Matching in the Smallest Large Market*

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Abstract

We study two-sided one-to-one matching with countably infinite agents. We find that the set of stable matching is non-empty if and only if agents' preferences admit a maximum on all subsets. We generalize the Deferred Acceptance algorithm to find the man-optimal and woman-optimal stable matchings and show that the set of stable matchings is a complete lattice under the preferences induced by men (or women). Unlike in finite models, the set of matched agents may vary across stable matchings. We discuss implications for dynamic matching markets.

1 Introduction

This paper analyzes a marriage market with countably infinite agents. This extension provides a benchmark for the study of large markets (infinite markets), and for the study of finite markets opening infinitely many times (dynamic finite markets). The literature on large markets has focused on the case of a continuum of agents, or on the analysis of finite markets with number of agents growing to infinity - but to our knowledge no other paper analyzes two-sided one-to-one matching with countably infinite agents. If one considers a finite market opening every year, in which agents can prefer being matched with agents entering the market in the future (or that entered in the past), then a complete description of the preferences must involve all the years in which the market opens. If the market is supposed to be opening forever, then the "smallest" model of such market is one with countably infinite agents, more precisely a countably infinite union of finitely many agents entering the market in a given year. In this

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sense our model provides a benchmark for the study of dynamic markets, and the results obtained here can be used for the design of matching in a dynamic environment (in particular the lattice structure of the set of stable matching, and the failure of the Rural Hospital Theorem).

This paper shows that when agents are countably infinite, the set of stable matchings is non-empty and satisfies the lattice structure observed in the finite case - provided agents' preferences admit a maximum on each subset. This is not a very stringent assumption, and in particular it is met in matching environments in which all agent finds only finitely many agent of the opposite set *better than being single*. Contrary to the finite case, we show that in our setting the set of "matched" men or women is not constant across stable matching: in other words there can be agents who are single in some stable matching, and matched in other. Hence the so-called Rural Hospital Theorem fails.

We show that if all agents' preferences satisfy our assumption on the maximum, there exists a stable matching in the market, and we show that besides being sufficient this assumption is necessary, too. In other words, if one agent's preferences do not admit maximum on some subset of the agents on the opposite side, then there might not exist any stable matching. The existence of a stable matching cannot be obtained directly by applying the Deferred Acceptance (henceforth DA) algorithm proposed by Gale and Shapley [1962], because such algorithm relied on the finiteness of the number of agents to be well-defined. For this reason, we provide a modification of it (the "generalized Deferred Acceptance algorithm") that is well-defined with countably many agents, and yields a stable matching.

After proving that the set of stable matching is non-empty, we show that it is a complete lattice under the partial order of "men-preferred" matching.[1] A matching is said to be "men-preferred" to another matching whenever all men are matched to a (weakly) better woman under the former. The generalized DA algorithm in which men propose to women is easily proved to be the men optimal stable matching, and analogously one can see that the matching obtained by women proposing is men pessimal. Given any two stable matching, we can define the operation of meet and join, and show that they yield a well-defined stable matching, and thus proving that the set of stable matching constitutes a complete lattice.[2]

[1] Given the symmetry in the problem, the same holds under the "women-preferred" order. We present our results only with respect of men for the sake of brevity.

[2] Alternatively, the same result could have been obtained by framing the problem as a

Contrarily to what happens for finite markets, we find that the set of matched agents is not constant across stable matchings. We provide a simple example in which a man is matched under the "man-optimal" stable matching, and he is single under the "woman-optimal" stable matching. We further provide necessary and sufficient conditions under which this happens and discuss the implications that this has on the manipulability result of the finite case.[3] This failure of the Rural Hospital Theorem distinguishes our model from the finite one, and in fact from the other extensions to infinitely many agents modeled with a continuum set. As a matter of fact, in the models with a continuum of agents the measure of agents is finite, and this ensures that the measure of matched agents is constant across stable matchings. A similar argument, in our setting would not work in that the *measure* of matched agents can be constantly infinite across stable matchings, without the *set* of matched agents being constant.

This result has consequences for the related literature in large matching markets, and dynamic matching markets. Firstly, if one consider the market with countably many agents as the limit of finite markets, then we showed a discontinuity in the limit. Secondly, when modeling finite markets that open infinitely many times if agents include in their preferences agents who enter the market in different years,[4] then we might have that the Rural Hospital theorem does not hold in the complete market (even though it does in any finite market with "truncated" preferences). We elaborate on this in Section 5, where we show how the failure of the Rural Hospital theorem yields a criterion for choosing between matching mechanisms in a dynamic matching market (Theorem 5.2).

The rest of the paper is structured as follows: in Section 2 we introduce the model, and in Section 3 we analyze the properties of the set of stable matching that hold for both finite and infinite marriage models. In Section 4, we show the main difference between the finite model and the infinite one, that is the failure

fixed point problem as firstly noted by Adachi [2000] and then invoking the Tarski fixed point theorem (see Tarski [1955]). This approach would have been more abstract, so we prefer to obtain the results directly, without relying on stronger mathematical theorems.

[3]In the study of the finite marriage market, one obtained that whenever the set of stable matching is not a singleton, at least one agent has incentive to misreport his/her preference under any stable mechanism. This result was proved using the Rural Hospital theorem, and we show that whenever the Rural Hospital theorem fails, the result on manipulability might fail too. In some sense, this shows that the Rural Hospital theorem is necessary and sufficient for manipulability.

[4]An example of this would be that of a high-school graduate who considers taking a year off before going to college. A complete description of her preferences would then include a college assignment this year and an assignment one year in the future.

of the Rural Hospital Theorem. In Section 5 we summarize how the results obtained apply to a dynamic market, and in Section 6 we conclude.

Related Literature

The model we analyze was initially proposed with finitely many agents by Gale and Shapley [1962], who also introduced the Deferred Acceptance algorithm. In this paper, we show that a modification of the same algorithm yields a stable matching also when there are countably many agents. Some of the further extensions and modifications of that model include Abdulkadiroglu et al. [2005], Abdulkadiroglu and Sonmez [2005], Alkan [1988], Ostrovsky [2008], Roth [1984], Roth [1986], Peranson and Roth [1999], Roth et al. [2004], Shapley and Shubik [1971], etc. A more detailed review of such applications can be found in Roth [2008].

Many of the proofs of the results that we obtain in Section 3 are adapted from Roth and Sotomayor [1992], who provide an extensive analysis of the marriage model with finitely many agents. A different approach to the problem of many to one matching is given in Hatfield and Milgrom [2005]. While our notation follows that of Roth and Sotomayor [1992], it is worth noting that our results of Section 3 could be obtained by adapting Hatfield and Milgrom [2005]to our case of countably many agents.

Our results are related to the extensions of the classical model to large markets. The approaches adopted in the literature are essentially two-fold: Che et al. [2015] and Azevedo and Leshno [2016] model the set of agents as a continuum space, whereas Immorlica and Mahdian [2005] and Kojima and Pathak [2009] analyzed finite markets with a number of agents increasingly large. Our theory is also related to the theory of dynamic matching markets, that is finite matching markets opening every year, possibly infinitely many times. Kurino [2008] defines dynamic matching as a sequence of one-period matching, and the same approach is taken by Kurino [2014] and Kadam and Kotowski [2016]. Our approach is markedly different, in that the object of interest is a unique matching in the infinite market. Other papers in the literature in dynamic matching markets include Monte and Tumennasan [2015], who analyze allocations in multiple markets (for which a special case is intertemporal allocation), and Abdulkadiroglu and Loerscher [2007] who study one-sided matching market opening for two periods.

2 The Model

The model is analogous to the finite model proposed by Gale and Shapley [1962], with the extension that now the set of men and women is countably infinite, $|M| = |W| = +\infty$.

Let $M = \{m_0, m_1, m_2, \dots\}$ be the set of men and $W = \{w_0, w_1, w_2, \dots\}$ the set of women. Each man m_j (resp. woman w_n) has a *strict total preference* $>_{m_j}$ on the set $W \cup \{m_j\}$ (resp. $M \cup \{w_n\}$), i.e. an irreflexive, asymmetric, and transitive relation on the set $W \cup \{m_j\}$ (resp. $M \cup \{w_n\}$). By writing:

$$w_1 >_{m_j} w_2 >_{m_j} m_j >_{m_j} w_3 \tag{2.1}$$

we mean that m_j prefers w_1 to w_2, and would rather be single than being matched to w_3.

We assume that all agents have a preferred element in each subset, formally:

Assumption 2.1. For each m_j, (resp. w_n) and for every $A \subseteq W \cup \{m_j\}$ (resp. $A \subseteq M \cup \{w_n\}$), $\exists x \in A \cup \{m_j\}$ (resp. $A \cup \{w_n\}$) such that:

$$x >_{m_j} y, \quad \forall y \in A \setminus \{x\} \quad \text{(resp. } >_{w_n}\text{)}.$$

Notice that we assume that the maximum for agent m_j exists on any subset $A \subset W \cup \{m_j\}$. This assumption is stronger than just requiring the existence of a "global maximum" on $W \cup \{m_j\}$. For example, we could have a man preferring w_0 over all other women, and preferring w_j to $w_{j'}$ whenever $j > j'$, $j' \neq 0$. While such preferences admit a global maximum (w_0), there exists no maximum on the set $\{w_1, w_2, \dots, w_n, \dots\}$. As Remark 3.1 below shows, assuming the existence of global maximum is not sufficient for the existence of a stable matching.

Observe also that Assumption 2.1 is trivially met when the set A is finite - since we assume the preferences to be complete. On the other hand when A is infinite a strict preference might not admit a maximum on A. We will show that this assumption is both necessary and sufficient for the existence of a stable matching.

If an agent's preferences satisfy Assumption 2.1, we can represent them by an infinite sequence. Then, besides the notation in (2.1), we will also write:

$$P(m_j) : w_{j_1}, w_{j_2}, \dots, w_{j_l}, m_j, w_{j_{l+1}}, \dots, w_{j_m}, \dots; \tag{2.2}$$

to indicate that m_j's most preferred agent is w_{j_1}, and that in general w_{j_k} is preferred to $w_{j_{k'}}$ for all $k < k'$. Each woman ranked *after* m_j is worse than being single, and we will truncate such list to m_j, because the ranking of women thereafter is irrelevant for the scope of our paper (as it will be clear from the definition of stable matching below, Definition 2.5). Then (2.2) will be written as:

$$P(m_j) : w_{j_1}, w_{j_2}, \ldots, w_{j_l},$$

with the understanding that a woman not listed in $P(m_j)$ is *worse than being single*, or *unacceptable*.

Remark 2.2. Observe that if a man m has finitely many acceptable women, then his preferences satisfy Assumption 2.1. While our model allows also for infinitely many acceptable partners, all the results we will obtain equally hold whenever agents find only finitely many agents acceptable (a very natural assumption in many applications, in particular in dynamic matching markets).

Formally, we can define a marriage market as follows:

Definition 2.3 (Marriage Market)**.** A Marriage Market is defined by a triple $(M, W, \mathbf{P})$ where:

- $M = \{m_0, m_1, \ldots\}$ is the set of men;
- $W = \{w_0, w_1, \ldots\}$ is the set of women;
- $\mathbf{P} = (P(x))_{x \in M \cup W}$ where for each $x \in W$ (resp. $x \in M$):
  - $P(x)$ is an irreflexive, asymmetric, and transitive relation on $W \cup \{x\}$ (resp. $M \cup \{x\}$);
  - $P(x)$ satisfies Assumption 2.1;

A matching is a set of pairs of agents, each pair consisting of one man and one woman, with unpaired agents remaining single. Mathematically, a matching is defined as:

Definition 2.4 (Matching)**.** A matching μ is a function:

$$\mu : M \cup W \to M \cup W,$$

such that:

1. for all $x \in M \cup W$, $\mu(\mu(x)) = x$;

2. if $x \neq \mu(x)$, then $\mu(x) \in M \Leftrightarrow x \in W$;

We interpret μ as mapping an agent x to itself if and only if the agent is left single. The second property then requires that if an agent is not single, then s/he is matched to an agent in the other set.

We will study the subset of matching that satisfy stability:

Definition 2.5 (Stable Matching)**.** A matching μ is stable in the marriage market $(M, W, \mathbf{P})$ if:

- *(individual rationality)*: there exists no x such that:

$$x >_x \mu(x);$$

- *(non-existence of blocking pairs)*: there does not exist any pair (m, w) such that:

$$m >_w \mu(w) \quad \text{and} \quad w >_m \mu(m).$$

Individual rationality means that no agent is matched to an agent that s/he likes worse than being single. Non-existence of blocking pairs means that there does not exist any pair that would rather be matched to each other than being matched according to μ (to put it in other words: there does not exist any blocking pair if anytime an agent x prefers y to his/her match, then y prefers her/his match to x).

3 Analysis of the set of Stable Matching

For each marriage market $(M, W, \mathbf{P})$, we will be interested in the set of stable matching. Before passing to this analysis, let us show by means of an example that Assumption 2.1 is necessary for the existence of a stable matching.

Remark 3.1. If the preferences $P(x)$ of an agent x do not satisfy Assumption 2.1 then there exist a market $(M, W, \mathbf{P})$ in which there are no stable matching.

Proof. Suppose that the preference of an agent does not satisfy Assumption 2.1, without loss let such agent be a woman $w \in W$. This means there exist a subset $\tilde{M} \subseteq M$ of *acceptable men* on which $>_w$ admits no maximum. Mathematically:

$$\forall m \in \tilde{M},\, \exists m' \in \tilde{M} \quad \text{s.t.} \quad m' >_w m, \tag{3.1}$$

and $m >_w w$, for all $m \in \tilde{M}$.

Now construct the set of preferences for men in the following way:

(Condition 1) if $m \in M \setminus \tilde{M}$, let $m >_m w$, - i.e. all men in $M \setminus \tilde{M}$ find w unacceptable;

(Condition 2) if $\tilde{m} \in \tilde{M}$ let $w \geq_{\tilde{m}} w'$ for all $w' \in W$ - i.e. all men in $\tilde{M}$ have w as their best choice;

Let us show that if preferences are so defined, there exists no stable matching. If a matching μ leaves w single, it is clearly not stable as taking any $\tilde{m} \in \tilde{M}$ we have that $(\tilde{m}, w)$ constitutes as blocking pair. If instead w is matched to a man m, than if $m \in M \setminus \tilde{M}$ the matching is not individually rational, because of Condition 1; if $m \in \tilde{M}$ then by mean of property (3.1) we must have that there exists a $m' \in \tilde{M}$ such that $m' >_w m$. But then notice that (m', w) constitute a blocking pair, because m' ranks w as his first choice, because of Condition 2. □

If preferences satisfy Assumption 2.1, instead, there exists at least a stable matching. In the following subsection, we prove existence by showing how the algorithm introduced by Gale and Shapley [1962] extends to our case of infinitely many agents.

3.1 Generalized Deferred Acceptance algorithm

The man proposing Deferred Acceptance algorithm (from now on "DA algorithm"), as defined for finite markets, consists in the following procedure:

1. All men propose to the best acceptable woman in their preference list (if any);
2. All women who got at least an acceptable proposal, retain the best one, and reject the remaining ones. If a woman receives no acceptable proposal she rejects them all;
3. If a man has been rejected, in the second step of the algorithm he proposes to the second best acceptable woman (if any);
4. All women with multiple proposals (of which at most one can be the *retained one* from the previous period) pick the best one;
5. and so on…

Remark 3.2. The Deferred Acceptance algorithm can be defined with men proposing, and with women proposing. We will give the definition for men, but the one for women is analogous.

A key property of the DA algorithm in markets with finitely many agents is that it ends in finitely many steps - as no man can propose to the same woman more than once (see Gale and Shapley [1962]). Hence the matching provided by the algorithm is well-defined, and a simple argument proves it must be stable. Obviously, when dealing with countably many agents the DA algorithm might not end in finitely many steps (even if we assume that each agent finds only finitely many agents acceptable):

Example 1. *Consider the following preference scheme:*

$$\begin{array}{ll} P(m_1): w_1 & P(w_1): m_1, m_2 \\ P(m_2): w_1, w_2 & P(w_2): m_2, m_3 \\ P(m_3): w_2, w_3 & P(w_3): m_3, m_4 \\ \quad\cdots\cdots & \quad\cdots\cdots \\ P(m_i): w_{i-1}, w_i & P(w_i): m_i, m_{i+1} \\ \quad\cdots\cdots & \quad\cdots\cdots \end{array}$$

Following the algorithm described above, in the first step w_1 gets proposed to by m_1 and m_2, and rejects m_2 who then is left unmatched. In the second step m_2 proposes to w_2, who was withholding the proposal of m_3, but prefers m_2 and thus rejects m_3, who is temporarily left unmatched. In general, at the i-th step, we have that the tentative matching leaves unmatched m_{i+1}, who was rejected by w_i and the proposes to w_{i+1}, who then rejects m_{i+2}. Then in this market the DA algorithm does not end in finitely many iterations.

Even though the original algorithm is infinite, it is plain to see from the same example that we can tweak the original algorithm to find a generalized version that is well-defined also with countably many agents. The idea is to define a "limit" matching that as we will see in Theorem 3.5 below is stable for any set of preferences.

Definition 3.3 ((Man Proposing) - Generalized DA algorithm)**.** Consider the DA algorithm as defined in the finite market. If the algorithm finishes in a finite number of steps, call μ_M the matching obtained when in the first step in which there are no rejections.

If the algorithm is infinite, define $\mu^{(j)}(\cdot)$ to be the *tentative* matching arising in the $j-$step of the algorithm and then define the limit function μ_M as follows:[5]

$$\mu_M(m) := \begin{cases} \lim_{j\to\infty} \mu^{(j)}(m) & \text{if} \quad (\mu^{(j)}(m))_{j=1}^{+\infty} \quad \text{is eventually constant;} \\ m & \text{otherwise.} \end{cases}$$

And complete the matching in the natural way: if $\mu_M(m) = w$ for some m and w, then let $\mu_M(w) = m$. Otherwise let $\mu_M(w) = w$.

Remark 3.4. Observe that we implicitly used Assumption 2.1 in the definition of the generalized DA algorithm in two parts. First, once a man's proposal is rejected, the man passes to the next best woman and if his preferences do not satisfy Assumption 2.1, then there might not exist a *next* woman. Secondly, we assumed that a woman receiving multiple proposals picks the best one - and hence we implicitly assumed that according to her preferences the *best* proposal exists.

Theorem 3.5. *The μ_M so defined is a stable matching.*

Proof. The longer proofs are in B. □

As a corollary, we get that the set of stable matching is never empty, as μ_M is well defined for any set of preferences. Similarly, if we instead used the generalized woman proposing DA algorithm we would have found a stable matching that we define μ_W.

3.2 Ranking of Stable Matching

In general, the set of stable matching is not a singleton - and it has been shown that in the model with finitely many agents it is possible to define an order on the set of matching under which the structure of stable matching is a lattice.

Similarly to what is done in the literature, we define the following (partial) order, that formalizes what it means to say that all men prefer matching μ to μ':

[5]We consider the limit $\lim_{j\to\infty} \mu^{(j)}(m)$ only for sequences $(\mu^{(j)}(m))_{j=1}^{+\infty}$ that are eventually constant and therefore its meaning is independent of the topology one considers on the discrete W.

Definition 3.6 ($>_M$ order)**.** Given any two stable matching μ and μ' we say that:

$$\mu >_M \mu' \Leftrightarrow \begin{cases} \mu(m) \geq_m \mu'(m) & \forall m \in M; \\ \mu(m) >_m \mu'(m) & \exists m \in M; \end{cases}$$

In the symmetric way we can define the women order $>_W$. Whenever $\mu >_M \mu'$ we will say that μ *is preferred to* μ' *by men* or that μ is men-preferred to μ'. It is clear that such strict order relation is irreflexive, antisymmetric, and transitive. Also, this order is, in general, only a partial order[6].

A result that carries through in our study of countably many agents is that under $>_M$, the matching μ_M is the maximal element within the set of stable matching, that is:

Theorem 3.7 (Optimality of μ_M (under $>_M$))**.** *For any μ stable, we have that $\mu_M \geq_M \mu$.*

And obviously, the symmetric statement for women also holds:

$$\mu_W \geq_W \mu, \quad \text{for all } \mu \text{ stable.}$$

Another well-known result that we can prove in our setting is that the ranking $>_M$ and $>_W$ are opposite orders, in the sense specified in the following lemma:

Lemma 3.8. *Let μ and μ' be stable matching of the marriage market $(M, W, \mathbf{P})$, we have that:*

$$\mu >_M \mu' \Leftrightarrow \mu' >_W \mu.$$

This lemma implies a simple corollary:

Corollary 3.9. *For all μ stable we have that:*

$$\mu_M \geq_M \mu \geq_M \mu_W,$$

so in particular μ_W is man pessimal.

Proof. The first inequality follows from Theorem 3.7 for men, while the second inequality follows with the same Theorem for women, jointly with Lemma 3.8. □

[6] That is to say: there may exist μ and μ' such that neither $\mu >_M \mu'$ nor $\mu' >_M \mu$, because some men are better matched under μ and some other under μ'.

Remark 3.10. The properties proved thus far (and the ones of the next subsection) carry through from the finite case to our setting with countably many agents essentially because their proof relied on the definition of stability only. Whenever the result can be obtained by assuming the counterpositive (i.e. the logical negation of the thesis) and finding a blocking pair, the same kind of argument can be replicated here – and this is essentially what we do in the proofs of the appendix. The same does not hold for the Lattice Theorem, and the Rural Hospital Theorem.

3.3 Lattice Theorem

In this subsection, we will show that the set of stable matching endowed with the men-preferred order has the algebraic structure of a complete lattice. Corollary 3.9 showed that there exists a max and a min in the set of stable matching ordered with the man-preferred relation. In what follows, we will show that *any* family of stable matching admits a sup and an inf according to the order just defined.

A lattice is couple $(A, \succeq)$ where A is a set, and $\succeq$ is a partial order on A. We call $(A, \succeq)$ a lattice if it is closed under the operation of meet ($\vee$) and join ($\wedge$) - which can be thought of as the inf and sup on the reals.

Formally, given any two elements $a, b \in A$, the element $c := a \vee b$ is defined by the following properties: (i) $c \succeq a$ and $c \succeq b$; (ii) if $c' \succeq a$ and $c' \succeq b$, then $c' \succeq c$. In other words, $a \vee b$ is the least of the elements larger than a and b. Symmetrically, we can define the join of a and b, $c := a \wedge b$, as the element such that: (i) $c \preceq a$ and $c \preceq b$; (ii) if $c' \preceq a$ and $c' \preceq b$ then $c' \preceq c$.

In the context of stable matching, the order we use is the men-preferred order (Definition 3.6), and the operations of $\vee$ and $\wedge$ can be explicitly defined as follows:

$$\mu \vee \nu(x) := \begin{cases} \max_{\geq_x}\{\mu(x), \nu(x)\} & \text{if } x \in M \\ \min_{\geq_x}\{\mu(x), \nu(x)\} & \text{if } x \in W \end{cases} \tag{3.2}$$

and:

$$\mu \wedge \nu(x) := \begin{cases} \min_{\geq_x}\{\mu(x), \nu(x)\} & \text{if } x \in M \\ \max_{\geq_x}\{\mu(x), \nu(x)\} & \text{if } x \in W \end{cases}. \tag{3.3}$$

For any man m, $\mu \vee \nu(m)$ is defined as the best of the matching of m - according to his own preferences. For all women, instead, the matching $\mu \vee \nu(w)$ is the *worst* of the two. It is then clear that for all man m:

$$\mu \wedge \nu(m) \leq_m \mu(m), \nu(m) \leq_m \mu \vee \nu(m).$$

What is not obvious is the fact that both $\mu \vee \nu$ and $\mu \wedge \nu$ are stable matching - i.e. the set of stable matching is closed under the operation of $\vee$ and $\wedge$:

Theorem 3.11. *Let μ and ν be two stable matching on $(M, W, \mathbf{P})$, the functions $\vee$ and $\wedge$ defined in* (3.2) *and* (3.3) *yield stable matching.*

Furtheremore, if a set endowed with a partial order admits meet and join of arbitrary families of elements (instead of just two), then it is called a *complete lattice*. As a matter of fact, the proof of Theorem 3.11 defines the sup $\lambda = \mu \vee \nu$, but a similar argument can be done with the generalized sup $\lambda := \bigvee_{i \in I} \nu_i$, hence we have the following corollary:[7]

Corollary 3.12. *The set of stable matching with the $\vee$ and $\wedge$ operation is a complete lattice.*

Remark 3.13. All the results obtained so far could alternatively be obtained by framing the problem in a more abstract space, and reducing the problem of finding the set of stable matching to a fixed-point problem. This is the approach adopted in Hatfield and Milgrom [2005], where they study many-to-one matching in a finite setting. They introduce a notation under which a stable matching is a fixed point of an operator that matches couple of sets to couple of sets. Once this characterization is obtained, Tarski Fixed Point Theorem (see Tarski [1955]) implies non-emptiness of the set of stable matching, lattice structure of such set, and its completeness.

The same approach would work in our setting, essentially because Tarski Fixed Point Theorem works in very general settings, including one with countably infinite sets.

Such approach, though, would make the results less clear - in our opinion - and so we decided to provide a direct proof of all those results, without relying on stronger mathematical theorems.

Remark 3.14. In the finite setting, a *characterization* of the lattice structure was obtained by Gusfield et al. [1987], who proved that not only is the set of stable

[7]More precisely the proof of Theorem 3.11 assumed that the operation $\mu \vee \nu$ did not define a stable matching, and obtained that μ or ν then could not be stable. If instead one considers $\lambda := \bigvee_{i \in I} \nu_i$ (i.e. the sup of an arbitrary family) one can use the same argument to prove that if λ is not a stable matching, then $\exists\, i \in I$ such that ν_i is not stable.

matching a complete distributive lattice, but actually also the opposite holds: every finite distributive lattice is a set of stable matching for some matching market. This shows that set of stable matching and distributive lattices are effectively the same mathematical object.

We do not investigate the same statement in this paper, but we doubt that the same happens when the set of agents is countably infinite. In fact we conjecture that when agents are countably infinite the set of stable matching is either finite or uncountable. If this conjecture was true, then quite clearly it would not be true that every distributive lattice is a set of stable matching – as no countably infinite distributive lattices could be found as set of stable matching.

4 Rural Hospital Theorem

A well-known result in the theory of stable matching with finitely many agents is that the set of matched agents is constant across stable matching. Precisely, define the set of non-single men and women in a given matching μ as:

$$M(\mu) := \{m \in M \mid \mu(m) \neq m\},$$

$$W(\mu) := \{w \in W \mid \mu(w) \neq w\}.$$

Definition 4.1 (Rural Hospital Theorem)**.** We say that the rural hospital theorem holds if for all stable matching μ and μ':

$$M(\mu) = M(\mu'), \text{ and } W(\mu) = W(\mu'),$$

Whenever the set of agents is finite, it was proved that the Rural Hospital Theorem holds, see Roth [1986].

In our infinite framework, this need not be the case. Before analyzing why this anomaly arises, let us show an example of a marriage market $(M, W, \mathbf{P})$ in which there exist μ and μ' stable such that $M(\mu) \neq M(\mu')$.

Example 2 (Failure of the Rural Hospital Theorem)**.** *Consider the following* *preferences:*

$$\begin{array}{ll} P(m_1) : w_1 & P(w_1) : m_2, m_1 \\ P(m_2) : w_2, w_1 & P(w_2) : m_3, m_2 \\ P(m_3) : w_3, w_2 & P(w_3) : m_4, m_3 \\ \\ \ldots\ldots & \ldots\ldots \\ P(m_i) : w_i, w_{i-1} & P(w_i) : m_{i+1}, m_i \\ \ldots\ldots & \ldots\ldots \end{array}$$

In this example, there are two stable matching, the one obtained by man proposing DA algorithm, μ_M, and the one obtained by woman proposing, μ_W. It is plain to see that under the man-proposing algorithm we get $\mu_M(w_i) = m_i$ for all $i \in \mathbb{N}$; whereas when women propose we obtain $\mu_W(w_i) = m_{i+1}$ for all $i \in \mathbb{N}$, and $\mu_M(m_1) = m_1$.

Therefore we have that:

$$M(\mu_M) \neq M(\mu_W),$$

because $m_1 \in M(\mu_M) \setminus M(\mu_W)$.

The failure of the Rural Hospital Theorem implies another difference between the model with finitely many agents and the extension to infinitely many analyzed here. In the finite model, it has been proved that whenever the stable matching is not unique, under *any stable mechanism* at least one agent has incentive to misreport his/her preferences - assuming everybody else tells the truth (see Theorem 4.6 in Roth and Sotomayor [1992]). When there are countably many agents, instead, even when there are multiple stable matchings it might be that *no agent has incentives to misreport his/her preferences.* We show this in Appendix A.

Even though the Rural Hospital Theorem does not hold in our setting, the following result is still true:

Lemma 4.2. *Let μ and μ' be stable matching. If $\mu \leq_M \mu'$ then:*

$$M(\mu) \subseteq M(\mu').$$

In particular, for any μ we have that:

$$M(\mu_W) \subseteq M(\mu) \subseteq M(\mu_M);$$

$$W(\mu_M) \subseteq W(\mu) \subseteq W(\mu_W).$$

In particular, this lemma explains why in the finite case we could then conclude that the set of matched men is the same across stable matching, as if the set of men was finite, then the inclusions in Lemma 4.2 would imply that:

$$|M(\mu_W)| \leq |M(\mu)| \leq |M(\mu_M)|;$$

and

$$|W(\mu_M)| \leq |W(\mu)| \leq |W(\mu_W)|.$$

But trivially for any μ stable, $|W(\mu)| = |M(\mu)|$ – as each couple is made of one man and one woman – and then:

$$M(\mu) = M(\mu')$$

for all μ and μ' stable. The same is not true in the infinite case, because having two sets $A \subseteq B$ with $|A| = |B|$ does not imply that $A = B$ when A is countably infinite.[8]

To relate this result to the literature in large markets, it is useful to observe that in Azevedo and Leshno [2016] the Rural Hospital theorem was obtained by imposing an equilibrium condition of supply meeting demand (in measure). The finiteness of the measure of the continuum of agents, in their model, effectively plays the same role of the finiteness of the set of matched men and women. Hence in their setting they obtain that the Rural Hospital Theorem holds - differently from what found in our setting.

In our setting, the measure of matched agents is still constant across stable matching, if we measure the number of matched agents with the "counting" measure which arises naturally in our setting. In other words, if in a matching infinitely many agents are matched, then in all matching the set of matched agents is infinite (because of Lemma 4.2). Nonetheless, the Rural Hospital theorem can fail, because infinite sets can have the same cardinality as proper subsets, so even though there are always infinitely many matched agents, in

[8] As a trivial example, if A is the set of even numbers and B the set of natural numbers then $A \subsetneq B$, but they have the same counting measure, $|A| = |B|$.

some matching the set of matched men (or women) can be strictly larger.

We can characterize the instances in which the Rural Hospital theorem fails:

Proposition 4.3. *The Rural Hospital Theorem holds if and only if:*[9]

$$M = \mu_W \circ \mu_M(M).$$

Remark 4.4. Observe that the same statement could have been given for women - that is to say:

$$M = \mu_W \circ \mu_M(M) \Leftrightarrow W = \mu_M \circ \mu_W(W).$$

To see this, if by contradictions $M \supset \mu_W \circ \mu_M(M)$, pick $m \in M \setminus \mu_W \circ \mu_M(M)$, it is easy to check that $\mu_M(m) \in W$, and $\mu_M(m) \in W \setminus \mu_M \circ \mu_W(W)$.

The proposition provides necessary and sufficient conditions for the failure of the RH theorem using the fact that the two matching μ_M and μ_W are "extremal points" in the set of stable matching. In other words, say there exist a man m such that $\mu(m) \in W$ and $\mu'(m) = m$ for some μ, μ' stable matching. But then it must be that $\mu_M(m) \in W$ and $\mu_W(m) = m$, and hence it is possible to find a characterization based solely on μ_M and μ_W.

Remark 4.5. The characterization we found is not given in terms of the exogenous variables of the model, i.e. the preferences, but in terms of μ_M and μ_W which are endogenous. The link between the preferences and the matching μ_M and μ_W is given by the generalized DA algorithm and using that jointly with Proposition 4.3 it is possible to obtain a characterization of the failure of the Rural Hospital theorem in terms of the exogenous variables. A clearer connection between preferences and the failure of the theorem is hindered by the fact that different sets of preferences yield the same set of stable matching (and in particular the same μ_M and μ_W). Then the characterizations of the result will depend, implicitly or explicitly, on agents' preferences through the matching μ_M and μ_W.

As a corollary, we can provide some sufficient conditions for the Rural Hospital theorem to hold:

Corollary 4.6. *The Rural Hospital holds if one of the following conditions holds:*

[9] We write $f \circ g$ to mean the composition map: $x \mapsto f(g(x))$, and as always for any set A, $f(A) = \{y \mid y = f(x), \exists x \in A\}$.

- $|M(\mu)| < +\infty$ *for some* μ *stable;*
- *there exists a partition of* $M = \bigcup_j M_j$ *and* $W = \bigcup_j W_j$*, such that for all* j $|M_j|, |W_j| < +\infty$*, and for all* $m \in M_j$*,* m *finds acceptable only the women in* W_j*, mathematically:*

$$m >_m w, \quad \forall m \in M_j, w \in W \setminus W_j.$$

Proof. The proof of the first point is trivial, given Lemma 4.2. The second point is then obvious, because the infinite problem boils down to a sequence of independent finite problems, and for finite problems the Rural Hospital theorem holds. □

Notice how both of the conditions are sufficient but not necessary. To see this, consider the preferences obtained by changing those of Example 2. Let M and W be indexed on $\mathbb{Z}$ - that is a bi-infinite sequence.[10] Then let $P(m_i) : w_i, w_{i-1}$ and $P(w_i) : m_{i+1}, m_i$, for all $i \in \mathbb{Z}$.

It is easy to see how in this example there are two stable matching (μ_M matches m_i with w_i and μ_W matches m_i with w_{i-1}), and the Rural Hospital Theorem is met as $M(\mu_W) = M$ and $W(\mu_M) = W$. Nonetheless neither of the conditions of Corollary 4.6 are met.

4.1 Maximal stable matching

Whenever the Rural Hospital Theorem fails it is convenient to select a stable matching that maximizes the set of matched agents (in the sense of set inclusion). Formally we can define:

Definition 4.7 (Maximal Stable Matching)**.** A matching μ is maximal if for all μ' stable:

$$M(\mu) \cup W(\mu) \supseteq M(\mu') \cup W(\mu').$$

The following examples show two negative results: there might exist no stable matching that maximizes the set of matched agents (Example 3); and if there exists it might be different from the optimal matching μ_M and μ_W (Example 4). In Theorem 4.9 we will give a further description of the set of maximal matching, whenever it is not empty.

[10] Explicitly, let

$$M = \{\dots, m_{-i}, \dots, m_{-1}, m_0, m_1, \dots, m_i, \dots\}$$

and the same for W.

Example 3 (Non-Existence of maximal matching)**.** *Consider the following preferences:*

$$P(m_1) : w_3 \qquad P(w_1) : m_3$$
$$P(m_2) : w_4 \qquad P(w_2) : m_4$$
$$P(m_3) : w_5, w_1 \qquad P(w_3) : m_5, m_1$$

$$\ldots\ldots \qquad \ldots\ldots$$
$$P(m_i) : w_{i+2}, w_{i-2} \qquad P(w_i) : m_{i+2}, m_{i-2}$$
$$\ldots\ldots \qquad \ldots\ldots$$

It is easy to see that there are only two stable matching, μ_M and μ_W, and:

$$\{m_3, \ldots, m_i, \ldots\} = M(\mu_W) \subsetneq M(\mu_M) = \{m_1, \ldots, m_i, \ldots\},$$

and identically for women.

Example 4 (Existence of μ maximal, $\mu \neq \mu_M, \mu_W$)**.** *Consider the following preferences:*

$$P(m_1) : w_3, w_1 \qquad P(w_1) : m_3, m_1$$
$$P(m_2) : w_4, w_2 \qquad P(w_2) : m_4, m_2$$
$$P(m_3) : w_5, w_3, w_1 \qquad P(w_3) : m_5, m_3, m_1$$

$$\ldots\ldots \qquad \ldots\ldots$$
$$P(m_i) : w_{i+2}, w_i, w_{i-2} \qquad P(w_i) : m_{i+2}, m_i, m_{i-2}$$
$$\ldots\ldots \qquad \ldots\ldots$$

Similarly to the previous case, we have that:

$$\{m_3, \ldots, m_i, \ldots\} = M(\mu_W) \subsetneq M(\mu_M) = \{m_1, \ldots, m_i, \ldots\},$$

and identically for women.

In this case, though, there is another stable matching μ, defined by $\mu(m_i) = w_i$. Clearly, it is maximal in the sense of set inclusion of matched agents, as the set of matched agents is $M \cup W$.

To analyze the set of maximal matchings, let us define two subsidiary sets in terms of which we can provide a description of the set of maximal matchings.

Definition 4.8 (Man maximal and woman maximal matchings)**.** A stable matching μ is said to be man maximal if:

$$M(\mu) \supseteq M(\mu'),$$

for all μ' stable. The definition for women is analogous.

The following theorem describes the set of man and woman maximal matchings and how they relate to the set of maximal matching.

Theorem 4.9. *The set of man maximal matching is a non-empty sublattice of the set of stable matching (under the meet and join operations defined in 3.2 and 3.3, and the man-preferred or woman-preferred order) .*

Corollary 4.10. *The set of maximal matchings is a (possibly empty) sublattice of the set of stable matching.*

Proof of Corollary 4.10. Observe that the set of maximal matchings is the intersection of the set of man maximal agents, and the set of woman maximal agents. Then being the intersection of two sublattices, it is a (possibly empty) sublattice.[11] □

Proof of Theorem 4.9. Notice that the non-emptiness of either maximal set is trivial, since by Lemma 4.2 we have that:

$$M(\mu_M) \supseteq M(\mu), \quad \forall\, \mu \text{ stable},$$

and hence the set of man maximal matching always contains μ_M. To prove that the set of man maximal matching is also a sublattice, notice that if μ and μ' are man maximal, then in particular $M(\mu) = M(\mu')$. But if $m \in M(\mu) \cap M(\mu')$ then $m \in M(\mu \vee \mu')$ and $m \in M(\mu \wedge \mu')$. Hence:

$$M(\mu \vee \mu') = M(\mu \wedge \mu') = M(\mu') = M(\mu),$$

and then $\mu \vee \mu'$ and $\mu \wedge \mu'$ are maximal too. □

[11] Certain textbooks define a sublattice as a *nonempty* subset closed under meet and join operations. If one refers to that definition, then a correct statement would be to say that the set of maximal matching is *either empty or a sublattice* of the set of stable matching.

5 Application to Dynamic Matching Markets

An application which requires a model with countably infinite agents is given by dynamic matching markets. Even though any real market is finite, agents often have preferences for matches in different periods. When we then model the matching market of this year, we have to consider the preferences of this year's agents to possibly include agents entering next year. But then a complete description of the market would include next year's agents, who in turn might find acceptable agents entering the market two years from now. Proceeding in this fashion we get that the *smallest model* describing the preferences for this matching market must include infinitely many agents, from this year's till indefinitely in the future.

More formally, let M^t and W^t be the set of men and women entering the market at time t. Agents in M^t or W^t can find agents who entered before or who are going to enter in the future acceptable. We assume that the preferences of each agent are exogenously given and summarized in the preference vector $\mathbf{P}$. The object of interest is a matching μ that is stable in the market given by $(\bigcup_{t\in\mathbb{N}} M^t, \bigcup_{t\in\mathbb{N}} W^t, \mathbf{P})$.

Depending on the application, only certain preferences $\mathbf{P}$ might be considered. To illustrate this with an example, consider the matching high-school graduates to colleges.[12] Interpret period t as a given graduation year, M^t as the high-school students graduating in year t, and W^t as the college positions rendered available at time t. This application induces a structure on the set of preferences that each agents can hold. Students might find it acceptable to be admitted by colleges in further years, i.e. for $t' \geq t$; because they can either decide to apply the year they graduate, or postpone to applications to a year in the future. Conversely, a college which admits student in a given year t, will be able to admit only students who already hold a high-school diploma, and to rephrase this in terms of preferences, a college in year t will find acceptable a subset of the students who graduated in $t' \leq t$.

To formalize the idea of the example just described, and in general any application to dynamic matching markets, we define *preferences for the future/past*.

Definition 5.1. We say that a man $m \in M^t$ has *preferences for the future*, if he finds acceptable (a subset of the) women entering the market in $t' \geq t$.

[12] We abstract from the fact that this would be a many-to-one matching problem, to focus on the dynamic nature of the model.

Mathematically:

$$m >_m w, \quad \forall w \in W^s, s < t,$$

i.e. he finds *unacceptable* any woman who entered the market in the past.

On the other hand, he has *preferences for the past* if:

$$m >_m w \quad \forall w \in W^s, s > t,$$

and we use the same definitions for women.

When all agents on one side of the market have preferences that satisfy the properties just defined, we can give a description of the set of stable matching by using the theory developed in the previous section.

Theorem 5.2. *If all men and women have preferences for the future (or the past) then the dynamic matching market is isomorphic to a sequence of "disjoint" finite matching markets in which all agents find only women in their year acceptable.*[13]

If all men have preferences for the past, and women have preferences for the future, then the market is in general not isomorphic to a sequence of finite markets.

We can rephrase the statement of the theorem as follows. If the dynamic market one is considering induces preferences for men and women that have the same structure, (say, both for the future) then analyzing the dynamic market is equivalent to analyzing a sequence of finite markets. This isomorphism can be shown explicitly by modifying the preferences of all agents: let $\mathbf{P}$ be the original preferences, and $\tilde{\mathbf{P}}$ be the preferences where all agents find unacceptable any agent entering the market in any other period (whereas the ranking of agents in the same period are not changed). Then $(M, W, \mathbf{P})$ and $(M, W, \tilde{\mathbf{P}})$ have the same set of stable matching.

On the other hand, if the structure of preferences for men is opposite to that of women, then the set of stable matching in the dynamic market is (in general) different from the set of stable matching obtained by considering each period's market separately. In this second case, then, the analysis carried out in the previous sections gives a description of the set of stable matching in the infinite market.

[13] In this context isomorphic refers to the set of stable matchings. Formally, two markets $(M, W, \mathbf{P}_1)$ and $(M, W, \mathbf{P}_2)$ with different preferences are isomorphic if the set of stable matchings is the same.

Proof. We will prove the first statement by assuming that both men and women have preferences for the future.

Consider a man $m \in M^t$, by assumption the set of women acceptable by m is a subset $\bigcup_{s \geq t} W^s$. Since each woman in $\bigcup_{s>t} W^s$ has preferences for the future, she will not find m acceptable. So we can eliminate from m's preferences all the women entering the market in the future - without changing the set of stable matching. Doing the same with all the other men and women we obtain an isomorphic problem in which each man in M^t (resp. woman in W^t) finds only women in W^t (resp. men in M^t) acceptable. Clearly in this modified problem, each period t is independent from any other period t', so the conclusion follows.

For the second part, consider the Example 2 above, where the failure of the Rural Hospital Theorem was shown. Then divide the set of men and women in sets of two agents and defining $M^i := \{m_{2i}, m_{2i+1}\}$, and $W^i := \{m_{2i}, m_{2i+1}\}$. Then in the market thus defined women have preferences for the future, and men have preferences for the past - and plainly the market is not isomorphic to a sequence of finite markets, otherwise the Rural Hospital theorem would hold, thanks to Corollary 4.6. □

6 Conclusions

We studied one-to-one two sided matching with countably infinite agents, showing that the set of stable matching constitutes a non-empty lattice, if agents' preferences satisfy our assumption on the maximum. We showed that the Rural Hospital Theorem might fail in our setting and we described the implications that this has for dynamic matching markets.

As a direction for future research, it would be interesting to find conditions on the preferences of agents that are necessary and sufficient for the Rural Hospital Theorem does hold. The characterization we provide in 4.3 is given in terms of the maximal matchings μ_M and μ_W, thus effectively *endogenous variables*. Also, we could describe the set of maximal stable matchings, but it would be interesting to provide a mechanism that maximizes the set of matched agent in any matching market, whenever such maximizer exists.

As pointed out in the introduction, the failure of the Rural Hospital Theorem implies that the manipulability result observed in the finite result (Theorem A.1) may fail to hold in our case. Precisely, we show that in a market with multiple stable matchings, all agents might have no incentives to misreport their preferences. We provide a simple example in which this happens, but it would be interesting for future developments to understand if this happens all the time the Rural Hospital Theorem does not hold.

Finally, it would be interesting to study dynamic matching markets with strategic agents. The inefficiency due to the failure of the Rural Hospital Theorem might be worsened by agents' strategically misreporting their preferences to secure a matching in their period, instead of waiting. Furthermore, if one introduced uncertainty about the preferences of agents entering the market in the future, then agents' attitude toward risk would also need to be modeled. This extensions of the model studied in this paper would provide further insights on the strategic issues of matching markets opening repeatedly.

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A Non-Manipulability

In the marriage model with finitely many agents, we had the following result:

Theorem A.1 (Theorem 4.6 from Roth and Sotomayor [1992])**.** . *When any stable mechanism is applied to a marriage market in which preferences are strict and there is more than one stable matching, then at least one agent can profitably misrepresent his or her preferences, assuming the others tell the truth*

On the other hand, with countably many agents the failure of the Rural Hospital Theorem implies the failure of that results - as we show by mean of an example.

Consider the preferences of Example 2, and let the stable mechanism be given by the women proposing Deferred Acceptance algorithm.

Under such algorithm if every agent reports truthfully, the outcome is the matching μ_W defined by:

$$\mu_W(w_i) = m_{i+1} \quad \text{and} \quad \mu_W(m_1) = m_1.$$

Notice that no woman has strict incentives to misreport, as she is already matched to the best agent according to her preferences. Also, each man has no incentive to misreport:

- if man m_1 manipulates his preferences, he remains single, as no woman finds him acceptable - except for w_1, who is matched to m_2;
- if man m_j ($j \geq 2$) manipulates his preferences then:
  - if he truncates his preferences to his best choice, w_j, then he remains single;
  - if he adds further women to the list of preferences or shuffles them, he remains matched to w_{j-1} - because she is the only one proposing to him.

B Proofs

Remark B.1. Most of the proofs in this Appendix are adapted from Roth and Sotomayor [1992], where the authors provide the same results for the finite marriage markets. Some of those proofs rely uniquely on the existence and definition of blocking pair, and then can be applied directly to our setting too.

Proof of Theorem 3.5. First off, let us prove that such function is in fact a matching: we need to prove that either a man is single, or he is matched to a woman (for the women it is trivial). If m is not single, then $\mu_M(m) = x \neq m$. But then, according to the algorithm, it means that $(\mu^{(j)}(m))_j$ is eventually x, and we know that $\mu^{(j)}(m) \in W \cup \{m\}$ for all j, so $x \in W$. Identically, if a woman is not single under μ_M, it means that $\mu^{(j)}(w) \in M$ for infinitely many j.

Also, it cannot be the case that $\mu_M(m) = \mu_M(m') \in W$, for some $m \neq m'$. If this was the case, then $\mu^{(j)}(m)$ and $\mu^{(j)}(m')$ eventually agree, so in particular $\exists K \in \mathbb{N}$ such that for all $k' \geq K$, $\mu^{(k')}(m) = \mu^{(k')}(m')$. But this can never happen in the algorithm defined, as in step K the woman $w = \mu^{(K)}(m)$ would have rejected either m or m', who then cannot re propose to her. Therefore, we get an absurd.

Let us prove that such a matching is also stable. In order to prove it, observe first that:

$$\mu^{(j)}(w) \geq_w \mu^{(l)}(w), \quad \forall j \geq l, \tag{B.1}$$

which means that as the algorithm proceeds each woman can only do better (according to her preferences). This follows directly from the fact that the women withhold the best proposal they got so far, so either $\mu^{(j+1)}(w) = \mu^{(j)}(w)$ (if they reject the new proposal they got, if any), or $\mu^{(j+1)}(w) >_w \mu^{(j)}(w)$ (if they decide to reject the previous pick and withhold the new proposal).

Now, let us prove that the matching obtained is stable:

- no agent is matched with an unacceptable agent: this is clear on the side of the men, since they only propose to acceptable women; and similarly easy for women, as they accept only proposal by acceptable men;

- no pair blocks the matching: suppose there exists a pair (m, w) such that $w >_m \mu_M(m)$. Then according to the algorithm, we would have that m proposed to w at some step j, and at some step $j' \geq j$, w rejected m in favor, say, of m'. But then, using equation (B.1) we have that:

  $$\mu_M(w) >_w m,$$

  so the pair (m, w) could not block the matching.

□

Proof of Theorem 3.7. In order to prove such theorem, it is convenient to introduce an expression we will use frequently.

Definition B.2 (Achievable Matching)**.** A woman w is achievable by a man m in a marriage market $(M, W, \mathbf{P})$ if there exists μ stable such that $\mu(m) = w$.

Now observe that the statement of the theorem is equivalent to saying that: *along the man proposing Deferred Acceptance algorithm no man is rejected by an achievable woman.* As a matter of fact if this is the case, it means that for any m, $\mu_M(m) \geq \mu(m)$, for any μ stable, which is the thesis.

We will prove the theorem by induction on the steps of the algorithm.

Suppose that in the first j steps of the algorithm no man has been rejected by an achievable woman, we argue by contradiction supposing that in the step $j+1$ some man is rejected by an achievable woman. Suppose that this achievable woman w rejects m in favor of m': this implies that:

$$m' >_w m. \tag{B.2}$$

Now, since m' was not rejected by any achievable woman (by the induction hypothesis), we must have that:

$$w >_{m'} \mu(m'), \quad \forall \mu \text{ stable.} \tag{B.3}$$

Now consider $\tilde{\mu}$ to be the stable matching that matches m to w (such matching must exist, since by assumption w is achievable by m). Now, notice that by equations (B.2) and (B.3) we have that (m', w) is a blocking pair for $\tilde{\mu}$ (as (B.3) holds *for any* stable matching), thus we get the contradiction that implies the thesis. □

Proof of Lemma 3.8. Observe that the proof of $\Rightarrow$ and $\Leftarrow$ are symmetric, therefore it is enough to prove that for any μ and μ' stable:

$$\mu >_M \mu' \Rightarrow \mu' >_W \mu.$$

Let us argue by contradiction: suppose that $\mu >_M \mu'$, but for a woman w:

$$\mu(w) >_w \mu'(w). \tag{B.4}$$

This implies that under μ, w is matched to a man (otherwise she would be indifferent between the two outcomes, as $\mu(w) = w = \mu'(w)$). Let $m = \mu(w)$. Observe that since $\mu >_M \mu'$ we also have, for man m:

$$\mu(m) \geq_m \mu'(m),$$

and actually even more, $\mu(m) >_m \mu'(m)$, as if this was not the case, then $\mu'(m) = \mu(m) = w$. But being μ and μ' matching we would then have that $\mu'(w) = m = \mu(w)$, thus contradicting equation (B.4) - which is strict, so it rules out indifference. Therefore we can conclude that:

$$\mu(m) >_m \mu'(m). \tag{B.5}$$

But then notice that using equations (B.4) and (B.5) (and the fact that $\mu(w) = m$) we have that (m, w) constitutes a blocking pair for μ', hence μ' is not stable, which contradicts the hypothesis of μ and μ' being both stable. □

Proof of Theorem 3.11. Since the proofs are identical, we will only prove that $\lambda := \mu \vee \nu$ is a stable matching. The fact that λ is the least upper bound is trivial - given it's definition - hence we have to prove:

1. λ is a matching, i.e. if $\lambda(m) = w$, then $\lambda(w) = m$;[14]
2. λ is stable;

Let us prove both statements by contradiction:

1. suppose that $\lambda(m) = w$ but $\lambda(w) \neq m$. Suppose furthermore (wlog) that $\lambda(m) = \mu(m)$, which implies that:

$$w = \mu(m) >_m \nu(m).$$

If $\lambda(w) \neq m$, it means $\lambda(w) = \nu(w)$, and then:

$$m = \mu(w) >_w \nu(w).$$

But then notice that ν is instable because (m, w) are a blocking pair.

2. now to prove that the matching λ is stable, notice that:

   - individual rationality is trivial - given the individual rationality of μ and ν;
   - for blocking pairs, suppose there is one. I.e. suppose that:

$$w >_m \lambda(m) \tag{B.6}$$

$$m >_w \lambda(w) \tag{B.7}$$

suppose wlog that $\lambda(m) = \mu(m)$ and $\lambda(w) = \nu(w)$.[15] This in turn buys us that:

$$\lambda(m) = \mu(m) >_m \nu(m) \tag{B.8}$$

$$\mu(w) >_w \nu(w) = \lambda(w) \tag{B.9}$$

Observe then that by (B.8) and (B.6):

$$w >_m \lambda(m) >_m \nu(m),$$

[14] The further property that if $\lambda(m) \neq m$, then $\lambda(m) \in W$ is trivial.

[15] Clearly it cannot be that $\lambda(m) = \mu(m)$ and $\lambda(w) = \mu(w)$ otherwise μ would be unstable.

we are done if we prove that $m >_w \nu(w)$, but notice that:

$$m >_w \lambda(w) = \nu(w),$$

by (B.9) - and hence ν has a blocking pair too, absurd.

$\square$

Proof of Lemma 4.2. Notice that if $m \in M(\mu)$ and $\mu \leq_M \mu'$ then $m's$ match under μ' must be at least as good as that under μ, i.e. $\mu(m) \leq_m \mu'(m)$. Since $\mu(m) \in W$, by individual rationality of μ we must have that $\mu'(m) \in W$, too, hence $m \in M(\mu')$. We then obtain that $M(\mu) \subseteq M(\mu')$, and the second part of the statement follows directly. $\square$

Proof of Proposition 4.3. Suppose the Rural Hospital theorem does not hold. Then either $M(\mu_W) \subsetneq M(\mu_M)$, or $W(\mu_M) \subsetneq W(\mu_W)$. In the first case pick $m \in M(\mu_M) \setminus M(\mu_W)$: clearly $m \in M$, but $m \notin \mu_W \circ \mu_M(M)$, as $\mu_M(m) \in W$, and $m \notin \mu_W(W)$ (as $\mu_W(m) = m$ by hypothesis).

In the second scenario, there exists a woman $w \in W(\mu_W) \setminus W(\mu_M)$. Pick $m := \mu_W(w)$: let us prove that $m \in M \setminus \mu_W \circ \mu_M(M)$. We have that: μ_M maps $\mu_M(M)$ into $W \setminus \{w\}$, but then $m \notin \mu_W(W \setminus \{w\})$, thus $m \in M \setminus \mu_W \circ \mu_M(M)$.

Now, for the opposite implication, suppose the Rural Hospital theorem holds, then for every $m \in M(\mu_M)$, we have that $m \in M(\mu_W)$ (and similarly for women). Let us prove that then $M = \mu_W \circ \mu_M(M)$. Trivially, $M \supset \mu_W \circ \mu_M(M)$, as:

- if m is single under μ_M, then so he is under μ_W, hence $\mu_W(\mu_M(m)) = m$;
- if m is matched to w under μ_M, then w is matched to a man under μ_W, hence $\mu_W(\mu_M(m)) \in M$.

Now, suppose by contradiction that $\exists m \in M \setminus \mu_W \circ \mu_M(M)$.

Clearly, m cannot be single under μ_M, as if he were, then he would be single also under μ_W (because $\mu_M(m) \geq_m \mu_W(m)$ - and μ_W is individually rational), then $\mu_W \circ \mu_M(m) = m \Rightarrow m \in \mu_W \circ \mu_M(M)$.

If m is not single under μ_M, then he is neither under μ_W (by the Rural Hospital Theorem), and then $\exists w \in W$ such that

$$\mu_W(w) = m. \tag{B.10}$$

This woman is matched under μ_W, and therefore she is also matched by μ_M, and therefore $\exists \tilde{m} \in M$ such that

$$\mu_M(\tilde{m}) = w. \tag{B.11}$$

Observe then that by equations (B.10) and (B.11):

$$\mu_W \circ \mu_M(\tilde{m}) = m \Rightarrow m \in \mu_W \circ \mu_M(M),$$

which contradicts the hypothesis and then proves the theorem. □

Proof. Let us prove that $M(\mu_W) \subseteq M(\mu)$ for any μ stable, the other inclusions are proved in the very same way.

Pick $m \in M(\mu_W)$, since by Corollary 3.9 $\mu_W \leq_M \mu$, then in particular:

$$\mu_W(m) \leq_m \mu(m).$$

Since $\mu_W(m) \neq m$ by hypothesis, then also $\mu_W(m) >_m m$, which in turn implies that $\mu(m) >_m m$, and thus $m \in M(\mu)$. □