which the equilibrium payoffs of $G_M$ coincide with equilibrium payoffs of $G$. The

[19]In fact proposition 2 is a reinterpretation of a well-known result: consider a normal form game $G$ where the players are allowed to write contracts with each other. Then any individually rational correlated strategy profile for the game $G$ is obtainable as an equilibrium of the "game with contracts" (see Myerson 1991).

condition requires that if a principal changes his strategy, while the other principal does not, the agent's payoff must change.

**Definition 4** *A game $G$ is called **agent-generic** if for any $y_j \in Y_j$, $y_i \neq y_i'$ implies $u(y_i, y_j) \neq u(y_i', y_j)$*

The idea of the agent-genericity condition is that the agent's payoff function is sensitive to each principal's strategy. Moreover, this condition is sufficient to obtain that delegation does not add any equilibria.

**Proposition 3** *If $G$ is agent-generic, then $\Pi_{SPNE}(G_M) = \Pi_{NE}(G)$.*

The proof of proposition 3 is an immediate consequence of lemma 1. In fact, lemma 1 not only states that a necessary condition for an outcome to be supported by an equilibrium of $G_M$ (but not of $G$) is that the agent must be indifferent over at least two outcomes; it also says that this indifference must be over two outcomes generated by the same strategy of a principal. Roughly speaking, the agent must be indifferent over at least two elements of the same column (or row). Thus, lemma 1 implies that agent-generic games cannot have an equilibrium in $G_M$ that is not an equilibrium in $G$.

Proposition 3 is the complement of proposition 2. It shows the contrast with the situation of complete indifference of the agent. While delegating to an agent who is indifferent enlarges the set of equilibrium outcomes, delegation to an agent who is not indifferent enough does not add new equilibrium outcomes. In other words, propositions 2 and 3 together set an upper bound and a lower bound for the set of equilibrium payoffs sustainable by delegation as the agent's payoff function changes.

The results of this section are very sharp and can be interpreted in two ways. First, if principals cannot offer monetary transfers to the common agent, delegation is not likely to produce any equilibrium outcome different from those that can be achieved without delegation. The second message of this section is that in this simplified environment, the revelation principle is valid for a large class of preferences of the agent.

In the next section I will enrich the delegation game, allowing principals to offer monetary transfers to the agent contingent on the action chosen by the agent within the menu. This makes the model of the next section very similar to the models of private common agency. Notice that if the principals can offer monetary transfers to the agent, it is then possible to generate endogenously the required indifference of the agent. Nevertheless, lemma 1 also has important consequences in this version of the model.

## 4 Delegation with monetary transfers

In this section, I consider the complete model in which I allow the principals to offer menus of their own actions together with a monetary transfer to the agent conditional upon which element of the menu the agent selects. That is, I enrich the game with menus $G_M$ allowing each principal to associate any action within the menu with a (possibly different) monetary transfer to the agent. An element $m_i$ of the menu $M_i$ is now a couple $(y_i, t_i)$ where $y_i \in Y_i$ is one of the actions available to principal $i$ and $t_i \in \mathbb{R}$ is a monetary transfer from the principal to the agent. I allow the monetary transfer to be negative, but impose the restriction that at least one $m_i = (y_i, t_i) \in M_i$ is such that $t_i \geq 0$. More formally, the strategy space of principal $i$ can be written as $\mathcal{M}_i \cup \Delta(Y_i)$ where

$$\mathcal{M}_i \equiv \{(M_i, f_i) | M_i \subset Y_i, f_i \in \mathbb{R}^{M_i}, \text{ such that } \forall M_i, \exists m_i \in M_i : f_i(m_i) \geq 0\}$$

The payoffs of principals and the agent, if the elements $(y_i, t_i)$ and $(y_j, t_j)$ are chosen, have the following standard expressions

$$\widetilde{v}_i(y_i, y_j, t_i, t_j) = v_i(y_i, y_j) - t_i$$
$$\widetilde{u}(y_i, y_j, t_i, t_j) = u(y_i, y_j) + t_i + t_j$$

In the previous section it has been shown that without monetary transfers, for almost all preferences of the agent, each equilibrium outcome of the game with delegation is also an equilibrium outcome of the original game. The key to this result is that in order for the agent to have an incentive to punish all mixed strategy deviations of one principal, she must be indifferent among the elements of the menu that the other principal offers. Introducing monetary transfers change the results because that indifference can be created endogenously by the principals. Nevertheless, the condition required by lemma 1 restricts the outcomes that are supported by menus even with monetary transfers. The restrictions differ depending on the payoff function of the agent; in particular I analyze how the results differ between the cases of supermodular and submodular payoff functions.

In this section I restrict the analysis to the class of games $G$ that satisfy **negative externalities**, that is games for which the payoff function of each player is non-increasing in the strategy of the other player. More precisely, I assume that for any $j \in \{1, 2\}$, $(Y_j, \geq_j)$ is a linearly ordered set (a chain) so that $y'_j \geq_j y_j$ implies $v_i(y_i, y_j) \geq v_i(y_i, y'_j)$, for all $y_i \in Y_i$.

It must be noted that while I present the results for games with negative externalities, the results are readily applicable to games with positive externalities (the payoff function of each player is non-decreasing in the strategy of the other player).[20] The class of games that satisfy this condition includes Cournot duopoly, Betrand duopoly (both with homogeneous and differentiated goods) as well as the tragedy of commons and many others, while it does not include games such as pure coordination or battle-of-the-sexes.

Given the orders $\geq_j$ over $Y_j$ for which the game $G$ satisfies negative externalities I focus attention on payoff functions of the agent that are monotone in differences, that is on pay-

[20] It suffices to consider, for $j \in \{1, 2\}$, the dual ordered sets $(Y_j, \geq'_j)$ where $y'_j \geq'_j y''_j$ if and only if $y''_j \geq_j y'_j$.

off functions that have either increasing differences (supermodular) or decreasing differences (submodular).

The rest of the section is organized as follows: proposition 4 establishes that for the knife-edge case of payoff functions with constant differences, the role of delegation is maximal, that is, every individually rational outcome can be achieved as an equilibrium outcome of the delegation game. If the assumption of constant differences is violated, the results change dramatically and depend whether the agent's payoff function is supermodular or submodular. Lemma 2 establishes that if the agent's payoff function is strictly supermodular, only outcomes in which at least one of the two principals is using his best response strategy in $G$ can be supported as equilibria of the delegation game. Lemma 2 has no counterpart for the case of submodularity and this is the reason why the results in this section differ between these two cases. Propositions 5 and 6 provide respectively sufficient and necessary conditions for an outcome to be an equilibrium of the game with delegation for the cases of a strictly supermodular agent's payoff function, while for the case of a strictly submodular agent's payoff function, the same results are presented in proposition 7 and 8. Lastly, I discuss the consequences of these results on efficiency and present an example of an equilibrium in which the agent randomizes among elements of the menu $M_1 \times M_2$ and that Pareto-dominates the outcome of the original game.

Let me recall the definition of a function satisfying constant differences:

**Definition 5** *The function $u$ satisfies constant differences over $Y$, if for any $y_1, y_1' \in Y_1$ and any $y_2, y_2' \in Y_2$, $u(y_1, y_2) - u(y_1, y_2') = u(y_1', y_2) - u(y_1', y_2')$*

**Proposition 4** *If $u$ satisfies constant differences, then any individually rational outcome $y \in Y$ is supportable as an equilibrium outcome of the game $G_M$.*

The above proposition tells us that if principals can offer monetary transfers to the agent and $u$ satisfies the assumption of constant differences then "everything is achievable". This "static version" of the folk theorem is a recurring theme for models that fall between non-cooperative and cooperative game theory.[21] Moreover, this result highlights the particular role that the payoff functions satisfying constant differences have in this analysis: on one hand, efficiency can be reached by means of delegation, notwithstanding the imposed limits on contracting. On the other hand, the role of mixed strategy deviations is null.

### Supermodular Agent Preferences

Since I focus attention on payoff functions of the agent that are monotone in differences, if the assumption of constant differences is violated, there are two cases left: the class of payoff

[21]See for example Fershtman, Judd and Kalai (1991) who obtain it in a model where each principal has his own agent.

functions $u$ with strictly increasing differences ($u$ strictly supermodular), and the class of payoff functions $u$ with strictly decreasing differences ($u$ strictly submodular). Let me recall the following definition:[22]

**Definition 6** *If the function $u(y_i, y'_j) - u(y_i, y_j)$ is increasing, decreasing, strictly increasing or strictly decreasing in $y_i$ for all $y'_j \geq_j y_j$ in $Y_j$, then $u(y_i, y_j)$ has respectively increasing differences, decreasing differences, strictly increasing differences or strictly decreasing differences over $(Y_i, Y_j)$.*

Since in the framework of our model, the definition of increasing differences coincides with the one of supermodularity, I use those terms interchangeably.[23]

**Lemma 2** *Assume that $u$ is strictly supermodular; if $(y_i, y_j)$ is supported by a subgame perfect equilibrium of the delegation game, then $y_i = BR_i(y_j)$ or $y_j = BR_j(y_i)$.*

Lemma 2 establishes that a necessary condition for an outcome $(y_i, y_j)$ to be supported by an equilibrium of the delegation game is that at least one of the two principals is playing a best response strategy in the original game. Clearly, if this is true for both principals, the outcome $(y_i, y_j)$ is a Nash equilibrium of the original game. Therefore lemma 2 implies that if an equilibrium of the delegation game supports an outcome $(y_i, y_j)$ that is *not* a Nash equilibrium of the original game, then this outcome is such that one principal is playing his best response strategy, while the other is not. This "asymmetry" is best appreciated, if the original game is symmetric. As an immediate corollary of lemma 2, it can be established that a symmetric equilibrium of the delegation game for a symmetric game $G$ must be a Nash equilibrium of the original game.

Lemma 2 does not hold true if we do not allow the principals to have mixed deviations. One example of this was given in section 2 in the prisoners' dilemma with strictly increasing differences where the (symmetric) cooperative outcome is achievable if the principals cannot deviate to mixed strategies, but it cannot be achieved if mixed deviations are allowed.[24] In fact lemma 2 is obtained using lemma 1 (it is a consequence of allowing principals to deviate using a mixed strategy), and simplifies considerably the task of identifying the equilibrium outcomes of the game with delegation.

Lemma 2 provides a much stricter upper bound over the outcomes that can be supported by an equilibrium of the game with delegation.[25] To restrict further the bounds we need to look at the set of outcomes $(y_i, y_j)$ such that $y_i \in BR_i(y_j)$ and $y_j \notin BR_j(y_i)$ and identify those which can be supported by an equilibrium of the delegation game. Proposition 5 and

[22] See Topkis (1998) section 2.6.
[23] See Theorem 2.6.1 and Corollary 2.6.1 in Topkis (1998).
[24] The case $u_{CC} + u_{DD} > u_{CD} + u_{DC}$ corresponds to the increasing differences case.
[25] Recall that proposition 4 identifies the upper bound as the set of individually rational outcomes.

6 achieve this goal; proposition 5 provides sufficient conditions and proposition 6 necessary conditions for an outcome to be supported as an equilibrium of the delegation game.

**Proposition 5** *Assume that $u$ is strictly supermodular and let $(y_i^\star, y_j^\star)$ be an individually rational outcome such that $y_i^\star = BR_i(y_j^\star)$. If the following two conditions hold then $(y_i^\star, y_j^\star)$ can be supported as a subgame perfect Nash equilibrium of $G_M$.*

1. *No principal has a* ***lower strategy deviation*** *in $G$; that is $y_j^\star >_j y_j$ implies $v_j(y_i^\star, y_j) \leq v_j(y_i^\star, y_j^\star)$;*
2. *$\exists y_i^p \in Y_i$ such that $\forall \delta_j \in \Delta(Y_j)$ for which there exists a $y_j' <_j y_j^\star$ in the support of $\delta_j$, $\sum_{y_j} v_j(y_i^\star, y_j)\delta_j(y_j) > v_j(y_i^\star, y_j^\star)$ implies*
   *(a) $\sum_{y_j}[u(y_i^p, y_j) - u(y_i^\star, y_j)]\delta_j(y_j) \geq u(y_i^p, y_j^\star) - u(y_i^\star, y_j^\star)$,*
   *(b) $\sum_{y_j} v_j(y_i^p, y_j)\delta_j(y_j) \leq v_j(y_i^\star, y_j^\star)$.*

The conditions above are sufficient and quasi-necessary as the next proposition clarify:

**Proposition 6** *Assume that $u$ is strictly supermodular and let $(y_i^\star, y_j^\star)$ be an individually rational outcome such that $y_i^\star = BR_i(y_j^\star)$. The outcome $(y_i^\star, y_j^\star)$ can be supported as a subgame perfect Nash equilibrium of $G_M$ only if:*

1. *No principal has a* ***lower strategy deviation*** *in $G$; that is $y_j^\star >_j y_j$ implies $v_j(y_i^\star, y_j) \leq v_j(y_i^\star, y_j^\star)$;*

*2bis. $\forall \delta_j \in \Delta(Y_j)$ for which there exists a $y_j' <_j y_j^\star$ in the support of $\delta_j$ such that $\sum_{y_j} v_j(y_i^\star, y_j)\delta_j(y_j) > v_j(y_i^\star, y_j^\star)$, $\exists y_i^p \in Y_i$ such that*
   *(a) $\sum_{y_j}[u(y_i^p, y_j) - u(y_i^\star, y_j)]\delta_j(y_j) \geq u(y_i^p, y_j^\star) - u(y_i^\star, y_j^\star)$,*
   *(b) $\sum_{y_j} v_j(y_i^p, y_j)\delta_j(y_j) \leq v_j(y_i^\star, y_j^\star)$.*

The condition 1 in proposition 5 and 6 is easy to understand: with $u$ strictly supermodular, the incentive of the agent to punish a deviation to a "lower" strategy is incompatible with the agent choosing the equilibrium outcome from the available alternatives in the menu. Condition 1 would still be a necessary condition even if we restrict principals to deviate only by using pure strategies. Moreover, since lemma 2 restricted the set of outcomes that can be supported by an equilibrium of the delegation game to outcomes where one principal is using a best response, then condition 1 also becomes a sufficient condition if we only consider pure deviations for principal $j$.

Condition 2 in proposition 5 (and 2bis in proposition 6) comes to play a role because principal $j$ might deviate to a mixed strategy with a support that contains $y_j <_j y_j^\star$. It guarantees that the outcome $(y_i^\star, y_j^\star)$ is robust to such deviations. If the support of the mixed strategy deviations contains lower strategies $(y_j <_j y_j^\star)$, then supermodularity of the agent's payoff function is not enough to guarantee that the agent has an incentive to choose a "punishment" $y_i^p$ from the menu of the non-deviating principal $i$. It is necessary to introduce a condition that compares the payoff function of the agent and that of principal $j$ (the principal who has a profitable deviation in $G$). Proposition 5 has an immediate and important corollary.[26]

**Corollary 2** *Assume that $u$ is supermodular and that the outcome $(y_i^1, BR_j(y_i^1))$ is individually rational. Then $(y_i^1, BR_j(y_i^1))$ is supported as a subgame perfect Nash equilibrium of the delegation game $G_M$.*

**Proof.** Since $y_i^1$ is the lowest element in $Y_i$ (that is for any $y_i \in Y_i$ we have $y_i \geq_i y_i^1$) condition (1) of proposition 5 is satisfied. Also for any $\delta_i \in \Delta(Y_i)$ the support of $\delta_i$ cannot contain any $y_i <_i y_i^1$ so that also condition (2) of proposition 5 is trivially satisfied. ■

One way to interpret Proposition 5 and 6 is that they provide a lower and upper bound for the outcomes that can be supported by an equilibrium of the game with delegation, when the agent payoff function is strictly supermodular. I first focus on the upper bound. Take a game $G$ and an individually rational outcome $(y_i^\star, y_j^\star)$ such that $y_j^\star = BR(y_i^\star)$ and that no principal has a lower strategy deviation. If $(y_i^\star, y_j^\star)$ is not supported by an equilibrium of the game with delegation, can we find another strictly supermodular agent payoff function such that $(y_i^\star, y_j^\star)$ is supported by an equilibrium of the delegation game? Proposition 5 and 6 suggest that the answer is in the affirmative.

In other words in the class of the strictly supermodular agent payoff functions, the upper bound of the $SPNE(G_M)$ is given by all the individually rational outcomes $(y_i^\star, y_j^\star)$ such that $y_j^\star = BR(y_i^\star)$ and that no principal has a lower strategy deviation.

An analogous reasoning can be carried out for the lower bound of the $SPNE(G_M)$. If an outcome is supported by a subgame perfect Nash equilibrium of $G_M$, can we find another strictly supermodular agent payoff function such that $(y_i^\star, y_j^\star)$ is not supported by an equilibrium of the delegation game? We already know from proposition 1 that this is not possible if $(y_i^\star, y_j^\star)$ is a Nash equilibrium of the game without delegation. Corollary 2 excludes two more outcomes.

In other words in the class of the strictly supermodular agent payoff functions, the lower bound of the $SPNE(G_M)$ is composed by the set of outcomes supported by the Nash equilibrium of the original game (proposition 1) and by the two outcomes $(y_i^1, BR_j(y_i^1))$ and $(BR_i(y_j^1), y_j^1)$ as long as they are individually rational (Corollary 2).

It is worth stressing the meaning of this finding. Given the assumption of negative externalities the outcome $(y_i^1, BR_j(y_i^1))$ is not only Pareto-efficient but also the outcome that

[26] I denote with $y_i^1$ the first (that is lowest) strategy for principal $i$ given the order $\geq_i$.

gives the highest payoff to principal $i$ among all the outcomes $y \in Y$. If this outcome is individually rational, then it is always possible to support it as an equilibrium with delegation. This asymmetric outcome is the only one that is supported by any supermodular agent payoff function. This finding also bears remarkable consequences for the validity of the revelation principle in common agency. In section 3, it was shown that (without monetary transfers) the introduction of mixed deviations was enough to restore the validity of the revelation principle for a large class of preferences of the agent. In this section, it has been shown that when the complete model is considered, there are always outcomes that cannot be supported as equilibria of the original game, but are supported by the principals offering menus. It must be added that the condition of individual rationality for the outcomes $(y_i^1, BR_j(y_i^1))$, and $(BR_i(y_j^1), y_j^1)$, translates in different ways, in various application of the model. For example in the prisoners' dilemma, the two outcomes $(C, D)$ and $(D, C)$ cannot be supported as they are not individually rational. Conversely, in the duopoly example in section 5, it means that (with a concave cost function) it is always possible to support an "exclusion" equilibrium, in which the firm $i$ produces zero output and the firm $j$ produces the monopoly outcome.

### Submodular Agent Preferences

The case of submodular preferences differs in many respects from the case of supermodularity. One crucial difference is that allowing the principals to have mixed strategy deviations does not restrict the outcomes supportable as equilibria in the same way. With $u$ submodular, an outcome $(y_i, y_j)$ can be supported by an equilibrium of the delegation game even if neither principal is using a best response strategy. In other words, there is no counterpart of lemma 2 when $u$ is submodular. To see why this is the case, consider an outcome $(y_i, y_j)$ such that both principals have a profitable deviation in $G$. As an immediate corollary of lemma 1, for each principal there exists an action $y_i^p$ that acts as a punishment and is part of the equilibrium menu that supports the outcome $(y_i, y_j)$. But in order to be a punishment, $y_i^p$ must be higher than $y_i$ (and the same is true for the other principal). This, together with the fact that the agent must be indifferent between the element of the menus that is chosen in equilibrium and those that can be used as punishment (lemma 1), implies a contradiction when $u$ is supermodular, but not when $u$ is submodular.

This does not mean that for the case of $u$ submodular, the introduction of mixed strategy deviations for the principals does not restrict the outcomes that can be supported as an equilibrium of $G_M$. The restriction is rather of second order, as the next propositions clarify. Proposition 7 provides sufficient conditions and proposition 8 necessary conditions for an outcome to be supported as an equilibrium of the delegation game.

**Proposition 7** *Assume that $u$ is strictly submodular and let $(y_i^\star, y_j^\star)$ be an individually rational outcome. If the following two conditions hold then $(y_i^\star, y_j^\star)$ can be supported as a subgame* *perfect Nash equilibrium of $G_M$.*

1. *No principal has a* ***higher strategy deviation*** *in $G$; that is $y_i > y_i^\star$ implies $v_i(y_i, y_j^\star) \leq v_i(y_i^\star, y_j^\star)$;*

2. *$\exists y_i^p \in Y_i$ such that $\forall \delta_j \in \Delta(Y_j)$ for which there exists a $y_j' >_j y_j^\star$ in the support of $\delta_j$, $\sum_{y_j} v_j(y_i^\star, y_j)\delta_j(y_j) > v_j(y_i^\star, y_j^\star)$ implies*

   *(a) $\sum_{y_j}[u(y_i^p, y_j) - u(y_i^\star, y_j)]\delta_j(y_j) \geq u(y_i^p, y_j^\star) - u(y_i^\star, y_j^\star)$,*

   *(b) $\sum_{y_j} v_j(y_i^p, y_j)\delta_j(y_j) \leq v_j(y_i^\star, y_j^\star)$.*

Also in this case, the conditions above are sufficient and quasi-necessary:

**Proposition 8** *Assume that $u$ is strictly submodular and let $(y_i^\star, y_j^\star)$ be an individually rational outcome. The outcome $(y_i^\star, y_j^\star)$ can be supported as a subgame perfect Nash equilibrium of $G_M$ only if:*

1. *No principal has a* ***higher strategy deviation*** *in $G$; that is $y_i > y_i^\star$ implies $v_i(y_i, y_j^\star) \leq v_i(y_i^\star, y_j^\star)$;*

*2 bis. $\forall \delta_j \in \Delta(Y_j)$ for which there exists a $y_j' >_j y_j^\star$ in the support of $\delta_j$ and such that $\sum_{y_j} v_j(y_i^\star, y_j)\delta_j(y_j) > v_j(y_i^\star, y_j^\star)$, $\exists y_i^p \in Y_i$ such that*

   *(a) $\sum_{y_j}[u(y_i^p, y_j) - u(y_i^\star, y_j)]\delta_j(y_j) \geq u(y_i^p, y_j^\star) - u(y_i^\star, y_j^\star)$,*

   *(b) $\sum_{y_j} v_j(y_i^p, y_j)\delta_j(y_j) \leq v_j(y_i^\star, y_j^\star)$.*

Condition 1 of proposition 7 and 8 excludes outcomes for which any principal has a higher strategy deviation: with $u$ submodular, the incentive to punish a deviation to a higher strategy is incompatible with the agent choosing the equilibrium strategies within the menu. Condition 1 does not depend on the ability of principals to use mixed strategy deviations whereas condition 2bis is necessary because principals could deviate by using a mixed strategy whose support includes pure strategies higher than the equilibrium strategy. As in the case of $u$ supermodular, proposition 7 and 8 can be interpreted as providing a lower bound and an upper bound to $SPNE(G_M)$. The lower bound is composed by the set of outcomes supported by the Nash equilibrium of the original game and by the two outcomes $(y_i^I, BR_j(y_i^I))$, and $(BR_i(y_j^J), y_j^J)$ as long as they are individually rational as it is stated in the following corollary of proposition 7.[27]

[27] I denote with $y_i^I$ the last (that is highest) strategy for principal $i$ and $y_i^J$ the highest strategy for principal $j$, given the orders $\geq_i$ and $\geq_j$.

**Corollary 3** *Assume that $u$ is submodular and that the outcome $(y_i^I, BR_j(y_i^I))$ is individually rational. Then $(y_i^I, BR_j(y_i^I))$ is supported as a subgame perfect Nash equilibrium of the delegation game $G_M$.*

**Proof.** Since $y_i^I$ is the highest element in $Y_i$ (that is for any $y_i \in Y_i$ we have $y_i^I >_i y_i$) condition (1) of proposition 7 is satisfied. Also for any $\delta_i \in \Delta(Y_i)$ the support of $\delta_i$ cannot contain any $y_i >_i y_i^I$ so that also condition (2) of proposition 7 is trivially satisfied. ■

The upper bound of $SPNE(G_M)$ is potentially very large: it is composed of all the individually rational outcomes for which no principal has higher strategy deviations. Recall that for $u$ supermodular the upper bound is composed of all the individually rational outcomes for which no principal has lower strategy deviations *and* at least one principal is using his best response in $G$.

Let me stress the relevance of this finding: one of the objectives of the paper is to test the robustness of the models that use only pure strategy to the introduction of mixed strategy deviations. Propositions 5 through 8 allow us to draw the following conclusions: if the payoff function of the agent is supermodular, then it is unlikely that outcomes supported by equilibria in $G_M$ are robust to the introduction of mixed strategy deviations. Conversely, for submodular agent's payoff functions, the answer depends on the relation between the payoff function of the principals and that of the agent.[28]

Efficiency Considerations and Equilibria in which the Agent Randomizes

One of the themes of this paper is to asses whether delegation to a common agent may improve the efficiency of the equilibrium outcome of a game. In particular, assume that the original game $G$ has a unique equilibrium outcome $(y_i^\star, y_j^\star)$. Does there exist an outcome $(y_i, y_j)$ supported by an equilibrium of the delegation game that Pareto-dominates $(y_i^\star, y_j^\star)$?

For $u$ strictly supermodular, delegation to an agent can always result in an equilibrium outcome $(y_i, y_j)$ such that one principal chooses his lowest action (that is $y_i = y_i^1$) and the other principal chooses his best response to it (that is $y_j = BR_j(y_i^1)$). As already pointed out, this outcome gives the highest payoff to one of the two principals (the one who is best responding) while the payoff of the other principal is ambiguous; if the original game $G$ is such that $v_i(y_i^1, BR_j(y_i^1)) \geq v_i(y_i^\star, y_j^\star)$, then delegation to a common agent can achieve a Pareto improvement over the original game. The same reasoning holds if one considers the other outcomes that can be supported as an equilibrium of the delegation game. By proposition 6, the upper bound of this set is given by the outcomes $(y_i, y_j)$, such that one principal is using his best response and the other principal does not have a lower deviation. This characterization

[28]One way to express this concept is the following: for $u$ submodular the upper bound of $SPNE(G_M)$ coincides with the set of equilibrium outcomes of a model without mixed deviations for the principals; for $u$ supermodular, the upper bound of $SPNE(G_M)$ is strictly included in the set of equilibrium outcomes of a model without mixed deviations for the principals.

in itself does not exclude the possibility that some of the equilibria with delegation might Pareto-dominate the equilibria in the original game. In general this will depend on the game $G$. In the duopoly example in section 5, none of the outcomes supported by delegation Pareto dominate the unique Cournot-Nash equilibrium of the original game, as the principal who is best-responding receives a payoff higher than the Cournot-Nash payoff, while the other principal receives a payoff lower than the Cournot-Nash one.

For $u$ strictly submodular, consider first the lower bound of $SPNE(G_M)$: the two outcomes that compose this set (apart from the Nash equilibrium of $G$) are such that one principal chooses his highest action (that is $y_i = y_i^I$) and the other principal chooses his best response to it (that is $y_j = BR_j(y_i^I)$). These outcomes have the feature that, for the principal $j$ (for whom the action $y_j$ is the best response in $G$) the payoff achieved is equal to the minimax value $\underline{v}_j$; therefore, such outcomes can never Pareto dominate the Nash equilibrium outcome.

Now I turn the attention to the upper bound of $SPNE(G_M)$. Notice that even if the set of outcomes that can be supported by an equilibrium of the game with delegation is large, such a set can be restricted considerably, if one is interested in the Pareto-efficient frontier. The reason is that any outcome $(y_i, y_j)$ such that neither one of the two principals has a deviation to a higher strategy, is Pareto-dominated by an outcome $(y_i, y_j')$, where $y_j' = BR(y_i)$. This implies that the Pareto-frontier of the set of outcomes supported by $SPNE(G_M)$ is included in the set of individually rational outcomes such that one principal is best responding and the other one does not have a deviation to a higher action (that is, $(y_i, y_j)$ such that $y_i = BR_i(y_j)$ and $v_j(y_i, y_j) \geq v_j(y_i', y_j)$ for all $y_i' >_i y_i$). As in the case of $u$ supermodular, this characterization does not exclude the possibility that some of the equilibria with delegation might Pareto-dominate the equilibria in the original game. Again I refer to section 5 for an example in which this is not the case.

Until now I focused my attention on equilibria in which the agent chooses a single outcome with probability one. It is well known that in common agency, the principals can use the agent as a correlating device (see for example Martimort and Stole 2002, Peters 2001, and 2003a) and this can create equilibria sustainable with menus but not with simple contracts. If $u$ is submodular restricting attention to equilibria in which the agent chooses an outcome with probability one is with loss of generality. That is, allowing the agent to randomize over the menus offered enlarges the set of payoff profiles supported by an equilibrium in the game with delegation. Moreover, some of these equilibria Pareto dominate the Nash equilibrium of the original game. I show this phenomenon by means of the example used in section 2 (prisoners' dilemma with an interested mediator). Consider the following payoffs of the agent: $u_{CC} = 3$, $u_{CD} = u_{DC} = 2$ and $u_{DD} = 0$. If each principal chooses to offer to the agent the following menu $M_i = \{(C, t_i^c = 0); (D, t_1^d = 2)\}$, the payoffs for the principals and the agent are

| | $(C,0)$ | $(D,2)$ |
|---|---|---|
| $(C,0)$ | $4,4,3$ | $0,4,4$ |
| $(D,2)$ | $4,0,4$ | $-1,-1,4$ |

Consider the following strategy of the agent $\sigma_A$: choose $\{(D,2),(C)\}$ with probability $\frac{1}{2}$ and $\{(C),(D,2)\}$ with probability $\frac{1}{2}$ if the principals have chosen the above menus $(M_1, M_2)$; if one of the principal deviates, choose $(D,2)$ from the non-deviating principal. The strategy profile $(M_1, M_2, \sigma_A)$ is a subgame perfect Nash equilibrium of the game with menus and the equilibrium payoffs are $(2,2,4)$.

The example shows that if the agent can randomize between the two elements of the menus that correspond to the outcomes $(C,D)$ and $(D,C)$ it is possible to achieve an equilibrium that Pareto dominates the Nash equilibrium of the original game and that is robust to both pure and mixed strategy deviations.[29] An interesting feature of this example is that in order to achieve this cooperative equilibrium the principals must pay a rent to use the agent as a randomizing device between the two asymmetric outcomes $(C,D)$ and $(D,C)$.

The existence of such correlation rents for the agent (and correlation costs for the principals) is a distinctive feature of the model presented in this paper and bears some interesting consequences. First, the agent receives a rent even if there is no asymmetric information between principals and agent. Second, these kind of equilibria can be sustained as long as the "correlation cost" (given by $u_{CD} - u_{DD}$) is not higher relative to the benefit of correlation.[30] This creates an interesting effect on efficiency.

# 5 Example: duopoly with common supplier

In this section, I present a simple model of Cournot duopoly where the two firms have a common supplier. This example is very similar to the models of private common agency in Martimort and Stole (2003) and of bidding games in Segal and Whinston (2003). This section serves, then, two purposes: on one hand, it illustrates the results obtained in the previous section by means of a simple example and on the other hand, it contrasts such results with the ones known in the literature.

Consider a duopolistic market in which the inverse demand function is given by $P(Q_1, Q_2) = max\{A - B(Q_1 + Q_2), 0\}$ where $P$ is the market price and $Q_i$ is the $i$th firm's output. Assume that both firms can buy any (integer) quantity $Q_i$ from a common manufacturer paying the

[29] The example is very close to the one in Peters (2003a) who shows that there exist equilibria in which the agent randomizes that are robust to both pure and mixed strategy deviations. The only difference is that in this example, the payoffs of the principals Pareto dominate the Nash equilibrium of the original game, while in the Peter's example the opposite is true.

[30] For a symmetric payoff function of the agent, the condition to have an equilibrium with correlation is $u_{CD} - u_{DD} < v_1(D,C) + v_1(C,D) - 2v_1(D,D)$.

per unit price $m$. The $i$th firm's profit function is given by $\Pi_i(Q_1, Q_2) = P(Q_1, Q_2)Q_i - mQ_i$. The common supplier's payoff function is $\Pi_A(Q_1, Q_2) = mQ_1 + mQ_2 - C(Q_1, Q_2)$ where $C(Q_1, Q_2)$ is the common agent's cost function. For the sake of simplicity, we further assume that $C(Q_1, Q_2) = \alpha(Q_1 + Q_2) + \beta(Q_1 + Q_2)^2$.

This model is an example of a game with externality $G$. The unique Nash equilibrium of the game $G$ is $Q_1^\star = Q_2^\star = \frac{(A-m)}{3B}$ and the equilibrium payoffs are $\Pi_1^\star = \Pi_2^\star = \frac{(A-m)^2}{9B}$ for the two firms and $\Pi_A^\star = m(\frac{2(A-m)}{3B}) - C(\frac{2(A-m)}{3B})$ for the common agent.[31]

In the resulting delegation game $G_M$, each of the two firms offers a menu $M_i$ to the common supplier, where the generic element $m_i \in M_i$ is composed of an integer quantity and a transfer, that is $m_i = (Q_i, t_i)$; the agent observes the menus $M_1$ and $M_2$ and chooses an element $m_1 \in M_1$ and $m_2 \in M_2$. In other words, the delegation game $G_M$ is equivalent to a model in which each competing firm $i$ offers a non linear price schedule $t_i(Q_i)$ to the manufacturer who then chooses to produce the quantities $(Q_1, Q_2)$ receiving monetary transfers from two firms for an amount $t_1(Q_1) + t_2(Q_2)$.[32]

This model differs from the ones presented in Martimort and Stole (2003) and Segal and Whinston (2003) in two important aspects: first, in their models the principals cannot offer mixed strategies. The second important difference regards the outside option of the agent. In the Martimort and Stole (2003) and Segal and Whinston (2003) models, as in all existing models of common agency, it is assumed that the agent has the option to reject the offers made by the principals. The interpretation is that while the principal has the bargaining power to make an offer, the agent can always ensure her reservation utility (the utility she receives rejecting both offers). In other words, the status quo is a no-trade outcome, to which the agent can decide to resort. In the model presented in this paper, the status quo can be seen as an agreement to trade at a given unitary price $m$. The agent cannot refuse to trade at a unitary price higher than $m$, but she can refuse to trade at a unitary price lower than $m$. This is why I assume that in $G_M$, the principals must offer acceptable menus, that is menus that contain at least one element $m'_i = (Q'_i, t'_i)$ with $t'_i(Q'_i) \geq mQ'_i$.

I present the results obtained in this model first for the case of $C(Q)$ (strictly) concave ($\beta < 0$), and then for the case of $C(Q)$ (strictly) convex ($\beta > 0$).[33]

I call the maximum collusive output any pair $(Q_1, Q_2)$, such that the total output is equal to the monopoly one and I denote with $\Pi^M$ the corresponding maximum collusive profits that the firms can achieve.

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[31] For simplicity, I am assuming that the parameters of the model are such that $\frac{(A-m)}{3B}$ is an integer.

[32] The resulting payoffs for firm $i$ are $\Pi_i(Q_1, Q_2) = P(Q_1, Q_2)Q_i - t_i(Q_i)$ and for the agent $\Pi_A(Q_1, Q_2) = t_1(Q_1) + t_2(Q_2) - C(Q_1, Q_2)$.

[33] The case of $C(Q)$ linear, ($\beta = 0$), is trivial as proposition 4 immediately implies that any individually rational outcome can be achieved as a $SPNE(G_M)$, and this result holds whether or not we allow mixed strategy deviations.

Strictly concave cost function

If $C(Q)$ is strictly concave we can apply the results obtained in section 4 obtaining:

**Corollary 4** *If the cost function of the agent $C(Q)$ is strictly concave ($\beta < 0$) then*

1. *The maximum collusive outcome can be obtained in equilibrium if and only if one of the two firms produce zero output and receive zero profits.*
2. *There is no equilibrium outcome of $G_M$ that Pareto improves over the Nash equilibrium of the original game.*

**Proof.**
[1] The if part follows immediately from corollary 2. The only if part follows lemma 2 from which we know that we can focus on outcomes in which $Q_j = BR(Q_i)$. Therefore we can write the total profits that can be achieved in a $SPNE(G_M)$ as a function of $Q_i$ only:

$$\Pi(Q_i) = \Pi_i(Q_i, BR(Q_i)) + \Pi_j(Q_i, BR(Q_i))$$

The result now follows from the fact that $\Pi$ is a decreasing function of $Q_i$.
[2] From proposition 6 we know that all the outcomes that can be supported as a $SPNE(G_M)$ must have one firm choosing $Q_i \leq \frac{(A-m)}{3B}$ and the other choosing $Q_j = \frac{A-m-BQ_i}{2B}$. This implies that $Q_i + Q_j = \frac{A-m}{2B} + \frac{1}{2}Q_i$. Therefore the profits of firm $i$ can be written as

$$\Pi_i(Q_i) = (A - B(\frac{A-m}{2B} + \frac{1}{2}Q_i))Q_i - mQ_i = (\frac{A-m}{2})Q_i - \frac{B}{2}Q_i^2$$

Therefore the profits are increasing in $Q_i$ as long as $Q_i < \frac{A-m}{2B}$. Since at $Q_i = \frac{A-m}{3B}$ the profits for firm $i$ are equal to the ones obtained in the Nash equilibrium of $G$ (that is $\Pi_i^\star = \frac{(A-m)^2}{9B}$), this implies that the profits for firm $i$ (the firm that is not best responding) are less than the profits obtained in the Nash equilibrium of $G$. ■

Strictly convex cost function

The case of a convex cost function is the one considered by both Martimort and Stole (2003) and Segal and Whinston (2003). In particular from these papers, we know that if we focus on symmetric outcomes ($Q_i = Q_j$), all the outcomes between the Cournot outcome and the competitive outcome (that is $(Q_i, Q_j)$ such that $P(Q_i + Q_j) = m$) are supportable as subgame perfect Nash equilibrium of the game $G_M$. Here I extend the analysis to asymmetric equilibria and check the robustness of these results to the introduction of mixed strategy deviations.

**Corollary 5** *If the cost function of the agent $C(Q)$ is strictly convex ($\beta > 0$), then an outcome $(Q_i, Q_j)$ can be supported by a $SPNE(G_M)$ if and only if $Q_i + Q_j \leq \frac{A-m}{B}$ and $Q_i \geq BR(Q_j)$, for all $i, j$.*

**Proof.**

[If:] If $Q_i + Q_j \leq \frac{A-m}{B}$, the profits of each of the two firms are greater than zero, and therefore the outcome $(Q_i, Q_j)$ is individually rational. Moreover, if for both $i$ and $j$, $Q_i \geq BR(Q_j)$, then no firm has a higher strategy deviation. Therefore condition 1 of proposition 7 is verified. Condition 2 of proposition 7 is always verified for the functional forms chosen in this example. In fact we have that the function $\Delta u(Q_i)$ is linear whereas the function $v_i(Q_i)$ is concave. In fact

$$\begin{aligned}\Delta u(Q_i) &= u(Q_i, Q_j^I) - u(Q_i, Q_j^\star) \\ &= m(Q_i + Q_j^I) - C(Q_i + Q_j^I) - m(Q_i + Q_j^\star) + C(Q_i + Q_j^\star) \\ &= (m - \alpha)(Q_j^I - Q_j^\star) - \beta({Q_j^I}^2 - {Q_j^\star}^2) - 2\beta(Q_j^I - Q_j^\star)Q_i\end{aligned}$$

and

$$\begin{aligned}v_i(Q_i, Q_j^\star) &= [A - B(Q_i + Q_j^\star)]Q_i - mQ_i \\ &= (A - BQ_j^\star - m)Q_i - BQ_i^2\end{aligned}$$

But then, for any $\delta_i$ with a support containing $Q_i < Q_i^\star$ and such that $\sum_{Q_i} v_i(Q_i, Q_j^\star)\delta(Q_i) > v_i(Q_i^\star, Q_j^\star)$, we have $\sum_{Q_i} \Delta u(Q_i)\delta(Q_i) \geq \Delta u(Q_i^\star)$; therefore condition 2 of proposition 7 is also verified.

[Only if:] If either $Q_i + Q_j > \frac{A-m}{B}$ or $Q_i < BR(Q_j)$, we have that condition 1 of proposition 8 is not satisfied and therefore we obtain that $(Q_i, Q_j)$ cannot be supported by a $SPNE(G_M)$.
■

As its proof clearly shows, corollary 5 is still valid even if we restrict principals to use only pure strategy deviations. This means that for the functional forms chosen for this model, the introduction of mixed strategy deviations do not change at all the set of equilibrium outcomes that are available with delegation. Let me clarify this result: one of the objectives of the paper is to assess whether the results obtained in common agency models are robust to the introduction of mixed strategies. The results of this section indicate that for the chosen functional forms, this is indeed the case. The message is more general though. On one hand, it is possible to choose payoff functions for the agent and the principals for which outcomes supported as equilibria of the model without mixed strategies are not equilibrium outcomes if principals are allowed to have mixed strategy deviations. On the other hand, with $u$ submodular (that is with a convex cost function of the agent), for any payoff functions of the principals, it is always possible to find a payoff function of the agent for which mixed strategy deviations have no role. This is not true for $u$ supermodular (concave cost function of the agent). Since Martimort and Stole (2003) and Segal and Whinston (2003, section 7) consider a supplier with convex cost function, it is likely that the equilibria that they find are robust to the introduction of mixed strategy deviations.

Corollaries 4 and 5 give a complete picture of the role of delegation to a common agent for the Cournot duopoly with a common supplier: if we focus only on symmetric outcomes, then the agent can never help the principals to improve over the outcome of the original game. On the other hand, if we consider asymmetric outcomes, the agent has always a role in enlarging the set of equilibrium outcomes. In particular, if the cost function is convex, the Stackelberg outcome is achievable, and if the cost function is concave, the exclusion outcome is achievable. None of these asymmetric outcomes Pareto dominate the Cournot outcome.

## 6 Conclusions

In this paper, I address the issue of whether and how delegating the choice of strategies to an interested mediator (agent) can improve upon the equilibrium outcome of two player games in strategic form. As can be expected, the role of the agent depends on her preferences and yet is somehow limited. This is not surprising: first of all, the agent does not have a proper action to take, i.e., she cannot affect the payoffs of the principals, unless they decide simultaneously and non-cooperatively to delegate part of their actions to the agent. Moreover, following the tradition of private common agency, I assumed a "delegation" technology (or better a "contracting" technology) that does not allow the principals to offer monetary transfers depending on the full action of the agent. In addition to these hurdles to delegation, I added a further obstacle: if a principal chooses not to delegate, I assumed that he can commit to a single mixed strategy.

In fact it might be surprising to some that even in these limiting conditions the agent still has a role in enlarging the set of outcomes achievable in equilibrium. One remarkable example is the persistence of asymmetric outcomes as equilibria of the delegation game. Another example of the successful role of the agent as a mediator was given in section 4, where (for the case of $u$ submodular) there exist equilibria that Pareto improve over the outcome of the original game (in which the agent correlates among asymmetric outcomes).

The results presented in this paper depend crucially on the assumption that principals can choose not to delegate and are able to commit to a single mixed strategy. This modeling choice is a departure from most models of common agency and I see it first of all as a simplifying assumption; once it is known that equilibria in menus in which principals are restricted to offer deterministic menus might not be robust to mixed strategy deviations, then it is reasonable to investigate how far this simplifying assumption brings us in characterizing the set of equilibrium outcomes achievable with delegation. There are other reasons that make this assumption appealing. First, with this assumption the original game without delegation is nested in the game with delegation. That is, all the strategies available to the players in the original game are still available to the players in the game with delegation. This has attractive consequences: for example, we obtain immediately the existence of an equilibrium in the delegation game for any finite game $G$. Secondly, this assumption allows us to consider the most favorable setting for the validity of the revelation principle with more than one principal: if even in this context, the revelation principle fails, there is no chance to recover it in a simple manner.

An important point to note is that while I allow principals to commit to a single mixed-strategy if they decide not to delegate, I do not allow them to offer menus in which one of the choices is a mixed strategy.[34] The question of what are the outcomes supported by such equilibria is interesting, yet it would dilute the message of this paper. Allowing such menus of mixed strategies might potentially enlarge the set of outcomes that can be supported as equilibria of the delegation game with respect to both the model in this paper and the model with only pure strategies. For this reason I leave this task to further research.

# 7 Appendix

**Proof of proposition 1.** Let $(\delta_1^\star, \delta_2^\star)$ be a Nash equilibrium of $G$. It is easy to show that there exists a subgame perfect Nash equilibrium of $G_M$ where neither principal delegates and both choose strategy $\delta_i^\star$. The strategy of the agent $\sigma_A$ in this subgame perfect Nash equilibrium is any strategy that is sequentially rational (that maximizes agent payoffs after any history). If a principal $j$ deviates by not delegating and choosing strategy $\delta_j'$, then payoff of principal $j$ is $\sum_{y_i \in Y_i} \sum_{y_j \in Y_j} v_j(y_i, y_j)\delta_i^\star(y_i)\delta_j'(y_j)$ that is smaller or equal than $\sum_{y_i \in Y_i} \sum_{y_j \in Y_j} v_j(y_i, y_j)\delta_i^\star(y_i)\delta_j^\star(y_j)$ for definition of Nash equilibrium. If a principal $j$ deviates by delegating to a menu $M_j$, the probability distribution $\mu(y)$ induced by the strategy of the agent $\sigma_A(M)$ can be written as a product between $\sigma_A(M)(y_j)$ and $\delta_i^\star$ obtaining again $V_j(\delta_i^\star, M_j, \sigma_A) = \sum_{y_i \in Y_i} \sum_{y_j \in Y_j} v_j(y_i, y_j)\delta_i^\star(y_i)\sigma_A(M)(y_j)$ that is smaller or equal than $\sum_{y_i \in Y_i} \sum_{y_j \in Y_j} v_j(y_i, y_j)\delta_i^\star(y_i)\delta_j^\star(y_j)$ for definition of Nash equilibrium. ■

**Proof of proposition 2.** Consider an arbitrary individually rational correlated strategy $\mu(y)$. It can be supported as a subgame perfect Nash equilibrium of $G_M$ by the following strategies of the principals: $M_i = Y_i$ for any $i = \{1, 2\}$ (that is both principals choose to delegate the complete menu). The strategy of the agent $\sigma_A^\star$ has the following 3 components: If $M = Y_1 \times Y_2$, then $\sigma_A^\star(M) = \mu(y)$. If only one principal $j$ deviates to $\delta_j \in \Delta(Y_j)$, then $\sigma_A^\star(M) = \arg\min_{\sigma_A \in \Delta(Y_i)} V_j(M_i, \delta_j, \sigma_A)$. If neither principal offers $M_i = Y_i$ then $\sigma_A^\star(M)$ is arbitrary. Since the agent is indifferent over all $Y$, any strategy of the agent is going to be a best response for any $M$, in particular the strategy $\sigma_A^\star$ is part of a $SPNE(G_M)$. Moreover given the strategy of the agent, an arbitrary deviating principal $j$ can get at most $\underline{v}_j$ and since $\mu(y)$ is individually rational the deviation is not profitable. ■

**Proof of lemma 1.** The proof is by contradiction. Assume, that $y^*$ is supported by a sub-

[34]For example, in the prisoners' dilemma with externality, one principal cannot offer a menu composed of $C$ and the mixed strategy $C$ with probability $\frac{1}{3}$, $D$ with probability $\frac{2}{3}$.

game perfect Nash equilibrium of $G_M$, $\{(M_i, M_j, \sigma_A)\}$, and that for principal $i$, there exist an action $y_i \in Y_i$ such that $v_i(y_i, y_j^*) > v_i(y_i^*, y_j^*)$. Since agent must find profitable to punish the pure strategy deviation $y_i$, this implies that the set $P_j(y_i, y_j^\star) \equiv \{y_j^p \neq y_j^\star \text{ such that } u(y_i, y_j^p) \geq u(y_i, y_j^\star)\} \cap M_j \neq \emptyset$. For any element $y_j^p \in P_j(y_i, y_j^\star)$ we have

$$u(y_i, y_j^p) \geq u(y_i, y_j^\star) \qquad (1)$$
$$u(y_i^\star, y_j^\star) \geq u(y_i^\star, y_j^p) \qquad (2)$$

Condition 1 above holds true by definition of the set $P_j(y_i, y_j^\star)$ and condition 2 is necessary for the agent to choose $(y_i^\star, y_j^\star)$ in equilibrium. Now if by contradiction $\forall y_j \in M_j \subset Y_j$ we have that $y_j \neq y_j^\star$ implies $u(y_i^\star, y_j) \neq u(y_i^*, y_j^*)$ then condition 2 must hold with strict inequality. This implies that for any $y_j^p \in P_j(y_i, y_j^\star)$ there exist a unique $q$ such that $0 \leq q < 1$ such that $qu(y_i^\star, y_j^\star) + (1-q)u(y_i, y_j^\star) = qu(y_i^\star, y_j^p) + (1-q)u(y_i, y_j^p)$. To indicate that such a $q$ depends on $y_j^p$, I denote with $q(y_j^p)$. Now denote with $q^\star$ the highest $q$ among these; that is $q^\star = \max_{y_j^p \in P_j} q(y_j^p)$. Since $Y_j$ is finite, the set $P_j(y_i, y_j^\star)$ contains at most a finite number of elements and therefore $q^\star < 1$. Now define $q' = q^\star + \epsilon$ with $0 < \epsilon < 1 - q^\star$ (so that also $q' < 1$) and consider the mixed strategy for principal $i$: $y_i^\star$ with probability $q'$ and $y_i$ with probability $(1-q')$. This mixed strategy for principal $i$ constitutes a profitable deviation from the outcome $(y_i^\star, y_j^\star)$ a contradiction with $(y_i^\star, y_j^\star)$ supported by the subgame perfect Nash equilibrium $\{(M_i, M_j, \sigma_A)\}$. To prove this last statement, first note that when principal $j$ chooses the menu $M_j$ and principal $i$ chooses (not to delegate) the mixed strategy $q'$, the agent will choose the element $y_j \in M_j$. In fact, by construction of $q'$, we have that $q'u(y_i^\star, y_j^\star) + (1-q')u(y_i, y_j^\star) > q'u(y_i^\star, y_j) + (1-q')u(y_i, y_j)$, for all the $y_j \in P_j(y_i, y_j^\star)$. Moreover for all the $y_j \notin P_j(y_i, y_j^\star)$ but $y_j \in M_j$ we have both $u(y_i^\star, y_j^\star) \geq u(y_i^\star, y_j)$ and $u(y_i, y_j^\star) \geq u(y_i, y_j)$ that imply $q'u(y_i^\star, y_j^\star) + (1-q')u(y_i, y_j^\star) > q'u(y_i^\star, y_j) + (1-q')u(y_i, y_j)$. To conclude the proof, note that then playing the mixed strategy $q'$ gives an expected payoff to principal $i$ equal to $q'v_i(y_i^\star, y_j^\star) + (1-q')v_i(y_i, y_j^\star)$ by assumption higher than $v_i(y_i^\star, y_j^\star)$. That is, the agent cannot punish the mixed strategy deviation $q'$ even if he can punish the pure strategy deviation $y_i$. ■

**Proof of proposition 3.** The proof is by contradiction. Assume $\Pi_{SPNE}(G_M) \neq \Pi_{NE}(G)$. By Proposition 1 this implies that there exist a $(v_1, v_2)$ such that $(v_1, v_2) \in \Pi_{SPNE}(G_M)$ and $(v_1, v_2) \notin \Pi_{NE}(G)$. We have two possible cases. Case 1: $(v_1, v_2)$ is supported by a single outcome $(y_1, y_2) \in Y$. Case 2: $(v_1, v_2)$ is supported by a non degenerate probability distribution $\mu$ over $Y$. Case 1: If $(v_1, v_2)$ is supported by a single outcome $(y_1^\star, y_2^\star) \in Y$ and $(y_1^\star, y_2^\star)$ is not a Nash Equilibrium of $G$, $\exists i \in \{1, 2\}$ such that the set $D_i(y_i^\star, y_j^\star) \equiv \{y_i \in Y_i | v_i(y_i, y_j^\star) > v_i(y_i^\star, y_j^\star)\} \neq \emptyset$. Therefore, by lemma 1, there exist an action $y_j \in M_j \subset Y_j$ such that $y_j \neq y_j^\star$ and $u(y_i^*, y_j) = u(y_i^*, y_j^*)$. That is a contradiction with the assumption of $u$ agent-generic. Case 2: If $(v_1, v_2)$ is supported by a non degenerate probability distribution $\mu$ over $Y$, we have two further subcases: in case 2a, there exist a $\exists i \in \{1, 2\}$ and a $(y_i, y_j)$

in the support of $\mu$, such that $v_i(y_i, y_j) > v_i$. Then principal $i$ could deviate to the singleton $y_i$ leaving the agent to choose among the elements of $y_i \times M_j$. Since $u$ is agent-generic the $\arg\max_{y_j \in M_j} u(y_i, y_j)$ is a singleton and since $(y_i, y_j)$ is in the support of $\mu$ the optimal choice for the agent is $(y_i, y_j)$. But this means that principal $i$ has a profitable deviation, a contradiction. In the case 2b, $\forall i \in N$ and $\forall y$ in the support of $\mu$, $v_i(y) = v_i$. This implies that if $y$ belongs to the support of $\mu$, $y$ cannot be a $NE(G)$; we can therefore apply the same reasoning as for the case 1, that is: there exist $\exists i \in N$ such that the set $D_i(y) \equiv \{y_i^d \in Y_i | v_i(y_i^d, y_j) > v_i(y_i, y_j)\} \neq \emptyset$. Therefore, by lemma 1, there exist an action $y_j^p \in M_j \subset Y_j$ such that $u(y_i, y_j^p) = u(y_i, y_j)$. That is a contradiction with the assumption of $u$ agent-generic. ■

**Proof of proposition 4.** Let $y = (y_i^\star, y_j^\star) \in Y$ be the outcome to be supported by a subgame perfect Nash equilibrium of $G_M$, $\{(M_i, M_j, \sigma_A)\}$. The two principals can offer the complete menus $M_i = \{y_i, t_i(y_i)\}_{y_i \in Y_i}$ and $M_j = \{y_j, t_j(y_j)\}_{y_j \in Y_j}$ where the monetary transfers $t_i(y_i)$ and $t_j(y_j)$ are defined as follows: $t_i(y_i) = u(y_i^\star, y_j^\star) - u(y_i, y_j^\star)$ and $t_j(y_j) = u(y_i^\star, y_j^\star) - u(y_i^\star, y_j)$. Notice that $t_i(y_i^\star) = 0$ and $t_j(y_j^\star) = 0$, so that both the menus $M_1$ and $M_2$ are feasible (that is $M_1 \in \mathcal{M}_1$ and $M_2 \in \mathcal{M}_2$). The strategy of the agent $\sigma_A$ has the following 3 components: if $M = M_1 \times M_2$, then agent chooses $\{(y_i^\star, t_i(y_i^\star)), (y_j^\star, t_j(y_j^\star))\}$. If only one principal $j$ deviates to $\delta_j \in \Delta(Y_j)$, then $\sigma_A(M) = \arg\min_{\sigma_A \in \Delta(Y_i)} V_j(M_i, \delta_j, \sigma_A)$. If neither principal offers $M_i = Y_i$ then $\sigma_A(M)$ is arbitrary. The strategy $\sigma_A$ is a best response for any $M$, and so part of a subgame perfect Nash equilibrium. This holds true because first, since $u$ satisfies constant differences, the agent is indifferent over the whole set of menus $M_i \times M_j$ and therefore choosing $\{(y_i^\star, t_i(y_i^\star)), (y_j^\star, t_j(y_j^\star))\}$ is part of the equilibrium strategy of the agent. Moreover if principal $j$ deviates to any $\delta_j \in \Delta(Y_j)$, the agent is indifferent over the set of choices that this deviation produces; that is $u(m_i, \delta_j)$ is constant for any $m_i \in M_i$. This is an immediate consequence of $u$ satisfying constant differences. Given the agent strategy, $\sigma_A$, neither principal has a profitable deviation. If principal $j$ deviates to $\delta_j$ his payoffs are equal to $V_j(M_i, \delta_j, \sigma_A) = \sum_{y_i \in Y_i} \sum_{y_j \in Y_j} v_j(y_i, y_j) \sigma_A(M)(y_i) \delta_j(y_j) = min_{\delta_i \in \Delta(Y_i)} \sum_{y_i \in Y_i} \sum_{y_j \in Y_j} v_j(y_i, y_j) \delta_i(y_i) \delta_j(y_j) \leq \underline{v}_j \leq v_j(y_1^\star, y_2^\star)$, where the last inequality holds true because $(y_1^\star, y_2^\star)$ is individually rational. ■

**Proof of lemma 2.** The proof is by contradiction. Assume there exists an outcome $(y_i^\star, y_j^\star)$ supported by a subgame perfect Nash equilibrium of $G_M$, and such that the sets $D_i(y_i^\star, y_j^\star)$ and $D_j(y_i^\star, y_j^\star)$ are both nonempty where $D_i(y_i^\star, y_j^\star) \equiv \{y_i \in Y_i | v_i(y_i, y_j^\star) > v_i(y_i^\star, y_j^\star)\}$ and $D_j(y_i^\star, y_j^\star) \equiv \{y_j \in Y_j | v_j(y_i^\star, y_j) > v_j(y_i^\star, y_j^\star)\}$. Since $(y_i^\star, y_j^\star)$ is supported by a subgame perfect Nash equilibrium of $G_M$, then for any $y_i \in D_i(y_i^\star, y_j^\star)$ there exist a $(y_j^p, t_j(y_j^p)) \in M_j$ such that $v_i(y_i, y_j^p) < v_i(y_i, y_j^\star)$. Moreover, for the assumption of negative externality, $y_j^p > y_j^\star$ and from lemma 1 $u(y_i^\star, y_j^\star, t_i(y_i^\star), t_j(y_j^\star)) = u(y_i^\star, y_j^p, t_i(y_i^\star), t_j(y_j^p))$. The same is true for the other principal and therefore we obtain that there exist $(y_i^\star, y_i^p) \in Y_i$, and $(y_j^\star, y_j^p) \in Y_j$ with

$y_i^\star > y_i^p$ and $y_j^\star > y_j^p$, such that

$$u(y_i^\star, y_j^\star, t_i(y_i^\star), t_j(y_j^\star)) = u(y_i^\star, y_j^p, t_i(y_i^\star), t_j(y_j^p)) \quad (1)$$

$$u(y_i^\star, y_j^\star, t_i(y_i^\star), t_j(y_j^\star)) = u(y_i^p, y_j^\star, t_i(y_i^p), t_j(y_j^\star)) \quad (2)$$

$$u(y_i^\star, y_j^\star, t_i(y_i^\star), t_j(y_j^\star)) \geq u(y_i^p, y_j^p, t_i(y_i^p), t_j(y_j^p)) \quad (3)$$

Condition 1 implies $t_j(y_j^p) - t_j(y_j^\star) = u(y_i^\star, y_j^\star) - u(y_i^\star, y_j^p)$ while condition 2 implies $t_i(y_i^p) - t_i(y_i^\star) = u(y_i^\star, y_j^\star) - u(y_i^p, y_j^\star)$. If we substitute these expressions in the condition 3 we obtain $u(y_i^\star, y_j^p) - u(y_i^\star, y_j^\star) \geq u(y_i^p, y_j^p) - u(y_i^p, y_j^\star)$ that is a contradiction with $u$ strictly supermodular. ■

**Proof of proposition 5.** If conditions (1) and (2) hold, then the outcome $(y_i^\star, y_j^\star)$ can be supported by a subgame perfect Nash equilibrium $(M_i, y_j^\star, \sigma_A)$, where the strategies for principals and the agent are the following: principal $j$ does not delegate choosing action $y_j^\star$; principal $i$ delegates the menu $M_i = \{(y_i^\star, t_i(y_i^\star)); (y_i^p, t_i(y_i^p)); (y_i^I, t_i(y_i^I))\}$, where $t_i(y_i^\star) = 0$, $t_i(y_i^p) = u(y_i^\star, y_j^\star) - u(y_i^p, y_j^\star)$ and $t_i(y_i^I) = u(y_i^\star, y_j^\star) - u(y_i^I, y_j^\star)$ and the agent strategy $\sigma_A$ requires the agent to choose $(y_i^\star, t_i(y_i^\star))$ from $M_i$ if principal $j$ chooses $y_j^\star$. If $y_i^I$ satisfies condition (2), in fact the menu $M_i$ only contains two elements $(y_i^\star, t_i(y_i^\star))$ and $(y_i^I, t_i(y_i^I))$; If $y_i^I$ does not satisfy condition (2), then the menu contains also the element $y_i^p$ for which condition (2) is satisfied. Notice that $t_i(y_i^\star) = 0$, and therefore $M_i$ is an acceptable menu. If principal $j$ offers $y_j^\star$, then the agent is indifferent between the elements of $M_i$ and therefore the strategy of the agent is optimal in equilibrium. Moreover since by assumption $y_i^\star = BR(y_j^\star)$, principal $i$ does not have a profitable deviation and therefore we only need to worry about possible deviations of principal $j$. We first focus on possible pure strategy deviations of principal $j$. Since $u$ is strictly supermodular, if principal $j$ deviates by choosing $y_j >_j y_j^\star$ the agent selects $(y_i^I, t_i(y_i^I))$ from $M_i$; conversely if principal $j$ deviates by choosing $y_j <_j y_j^\star$ the agent selects $(y_i^\star, t_i(y_i^\star))$ from $M_i$ (note that condition 2 implies $y_i^p >_i y_i^\star$). By condition (1) deviations to $y_j <_j y_j^\star$ are unprofitable because $v_j(y_i^\star, y_j) \leq v_j(y_i^\star, y_j^\star)$. Deviations to $y_j >_j y_j^\star$ are unprofitable because $(y_i^\star, y_j^\star)$ is individually rational and so $v_j(y_i^I, y_j) \leq v_j(y_i^\star, y_j^\star)$. Now consider mixed strategy deviations $\delta_j$; if the support of $\delta_j$ contains only actions $y_j$ such that $y_j > y_j^\star$, then again for supermodularity of $u$ the agent will select the element $(y_i^I, t_i(y_i^I))$ from $M_i$ and this is sufficient to punish the deviation $\delta_j$. If the mixed strategy deviation $\delta_j$ has support with elements $y_j$ with $y_j <_j y_j^\star$, then (because of condition 1) the support of $\delta_j$ will also contain elements $y_j' >_j y_j^\star$. Then in order for such a deviation $\delta_j$ to be profitable it is necessary to have $\sum_{y_j \in Y_j} v_j(y_i^\star, y_j)\delta_j(y_j) > v_j(y_i^\star, y_j^\star)$. But then for condition (2) we have that $\sum_{y_j \in Y_j} [u(y_i^p, y_j) - u(y_i^\star, y_j)]\delta_j(y_j) \geq u(y_i^p, y_j^\star) - u(y_i^\star, y_j^\star)$. This last inequality is sufficient to guarantee that the agent selects the element $(y_i^p, t_i(y_i^p))$ from menu $M_i$, making the deviation to $\delta_j$ unprofitable. ■

**Proof of proposition 6.** The proof is by contradiction. Assume that condition 1 is not satisfied and that $(y_i^\star, y_j^\star)$ is supported as a subgame perfect Nash equilibrium of the game $G_M$.

Then $\exists y_j \in Y_j$ with $y_j^\star >_j y_j$ and $v_j(y_i^\star, y_j) > v_j(y_i^\star, y_j^\star)$. This implies that there must exist a $(y_i^p, t_i^p) \in M_i$ such that $u(y_i^p, y_j)+t_i^p \geq u(y_i^\star, y_j)+t_i^\star$, and $u(y_i^\star, y_j^\star)+t_i^\star \geq u(y_i^p, y_j^\star)+t_i^p$ where the first inequality must hold otherwise $y_i$ represent a profitable deviation for principal $j$ and the second inequality must hold for the agent to choose $(y_i^\star, y_j^\star)$ as a part of the equilibrium strategy. Rearranging the two inequalities we have $u(y_i^p, y_j)-u(y_i^\star, y_j) \geq u(y_i^p, y_j^\star)-u(y_i^\star, y_j^\star)$. This last inequality contradicts the strict supermodularity of $u$ as $y_j^\star >_j y_j$ holds by assumption and $y_i^p >_i y_i^\star$ because of the assumption of negative externality. If condition 2 is not satisfied this means that there exist a mixed strategy deviation $\delta_j$ such that $\sum_{y_j \in Y_j} v_j(y_i^\star, y_j)\delta_j(y_j) > v_j(y_i^\star, y_j^\star)$ and for all the $y_i^p \in Y_i$ such that $\sum_{y_j \in Y_j} v_j(y_i^p, y_j)\delta_j(y_j) < v_j(y_i^\star, y_j^\star)$ we have $\sum_{y_j \in Y_j}[u(y_i^p, y_j) - u(y_i^\star, y_j)]\delta_j(y_j) < u(y_i^p, y_j^\star) - u(y_i^\star, y_j^\star)$. But this implies that for any menu $M_i$ offered by principal $i$ such that the agent finds profitable to select the element $y_i^\star$ when principal $j$ chooses $y_j^\star$, the agent will not choose a $y_i^p$ such that $\sum_{y_j \in Y_j} v_j(y_i^p, y_j)\delta_j(y_j) < v_j(y_i^\star, y_j^\star)$, if the principal $j$ deviates by choosing the mixed strategy $\delta_j$. But this implies that principal $j$ has a profitable mixed deviation, that is a contradiction. ■

**Proof of proposition 7.** If conditions (1) and (2) hold, then the outcome $(y_i^\star, y_j^\star)$ can be supported by a subgame perfect Nash equilibrium $(M_i, M_j, \sigma_A)$, where the strategies for principals and the agent are the following: principal $j$ delegates the menu $M_j = \{(y_j^\star, t_j(y_j^\star)); (y_j^p, t_j(y_j^p)); (y_j^I, t_j(y_j^I))\}$, where $t_j(y_j^\star) = 0$, $t_j(y_j^p) = u(y_i^\star, y_j^\star) - u(y_i^\star, y_j^p)$ and $t_j(y_j^I) = u(y_i^\star, y_j^\star) - u(y_i^\star, y_j^I)$ ; principal $i$ delegates the menu $M_i = \{(y_i^\star, t_i(y_i^\star)); (y_i^p, t_i(y_i^p)); (y_i^I, t_i(y_i^I))\}$, where $t_i(y_i^\star) = 0$, $t_i(y_i^p) = u(y_i^\star, y_j^\star) - u(y_i^p, y_j^\star)$ and $t_i(y_i^I) = u(y_i^\star, y_j^\star) - u(y_i^I, y_j^\star)$; the agent strategy $\sigma_A$ requires the agent to choose $m_i^\star = (y_i^\star, t_i(y_i^\star))$ from $M_i$ and $m_j^\star = (y_j^\star, t_j(y_j^\star))$ from $M_j$, if the menus offered by principals are $M_i$ and $M_j$. First, notice that among the elements $(m_i, m_j)$ of the menu $M_i \times M_j$ we have $u(m_i^\star, m_j^\star) = u(m_i', m_j^\star) = u(m_i^\star, m_j') > u(m_i', m_j')$, (for $m_i' = \{m_i^I, m_i^p\}$ and $m_j' = \{m_j^I, m_j^p\}$) where the equalities follow by the choice of element of the menus and the inequality from the assumption of $u$ strictly submodular. We first focus on possible pure strategy deviations. Since $u$ is stritly submodular, if principal $j$ deviates by choosing $y_j >_j y_j^\star$ the agent selects $m_i^\star = (y_i^\star, t_i(y_i^\star))$ from $M_i$ (notice that condition 2 implies $y_i^p >_i y_i^\star$); conversely if principal $j$ deviates by choosing $y_j <_j y_j^\star$ the agent selects $m_i^I = (y_i^I, t_i(y_i^I))$ from $M_i$. By condition (1) deviations to $y_j >_j y_j^\star$ are unprofitable because $v_j(y_i^\star, y_j) \leq v_j(y_i^\star, y_j^\star)$. Deviations to $y_j <_j y_j^\star$ are unprofitable because $(y_i^\star, y_j^\star)$ is individually rational and so $v_j(y_i^I, y_j) \leq v_j(y_i^\star, y_j^\star)$. The same is true for pure strategy deviations of the other principal $i$. Now consider mixed strategy deviations of principal $j$, $\delta_j$; if the support of $\delta_j$ contains only actions $y_j$ such that $y_j <_j y_j^\star$, then again for submodularity of $u$ the agent will select the element $(y_i^I, t_i(y_i^I))$ from $M_i$ and this is sufficient to punish the deviation $\delta_j$. If the mixed strategy deviation $\delta_j$ has support with elements $y_j$ with $y_j >_j y_j^\star$, then (because of condition 1) the support of $\delta_j$ will also contain elements $y_j' <_j y_j^\star$. Then in order for such a deviation $\delta_j$ to be profitable it is necessary to have $\sum_{y_j \in Y_j} v_j(y_i^\star, y_j)\delta_j(y_j) > v_j(y_i^\star, y_j^\star)$. But then for condition (2) we have that

$\sum_{y_j \in Y_j}[u(y_i^p, y_j) - u(y_i^\star, y_j)]\delta_j(y_j) \geq u(y_i^p, y_j^\star) - u(y_i^\star, y_j^\star)$. This last inequality is sufficient to guarantee that the agent selects the element $(y_i^p, t_i(y_i^p))$ from menu $M_i$, making the deviation to $\delta_j$ unprofitable. ■

**Proof of proposition 8.** The proof follows very closely the one of prop 6 and I include it for completeness. We need to prove that if either condition 1 or 2 are not satisfied $(y_i^\star, y_j^\star)$ cannot be supported as a subgame perfect Nash equilibrium of the game $G_M$. The proof is by contradiction. Assume first that condition 1 is not satisfied and that $(y_i^\star, y_j^\star)$ is supported as a subgame perfect Nash equilibrium of the game $G_M$. Then $\exists y_j \in Y_j$ with $y_j >_j y_j^\star$ and $v_j(y_i^\star, y_j) > v_j(y_i^\star, y_j^\star)$. This implies that there must exist a $(y_i^p, t_i^p) \in M_i$ such that $u(y_i^p, y_j) + t_i^p \geq u(y_i^\star, y_j) + t_i^\star$, and $u(y_i^\star, y_j^\star) + t_i^\star \geq u(y_i^p, y_j^\star) + t_i^p$ where the first inequality must hold otherwise $y_i$ represent a profitable deviation for principal $j$ and the second inequality must hold for the agent to choose $(y_i^\star, y_j^\star)$ as a part of the equilibrium strategy. Rearranging the two inequalities we have $u(y_i^p, y_j) - u(y_i^\star, y_j) \geq u(y_i^p, y_j^\star) - u(y_i^\star, y_j^\star)$. This last inequality contradicts the strict submodularity of $u$ as $y_j >_j y_j^\star$ holds by assumption and $y_i^p >_i y_i^\star$ because of the assumption of negative externality. If condition 2 is not satisfied this means that there exist a mixed strategy deviation $\delta_j$ such that $\sum_{y_j \in Y_j} v_j(y_i^\star, y_j)\delta_j(y_j) > v_j(y_i^\star, y_j^\star)$ and for all the $y_i^p \in Y_i$ such that $\sum_{y_j \in Y_j} v_j(y_i^p, y_j)\delta_j(y_j) < v_j(y_i^\star, y_j^\star)$ we have $\sum_{y_j \in Y_j}[u(y_i^p, y_j) - u(y_i^\star, y_j)]\delta_j(y_j) < u(y_i^p, y_j^\star) - u(y_i^\star, y_j^\star)$. But this implies that for any menu $M_i$ offered by principal $i$ such that the agent finds profitable to select the element $y_i^\star$ when principal $j$ chooses $y_j^\star$, the agent will not choose a $y_i^p$ such that $\sum_{y_j \in Y_j} v_j(y_i^p, y_j)\delta_j(y_j) < v_j(y_i^\star, y_j^\star)$, if the principal $j$ deviates by choosing the mixed strategy $\delta_j$. But this implies that principal $j$ has a profitable mixed deviation, that is a contradiction. ■

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