Appendix to:
Lost in Transit: 
Product Replacement Bias and Pricing to Market

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September 9, 2009

A Log-Likelihood in the Presence of Unobserved Heterogeneity in the Frequency of Price Change

We assume that product $i$ has a constant hazard of adjusting, $f_i$, in each month, where $f_i \sim \text{Beta}(a, b)$. Let us denote the product’s lifetime by $n_i$. These assumptions imply that the total number of price changes in a product’s lifetime is distributed according to the binomial distribution, $x_i \sim \text{Bin}(n_i, f_i)$. We assume, furthermore, that $f_i$ is distributed according to the beta distribution, $f_i \sim \text{Beta}(a, b)$.

Given this model, we can write the likelihood of observing a product with length $n_i$ and the total number of price changes $x_i$ as,

$$L = \prod_{i=1}^{I} \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} f_i^{n_i - x_i} (1 - f_i)^{n_i - x_i}$$

$$= \prod_{i=1}^{I} \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} f_i^{x_i} (1 - f_i)^{n_i - x_i}$$

We can integrate out the $f_i$'s to get,

$$L = \prod_{i=1}^{I} \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \left( \frac{n_i}{x_i} \right) ^{\Gamma(a + x_i)\Gamma(b + n_i - x_i)} \Gamma(a + b + n_i).$$

The log-likelihood function is, therefore,

$$\log L = n \log \Gamma(a + b) - n \log \Gamma(a) - n \log \Gamma(b)^n + \sum_{i=1}^{I} \left( \log n_i! - \log x_i! \right) - \log(n_i - x_i)! + \log \Gamma(a + x_i) + \log \Gamma(b + n_i - x_i) - \log \Gamma(a + b + n_i)].$$