Supplement to "Comment to Why Selective Colleges Should Become Less Selective-And Get Better Students "

(not for publication)

Emanuele Gerratana
SIPA, Columbia University

July 27, 2016

In what follows I provide the omitted proofs of the statements and calculations for the paper "Comment to Why Selective Colleges Should Become Less Selective-And Get Better Students."

An immediate consequence of Moldovanu and Sela (2001) is that if $\gamma(x)$ is linear or concave, then

$$\int_m^1 \gamma(x_{sq}(c))dF(c) > \int_m^1 \gamma(x_{mp}(c))dF(c) \tag{1}$$

Assumption 1.

$$\int_m^1 \gamma(x_{sq}(c))dF(c) > \int_m^1 \gamma(x_{mp}(c))dF(c)$$

implies

$$\int_m^1 c\gamma(x_{sq}(c))dF(c) > \int_m^1 c\gamma(x_{mp}(c))dF(c)$$

Corollary 1. *Assume that assumption 1 holds, and that $\gamma(x)$ is a strictly increasing function. Then the total students welfare is higher with the modified proposal than with the status quo*

Proof of Corollary 1. The total welfare of the students in the status quo game is given by

$$P\overline{U} - K\int_m^1 c\gamma(x_{sq}(c))dF(c) \tag{2}$$

and the total welfare of the students in the modified proposal game is given by

$$P\overline{U} - K\int_m^1 c\gamma(x_{mp}(c))dF(c) \tag{3}$$

We have that (3) is greater than (2) if and only if

$$\int_m^1 c(\gamma(x_{sq}(c)) - \gamma(x_{mp}(c)))dF(c) > 0 \tag{4}$$

If $\gamma(x) = x$, then (4) follows from proposition 2 of Moldovanu and Sela (2001) and assumption 1 and we are done. If $\gamma(x) \neq x$ we know again from Moldovanu and Sela (2001) that $x_{sq}(c) = \gamma^{-1}(\tilde{x}_{sq}(c))$ and that $x_{mp}(c) = \gamma^{-1}(\tilde{x}_{mp}(c))$ where $\tilde{x}_{sq}(c)$ and $\tilde{x}_{mp}(c)$ are the equilibrium effort functions that would arise in equilibrium if $\gamma(x) = x$. Therefore (4) holds if and only if

$$\int_m^1 c(\tilde{x}_{sq}(c) - \tilde{x}_{mp}(c))dF(c) > 0 \tag{5}$$

that follows again from proposition 2 of Moldovanu and Sela (2001) and assumption 1.

□

Derivation of equations (1) and (2) in the paper. Using proposition 1 of Moldovanu and Sela (2001)

$$x_{sq}(c) = A(c)\overline{U} \tag{6}$$

where $A(c) = (K-1)\int_c^1 \frac{1}{a}(1-F(a))^{K-2} \times F'(a)da$
with our parametrization, it becomes

$$A(c) = 9\int_c^1 \frac{1}{a}(2-2a)^8 \times 2da \tag{7}$$

that is equation (1) in the paper.

Again using proposition 1 of Moldovanu and Sela (2001)

$$x_{mp}(c) = \frac{1}{2}A(c)\overline{U} + \frac{1}{2}B(c)\overline{U} \tag{8}$$

where $A(c)$ is given by (7) and $B(c)$ is given by
$B(c) = (K-1)\int_c^1 \frac{1}{a}(1-F(a))^{K-3} \times [(K-1)F(a)-1]F'(a)da$
with our parametrization, it becomes

$$B(c) = 9\int_c^1 \frac{1}{a}(2-2a)^7 \times [9(2a-1)-1]2da \tag{9}$$

Therefore we obtain

$$x_{mp}(c) = \frac{1}{2}(9\int_c^1 \frac{1}{a}(2-2a)^8 \times 2da) + \frac{1}{2}(9\int_c^1 \frac{1}{a}(2-2a)^7 \times [9(2a-1)-1]2da) \tag{10}$$

that is equation (2) in the paper. □

Proof of Proposition 2. From proposition 1 in the paper (again from Moldovanu and Sela, 2001) we know that $x_{sq}(c)$ is a decreasing function of c and that $x_{sq}(1) = 0$. Moreover the expected utility of student with type c in the status quo, $EU_{sq}(x_{sq}(c), c)$ is also a decreasing function of c and $EU_{sq}(x_{sq}(1), 1) = 0$. Since $0 < f^{-1}(1) < x_T < \overline{x}_T$, and since $x_{sq}(c)$ is continuous in c, there exist a $c_1 \in (m, 1)$ such that $x_{sq}(c_1) = x_T$. This implies that for all $c < c_1$, $x_{sq}(c) > x_T$. Since for all $c < c_1$, we have $EU_{sq}(x_{sq}(c), c) > 0$, then we have $x_{sp}(c) = x_T < x_{sq}(c)$. Since $f^{-1}(1) < x_T$, student with type $c = 1$ prefers to choose $x = 0$ rather than $x = x_T$ in the Schwartz proposal game. Also since $x_T < \overline{x}_T < f^{-1}(m)$, the student of type $c = m$ prefers choosing $x = x_T$ rather than $x = 0$. Since the expected utility $EU_{sp}(x_s p(c), c)$ is continuous in c, we have that there exist $c_2 \in (m, 1)$ such that $f(x_T) = c_2$. Also, notice that we must have $c_1 < c_2$. Since c_2 is the marginal student for the threshold x_T, it implies that for all $c \in (c_2, 1)$, $x_{sp}(c) = 0 < x_{sq}(c)$. Finally , we need to show that for all $c \in (c_1, c_2)$, we have $x_{sp}(c) > x_{sq}(c)$. Since $c_2 = f(x_T)$, for all $c < c_2$, $x_{sp} = x_T$. Since $c > c_1$ $x_{sq}(c) < x_{sq}(c_1) = x_T$ therefore we obtained that $x_{sp}(c) > x_{sq}(c)$. □

Calculations to derive the threshold $x_T = 0.552338$. In the numerical example part 2, in the paper, I assume that the college sets a threshold $x_T = 0.552338$. Here I show that like stated in the paper "such a threshold induces a marginal student with cost parameter equal to $c' = \frac{28}{44}$ and an average ex-ante quality for the admitted students in line with the average ex-ante quality obtained in the modified proposal with $\alpha = 2$."

First, I show that in the numerical example part 1, the average quality of the admitted student is $c = \frac{25}{44}$. In general the quality of the best student out of N candidates is equal to

$$AV_1 = \int_m^1 xN(1-F(x))^{N-1} f(x)dx \tag{11}$$

and the average quality of the second best student is

$$AV_2 = \int_m^1 xN(N-1)F(x)(1-F(x))^{N-2} f(x)dx \tag{12}$$

For $m = \frac{1}{2}$ and uniform distribution, we have

$$AV_1 = \int_{\frac{1}{2}}^1 x2N(2-2x)^{N-1} dx \tag{13}$$

and

$$AV_2 = \int_{\frac{1}{2}}^1 x2N(N-1)(2x-1)(2-2x)^{N-2} dx \tag{14}$$

With 10 candidates, one spot in a selective college and a randomization over the two best candidates, the average ex-ante quality of the admitted college student is

$$\frac{1}{2}\int_{\frac{1}{2}}^1 x(20(2-2x)^9)dx + \frac{1}{2}\int_{\frac{1}{2}}^1 x180(2x-1)(2-2x)^8 dx = \frac{1}{2}\frac{6}{11} + \frac{1}{2}\frac{13}{22} = \frac{25}{44} \tag{15}$$

Next, we need to find the quality of the marginal student, c' such that the average ex-ante quality of the admitted college student with the Schwartz proposal is equal to $c = \frac{25}{44}$. In order to do this, we make the simplifying (but wrong) assumption that there will be always at least one student with threshold above c'. If this were the case the marginal student must be $c' = \frac{28}{44}$.[1]

Finally, we need to find the level of the threshold x_T that induces a marginal student with parameter $c' = \frac{28}{44}$. The threshold x_T that induces a marginal student with parameter c' is defined implicitly by the following equation

Cost of effort at the threshold x_T equal to gain from making effort x_T rather than zero.

That is

$$c'\gamma(x_T) = \sum_{j=0}^{N-1} \binom{N-1}{j} F(c)^{N-1-j}(1-F(c))^j \frac{1}{N-j} - \frac{(1-F(c))^N}{N} \tag{16}$$

[1] As discussed in the paper, see footnote 7, there is a positive probability, although small and vanishing as the number of candidates increases, that none of the candidate has parameter $c \le c'$. This event complicates the calculation of what is the marginal student c' that induces an average ex-ante quality $c = \frac{25}{44}$. In fact, it turns out that for the choice of parameters of the numerical example, it is impossible to find a threshold that induces an average ex-ante quality of student with the Schwartz proposal equal to $c = \frac{25}{44}$. The highest possible average quality is in fact lower than $c = \frac{25}{44}$. Since the use of these calculations is only to find a level of threshold x_T comparable with the ex-ante average quality of obtained in the modified proposal with $\alpha = 2$, we take the short-cut of assuming that the event "no student with $c \le c'$" has zero probability.

The left hand side of the equation (16) is the cost of exerting the effort threshold for the marginal student; the first term of the right-end side of the equation (16) is the probability of being admitted to college[2] for type c conditional to c being the worse quality student who exerts effort and the second term of the right-end side of the equation (16) is the probability of being admitted to college by not making any effort. This last term is very close to zero and it is calculated under the assumption that if no applicant has produced an effort e_T, the college chooses randomly over all the applicants. By employing the same simplifying assumption above, we impose that the probability of being admitted to college by exerting zero effort is equal to zero and in doing so the equation (16) becomes

$$c'\gamma(x_T) = \sum_{j=0}^{N-1} \binom{N-1}{j} F(c)^{N-1-j} (1-F(c))^j \frac{1}{N-j} \tag{17}$$

Now, by substituting the parameter values of the numerical example in the paper, that is $\gamma(x) = x$, $N = 10$ and $F(c) = 2c - 1$) we obtain

$$c' x_T = \sum_{j=0}^{9} \binom{9}{j} (2c-1)^{9-j} (2-2c)^j \frac{1}{10-j} \tag{18}$$

that for $c' = \frac{28}{44}$ gives us the value of $x_T = 0.552338$. □

[2]The value of being admitted is normalized to 1.