Surge Pricing and its Spatial Supply Response

Omar Besbes[1], Francisco Castro[1], and Ilan Lobel[2]

[1]Columbia Business School
[2]NYU Stern School of Business

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Abstract

We consider the pricing problem faced by a platform matching price sensitive customers to flexible supply units within a geographic area. This can be interpreted as the problem faced in the short-term by a ride-hailing platform trying to match supply and demand within a city. We propose a framework where a platform chooses prices for the different locations, and drivers respond by choosing where to relocate based on prices, travel costs and driver congestion levels. We derive structural results for supply equilibria and the equilibrium profits supply units garner, and show that the platform's problem can be reformulated as a function of these. We also show how the problem can be spatially decomposed based on attraction regions. We then analyze a demand shock scenario to highlight the key features of an optimal pricing policy and the supply response it generates. We characterize the optimal policy and the implications of the strategic nature of supply units. We show that the platform will use prices to create artificially damaged regions where driver congestion is artificially high in order to lure drivers towards more profitable locations for the platform. Furthermore, the optimal solution, while better balancing supply and demand around the region of the demand shock, also incentivizes some drivers to move away from the demand shock in search of better conditions.

Keywords: spatial pricing, revenue management, ride-hailing, strategic supply, market design.

1 Introduction

Pricing and revenue management have seen significant developments over the years in both practice and the literature. At a high level, the main focus has been to investigate tactical pricing decisions given the dynamic evolution of inventories, with prototypical examples coming from the airline, hospitality and retail industries (Talluri & Van Ryzin (2006)). With the emergence and multiplication of two-sided marketplaces, a new question has emerged: how to price when capacity/supply units are strategic and can decide when and where to participate. This is particularly relevant for ride-hailing platforms such as Uber and Lyft. In these platforms, drivers are independent contractors who have the ability to relocate strategically within their cities to boost their own profits. On the one hand, this leads to a more flexible supply. On the other hand, one is not able to simply reallocate supply across locations when needed, but rather a platform needs to ensure that incentives are in place for a "good" reallocation to take place. Consider the spatial pricing problem within a city faced by a platform that shares its revenues with drivers. Suppose there are different demand and supply conditions across the city. The platform may want to increase prices at locations with high demand and low supply. Such an increase would have two effects. The first effect is a local demand response, which pushes the riders who are not willing to pay a higher price away from the system. The second effect is global in nature, as drivers throughout the city may find the locations with high prices more attractive than the ones where they are currently located and maybe decide to relocate. In turn, this may create a deficit of drivers at some locations. In other words, prices set in *one region* of a city impact demand and supply at this region, but also potentially impact supply in *other regions*. This brings to the foreground the question of how to price in space when supply units are strategic.

The central focus of this paper is to understand the interplay between spatial pricing and supply response. In particular, we aim to understand how to optimally set prices across locations in a city, and what the impact of those prices is on the strategic repositioning of drivers. To that end, we consider a short-term model over a given timeframe where overall supply is constant. That is, drivers respond to pricing and congestion by moving to other locations, but not by entering or exiting the system. In our short-term framework, the platform's only tool for increasing the supply of drivers at a given location is to encourage drivers to relocate from other places. In turn, this time scale permits us to isolate the spatial implications on the different agents' strategic behavior. In this sense, our model can be thought of as a building block to better understand richer temporal-dynamic environments.

In more detail, we consider a revenue-maximizing platform that sets prices to match price-sensitive riders (demand) to strategic drivers (supply) who receive a fixed commission. In making their decisions, drivers take into account prices, supply levels across the city, and transportation costs. More formally, we consider a measure-theoretical Stackelberg game with three groups of players: a platform, drivers and potential customers. Supply and demand are non-atomic agents, who are initially arbitrarily positioned. We use non-negative measures to model how these agents are distributed in the city. All the players interact with each other in a linear city. Every location can admit different levels of supply and demand. The platform moves first, selecting prices for the different locations around the city. Once prices are set, the set of customers willing to pay such levels is determined. Then, drivers move in equilibrium in a simultaneous move game, choosing where to reposition based on prices, supply levels and driving costs. In fact, besides prices and transportation costs, supply levels across the city are a key element for drivers to optimize their repositioning. If too many other drivers are at a given location, a driver relocating there will be less likely to be matched to a rider, negatively affecting that driver's utility. The platform's optimization problem consists of finding optimal prices for all locations given that drivers move in equilibrium.

Main contributions. Our first set of contributions is methodological. We propose a general framework that encompasses a wide range of environments. Our measure-theoretical setup can be used to study spatial interactions in both discrete and continuous settings. In this general framework, we develop structural properties of the equilibrium utilities of drivers and prove that the city admits a form of spatial decomposition. Furthermore, we establish that the equilibrium utility of drivers admits a fundamental upper bound driven by the local driver congestion level. In turn, we leverage these properties to provide a crisp structural characterization of an optimal pricing solution and its corresponding supply response. This characterization provides a one-to-one mapping between the equilibrium utilities and the optimal prices and equilibrium flows.

In our second set of contributions, we shed light on the scope of prices as an incentive mechanism for drivers and provide insights into the structure of an optimal policy. To that end, we study a special family of cases in which a central location in the city, the origin, experiences a shock of demand. To put the optimal policy in perspective, we first characterize an optimal *local price response* policy, a pricing policy that only optimizes the price at the demand shock location. Such a policy increases prices at the demand shock location leading to an attraction region around the shock in which drivers move toward the origin.

Leveraging our earlier methodological results in conjunction with the derivation of new results, we characterize in quasi-closed form the optimal pricing policy and its corresponding supply response. The optimal policy admits a much richer structure. Quite strikingly, the optimal pricing policy induces movement toward the demand shock but potentially also *away* from the demand shock. The platform may create *damaged regions* through both prices and congestion to steer the flow of drivers toward more profitable regions. Compared to the *local price response* policy, the optimal solution or *global price response* incentivizes more drivers to travel toward the demand shock.

The optimal pricing policy splits the city into six regions around the origin (Figure 1). The mass

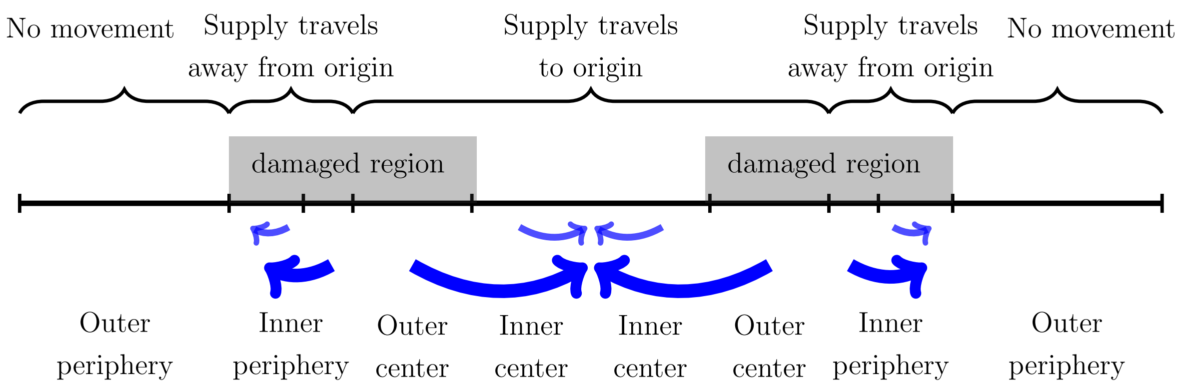

Figure 1: **The optimal solution creates six regions.**

of customers needing rides at the location of the shock is serviced by three subregions around it: the origin, the inner center and the outer center. The origin is the most profitable location and so the platform surges its price, encouraging the movement of a mass of drivers to meet its high levels of demand. These drivers come from both the inner and outer center. In the former, locations are positively affected by the shock, and some drivers choose to stay in them while others travel toward the origin. In the latter, drivers are too far from the demand shock and so the platform has to *deliberately damage* this region through prices (e.g., by using excessively high prices) to create incentives for drivers to relocate toward the origin. However, drivers in this region have an option: instead of driving toward the demand shock at the origin, they could drive away from it. This gives rise to the next region, the inner periphery. Consider the marginal driver, i.e., the furthest driver willing to travel to the origin. To incentivize the marginal driver to move to the origin, the platform is obligated to also damage conditions in the inner periphery. The optimal solution creates two subregions within the inner periphery. In the first, conditions are degraded through prices that make it unattractive for drivers. Drivers in this region leave toward the second region. That is, they drive in the direction opposite to the demand shock. The action of the platform in the second region is more subtle. Here, the platform does not need to play with prices. The mere fact that drivers from the first region run away to this area creates congestion, and this is sufficient degradation to make the region unattractive for the marginal driver. The final region is the outer periphery, which is too far from the origin to be affected by its demand shock.

We complement our analysis with a set of numerics that highlights that the optimal policy can generate significantly more revenues than a local price response. In other words, anticipating the global supply response and taking advantage of the full flexibility of spatial pricing plays a key role in revenue optimization.

2 Related Literature

Several recent papers examine the operations of ride-hailing platforms from diverse perspectives. We first review works that do not take spatial considerations into account. There is a recent but significant body of work on the impact of incentive schemes on agents' participation decisions. Gurvich et al. (2016) study the cost of self-scheduling capacity in a newsvendor-like model in which the firm chooses the number of agents it recruits and, in each period, selects a compensation level as well as a cap on the number of available workers. Cachon et al. (2017) analyze various compensation schemes in a setting in which the platform takes into account drivers' long-term and short-term incentives. They establish that in high-demand periods all stakeholders can benefit from dynamic pricing, and that fixed commission contracts can be nearly optimal. The performance of such contracts in two-sided markets is analyzed by Hu & Zhou (2017) who derive performance guarantees. Taylor (2017) considers how uncertainty affects the price and wage decisions of on-demand platforms when facing delay-sensitive customers and autonomous capacity. Nikzad (2018) focuses on the effect of market thickness and competition on wages, prices and welfare and shows that, in some circumstances, more supply could lead to higher wages, and that competition across platforms could lead to high prices and low consumer welfare.

There is also a literature on matching in ride-hailing without pricing. Feng et al. (2017) compare the waiting time performance, in a circular city, of on-demand matching versus traditional street-hailing matching. Hu & Zhou (2016) analyze a dynamic matching problem as well as the structure of optimal policies. In a related study, Ozkan & Ward (2016) develop a heuristic based on a continuous linear program to maximize the number of matches in a network. Afèche et al. (2017) study demand admission controls and drivers' repositioning in a two-location network, without pricing. They show that the value of the controls is large when both capacity is moderate and demand is imbalanced.

Most closely related to our work are papers that study pricing with spacial considerations. Castillo et al. (2017) takes space into account, but only in reduced form through the shape of the supply curve. This paper points out that surge pricing can help to avoid an inefficient situation termed the "wild goose chase" in which drivers' earnings are low due to long pick-up times. Banerjee et al. (2015) consider a queueing network where drivers do not make decisions in the short-term (no repositioning decisions) but they do care about their long-term earning. They prove that a localized static policy is optimal as long as the system parameters are constant, but that a dynamic pricing policy is more robust to changes in these parameters. Banerjee et al. (2016) find approximation methods to find source-destination prices in a network to maximize various long-run average metrics. Customers have a destination and react to prices, but supply units do not behave strategically. Bimpikis et al. (2016) focus on pricing for steady-state conditions in a network in which drivers behave in equilibrium and decide wether and when to provide service as well as where to reposition. They are able to isolate an interesting "balance" property of the network and establish its implications for prices, profits and consumer surplus. Buchholz (2017) structurally estimates a spatial equilibrium model to understand the welfare costs of taxi fare regulations. These papers investigate long-term behavior associated with spatial pricing. In contrast, our work examines how the platform should respond to short-term supply-demand imbalances given that the supply units are strategic.

From a methodological point of view, our work borrows tools from the literature on non-atomic congestion games. Our equilibrium concept is similar to the one used by Roughgarden & Tardos (2002) and Cole et al. (2003) to analyze selfish routing under congestion in discrete settings: in equilibrium, drivers only depart for locations that yield the largest earnings. We consider a more general measure-theoretical environment that can be traced back to Schmeidler (1973) and Mas-Colell (1984). Our work is also related to the literature on optimal transport (see Blanchet & Carlier (2015)). Once the platform sets prices, drivers must decide where to relocate. This creates a "flow" or a "transport plan" in the city from initial supply (initial measure) to post-relocation supply (final measure). However, in our problem, the final measure is endogenous.

Finally, some of our insights relate back to the damaged goods literature. Deneckere & McAfee (1996) explain that a firm can strategically degrade a good in order to price discriminate. In our setting the platform can damage some regions in the city through prices and congestion to steer drivers toward more profitable locations and thus increase revenues.

Our linear city framework relates to the class of Hotelling models (Hotelling 1990), which are typically used to study horizontal differentiation of competing firms. In contrast to this classical stream of work, we consider a monopolist who can set prices across all locations. Furthermore, these prices affect the capacity at each location and supply units can choose among all regions of the city to provide service.

3 Problem Formulation

Preliminaries. Throughout the paper, we will use measure-theoretic objects to represent supply, demand and related concepts. This level of generality will enable us to capture the rich interactions that arise in the system through a continuous spatial model. The continuous nature of space simplifies our solution, enabling us to express the solution to special cases of interest in quasi-closed form. To that end, we introduce some basic notation. For an arbitrary metric set $\mathcal{X}$ equipped with the Borel $\sigma-$algebra, we let $\mathcal{M}(\mathcal{X})$ denote the set of non-negative finite measures on $\mathcal{X}$. For any measure τ, we denote its restriction to a set $\mathcal{B}$ by $\tau|_{\mathcal{B}}$. The notation $\tau \ll \tau'$ represents measure τ being absolutely continuous with respect to measure τ'. The notation $\operatorname{ess\,sup}_{\mathcal{B}}$ corresponds to the essential supremum, which is the measure-theoretical version of a supremum that does not take into account sets of measure zero. To denote the support of any measure τ we use $\text{supp}(\tau)$. The notation $\tau - a.e.$ represents almost everywhere with respect to measure τ. For any measure τ in a product space $\mathcal{B} \times \mathcal{B}$, τ_1 and τ_2 will denote, respectively, the first and second marginals of τ. We use $\mathbf{1}_{\{\cdot\}}$ to denote the indicator function, and $S^{\circ}, \partial S, \overline{S}, S^c$ to represent the interior, boundary, closure and complement of a set S respectively. If $F(\cdot)$ is a cumulative distribution function, then $\overline{F}(q) = 1 - F(q)$. For consistency, we use masculine pronouns to refer to drivers and feminine ones to refer to customers.

3.1 Model elements

Our model contains four fundamental elements: a city, a platform, drivers and potential customers. We represent the city by a line interval $\mathcal{C} = [-H, H]$, for some $H \geq 0$ and a measure $\mathit{\Gamma}$ in $\mathcal{M}(\mathcal{C})$. We refer to this measure as the city measure and it characterizes the "size" of every location of the city. For example, if $\mathit{\Gamma}$ has a point mass at some location then that location is large enough to admit a point mass of supply and demand.

Demand (potential customers) and supply (drivers) are assumed to be infinitesimal and initially distributed on $\mathcal{C}$. We denote the initial demand measure by $\mathit{\Lambda}(\cdot)$ and the supply measure by $\mu(\cdot)$, with both measures belonging to $\mathcal{M}(\mathcal{C})$. For example, if μ is the Lebesgue measure on $\mathcal{C}$, then drivers are uniformly distributed over the city. Both the demand and supply measures are assumed to be absolutely continuous with respect to the city measure, i.e., $\mathit{\Lambda}, \mu \ll \mathit{\Gamma}$. Customers at location $y \in \mathcal{C}$ have their willingness to pay drawn from a distribution $F_y(\cdot)$. For all $y \in \mathcal{C}$, we assume the revenue function $q \mapsto q \cdot \overline{F}_y(q)$ is continuous and unimodular in q and that F_y is strictly increasing over its support $\left[0, \overline{V}\right]$, for some finite positive $\overline{V}$.

We model the interactions between the platform, the customers and the supply as a game. The first player to act in this game is the platform. The platform selects fares across locations and facilitates the matching of drivers and customers. Specifically, the platform chooses a measurable price mapping $p : \mathcal{C} \to [0, \overline{V}]$ so as to maximize its citywide revenues.

After prices are chosen, drivers select *whether* to relocate and *where* to do so. The relocation of drivers generates a flow/transportation of mass from the initial measure of drivers μ to some final endogenous measure of drivers. This final measure corresponds to the supply of drivers in the city after they have traveled to their chosen destination. The movement of drivers across the city is modeled as a measure on $\mathcal{C} \times \mathcal{C}$, which we denote by τ. Any feasible flow has to preserve the initial mass of drivers in $\mathcal{C}$. That is, the first marginal of τ should equal μ. Moreover, τ generates a new (after relocation) distribution of drivers in the city, which corresponds to the second marginal of τ, τ_2. Formally, the set of feasible flows is defined as follows

$$\mathcal{F}(\mu) = \{\tau \in \mathcal{M}(\mathcal{C} \times \mathcal{C}) : \tau_1 = \mu, \quad \tau_2 \ll \mathit{\Gamma}\}.$$

The first condition ensures consistency with the initial positioning of drivers, the second condition ensures that there is no mass of relocated supply at locations where the city itself has measure zero. In particular, given the latter, the Radon-Nikodym derivatives of τ_2 and $\mathit{\Lambda}$ with respect to $\mathit{\Gamma}$, $d\tau_2(y)/d\mathit{\Gamma}$ and $d\mathit{\Lambda}(y)/d\mathit{\Gamma}$, are well defined and for ease of notation we let, for any y in $\mathcal{C}$,

$$s^\tau(y) \triangleq \frac{d\tau_2}{d\mathit{\Gamma}}(y), \text{ and } \lambda(y) \triangleq \frac{d\mathit{\Lambda}}{d\mathit{\Gamma}}(y).$$

Physically, $s^\tau(y)$ represents the *post-relocation supply* at location y normalized by the size of location y, and $\lambda(y)$ corresponds to the potential demand at location y also normalize by the size of such location. Here and in what follows, we will refer to $s^\tau(y)$ and $\lambda(y)$ as the *post-relocation supply* and potential demand at y, respectively. We use the notation $\mathcal{C}_\lambda$ to represent the set of locations with positive potential demand in the city, i.e., $\mathcal{C}_\lambda = \{y \in \mathcal{C} : \lambda(y) > 0\}$.

Given the prices in place, the effective demand at a location y is given by $\lambda(y) \cdot \overline{F}_y(p(y))$, as at location y, only the fraction $\overline{F}_y(p(y))$ is willing to purchase at price $p(y)$. At the same time, the supply at y is given by $s^\tau(y)$. Therefore, the ratio of effective (as opposed to potential) demand to supply at y is given by

$$\frac{\lambda(y) \cdot \overline{F}_y(p(y))}{s^\tau(y)},$$

assuming $s^\tau(y) > 0$. Since a driver can pick up at most one customer within the time frame of our game, a driver relocating to y will face a utilization rate of $\min\left\{1, \lambda(y) \cdot \overline{F}_y(p(y))/s^\tau(y)\right\}$, assuming $s^\tau(y) > 0$. The effective utilization can be interpreted as the probability that a driver who relocated to y will be matched to a customer within the time frame of our game. In particular, if $s^\tau(y) > \lambda(y) \cdot \overline{F}_y(p(y))$, there is driver congestion at location y, and not all drivers will be matched to a customer. If $s^\tau(y) = 0$ at location y, we say the utilization rate is one if the effective demand at y is positive and zero if the effective demand is zero. Formally, the utilization rate at location y

is given by

$$R\Big(y, p(y), s^\tau(y)\Big) \triangleq \begin{cases} \min\Big\{1, \frac{\lambda(y)\cdot \overline{F}_y(p(y))}{s^\tau(y)}\Big\} & \text{if } s^\tau(y) > 0; \\ 1 & \text{if } s^\tau(y) = 0,\ \lambda(y)\cdot \overline{F}_y(p(y)) > 0; \\ 0 & \text{if } \lambda(y)\cdot \overline{F}_y(p(y)) = 0. \end{cases}$$

When deciding whether to relocate, drivers take three effects into account: prices, travel distance and congestion. The driver congestion effect (or utilization rate) is the one described in the paragraph above. We assume that the platform uses a commission model and transfers a fraction α in $(0, 1)$ of the fare to the driver. As a result, a driver who starts in location y and chooses to remain there earns utility equal to

$$U\Big(y, p(y), s^\tau(y)\Big) \triangleq \alpha \cdot p(y) \cdot R\Big(y, p(y), s^\tau(y)\Big). \tag{1}$$

That is, the utility is given by the compensation per ride times the probability of a match. We model the cost for drivers of repositioning from location x to location y through the distance between the locations, $|y - x|$. Therefore, a driver originating in x who repositions to y earns utility

$$\Pi\Big(x, y, p(y), s^\tau(y)\Big) \triangleq U\Big(y, p(y), s^\tau(y)\Big) - |y - x|. \tag{2}$$

When clear from context, and with some abuse of notation, we omit the dependence on price and the supply-demand ratio, writing $U(y)$ and $\Pi(x, y)$. We are now ready to define the notion of a supply equilibrium.

Definition 1 (Supply Equilibrium)**.** *A flow $\tau \in \mathcal{F}(\mu)$ is an equilibrium if it satisfies*

$$\tau\Bigg(\Big\{(x, y) \in \mathcal{C} \times \mathcal{C} : \Pi(x, y, p(y), s^\tau(y)) = \operatorname*{ess\,sup}_{\mathcal{C}} \Pi\Big(x, \cdot, p(\cdot), s^\tau(\cdot)\Big)\Big\}\Bigg) = \mu(\mathcal{C}),$$

where the essential supremum is taken with respect to the city measure Γ.

That is, an equilibrium flow of supply is a feasible flow such that essentially no driver wishes to unilaterally change his destination. As a result, the mass of drivers selecting the best location for themselves has to equal the original mass of drivers in the system.

The platform's objective is to maximize the revenues it garners across all locations in $\mathcal{C}$. From a given location y, it earns $(1-\alpha)\cdot p(y)\cdot \min\{s^\tau(y), \lambda(y)\cdot \overline{F}_y(p(y))\}$. The term $(1-\alpha)\cdot p(y)$ corresponds to the platform's share of each fare at location y, and the term $\min\{s^\tau(y), \lambda(y)\cdot \overline{F}_y(p(y))\}$ denotes the quantity of matches of potential customers to drivers at location y. If location y is demand constrained, then $\min\{s^\tau(y), \lambda(y)\cdot \overline{F}_y(p(y))\}$ equals $\lambda(y)\cdot \overline{F}_y(p(y))$, while if location y is supply constrained, then $\min\{s^\tau(y), \lambda(y) \cdot \overline{F}_y(p(y))\}$ amounts to $s^\tau(y)$. The platform's price optimization problem can in turn be written as

$$\begin{aligned} \sup_{p(\cdot),\, \tau \in \mathcal{F}(\mu)} \quad & (1-\alpha) \int_{\mathcal{C}} p(y) \cdot \min\{s^\tau(y), \lambda(y) \cdot \overline{F}_y(p(y))\}\, d\varGamma(y) \\ s.t. \quad & \tau \text{ is a supply equilibrium}, \\ & s^\tau = \frac{d\tau_2}{d\varGamma}. \end{aligned} \tag{$\mathcal{P}_1$}$$

Remark. Our model may be interpreted as a basic model to understand the short-term operations of a ride-hailing company. In particular, each driver completes at most one customer pickup within the time frame of our game and there is not enough time for the entry of new drivers into the system. In the present model, we do not account explicitly for the destinations of the rides. We do so in order to isolate the interplay of supply incentives and pricing. In that regard, one could view our model as capturing origin-based pricing, a common practice in the ride-hailing industry.

4 Structural Properties and Spatial Decomposition

A key challenge in solving the optimization problem presented in ($\mathcal{P}_1$) is that the decision variables, the flow τ and the price function $p(\cdot)$, are complicated objects. The flow τ, being a measure over a two-dimensional space, is obviously a complex object to manipulate. The price function will turn out to be a difficult object to manipulate as well in that the optimal price function will often be discontinuous. In order to analyze our problem, we will need to introduce a better-behaved object. This object, which will be central to our analysis, is the (after movement) driver equilibrium utility.

Drivers' utilities. For a given price function p and flow τ, we denote by $V_{\mathcal{B}}(x|\, p, \tau)$ the essential maximum utility that a driver departing from location x can garner by going to anywhere within a measurable region $\mathcal{B} \subseteq \mathcal{C}$. In particular, the mapping $V_{\mathcal{B}}(\cdot|\, p, \tau) : \mathcal{C} \to \mathbb{R}$ is defined as

$$V_{\mathcal{B}}(x|\, p, \tau) \triangleq \operatorname*{ess\,sup}_{\mathcal{B}} \varPi\left(x, \cdot, p(\cdot), s^\tau(\cdot)\right). \tag{3}$$

When $\mathcal{B} = \mathcal{C}$, we use V instead of $V_{\mathcal{C}}$. By the definition of a supply equilibrium, essentially all drivers departing from location x earn $V(x|\, p, \tau)$ utility in equilibrium.

We now show that the equilibrium utility $V_{\mathcal{B}}(\cdot|\, p, \tau)$ must be 1-Lipschitz continuous. Intuitively, drivers from two different locations x and y that consider relocating to $\mathcal{B}$ see exactly the same potential destinations. Hence, the largest utility drivers departing from x can garner must be greater or equal to that of the drivers departing from y minus the disutility stemming from relocating from

x to y, that is, $V_{\mathcal{B}}(x) \geq V_{\mathcal{B}}(y) - |x-y|$. Since this argument is symmetric, we deduce the 1-Lipschitz property.

Lemma 1. *(Lipschitz) Consider a measurable set $\mathcal{B} \subseteq \mathcal{C}$ such that $\Gamma(\mathcal{B}) > 0$. Let p be a measurable mapping $p : \mathcal{B} \to \mathbb{R}_+$, and let $\tau \in \mathcal{F}(\mu)$. Then, the function $V_{\mathcal{B}}(\cdot|\, p, \tau)$ is 1-Lipschitz continuous.*

We now introduce a reformulation of $(\mathcal{P}_1)$ that focuses on the equilibrium utility V and the post-relocation supply s^τ as the central elements. We then establish important structural properties of V and establish a spatial decomposition result that is based on the equilibrium behavior of drivers.

4.1 Reformulating the Platform's problem

In what follows, we define $\gamma \triangleq (1-\alpha)/\alpha$. In the next result, we establish that the platform's objective can be rewritten in terms of the utility function $V(\cdot|\, p, \tau)$ and the post-relocation supply s^τ, yielding an alternative optimization problem.

Proposition 1 (Problem Reformulation)**.** *The following problem*

$$\begin{aligned} \sup_{p(\cdot)\,,\,\tau \in \mathcal{F}(\mu)} \quad & \gamma \cdot \int_{\mathcal{C}_\lambda} V(x|\, p, \tau) \cdot s^\tau(x)\, d\Gamma(x) \qquad\qquad (\mathcal{P}_2) \\ s.t. \quad & \tau \text{ is an equilibrium flow,} \\ & V(x|\, p, \tau) = \operatorname*{ess\,sup}_{\mathcal{C}} \Pi\Big(x, \cdot, p(\cdot), s^\tau(\cdot)\Big), \quad s^\tau = \frac{d\tau_2}{d\Gamma}, \end{aligned}$$

admits the same value as the platform's optimization problem $(\mathcal{P}_1)$, and a pair (p, τ) that solves $(\mathcal{P}_2)$ also solves $(\mathcal{P}_1)$.

The first step in the proof of the proposition above is to rewrite the platform's objective in terms of the post-relocation supply $s^\tau(x)$ and the pre-movement utility function $U(x, p(x), s^\tau(x))$ (see Eq. (1)). This transformation is not particularly useful per se, since the function $U(x, p(x), s^\tau(x))$ is not necessarily well-behaved. The next step consists of establishing that $U(x, p(x), s^\tau(x))$ coincides with $V(x|\, p, \tau)$ whenever a location has positive post-movement equilibrium supply (see Lemma A-2 in the Appendix). Indeed, whenever the equilibrium outcome is such that a location has positive supply, the utility generated by staying at that location has to be equal to the best utility one could obtain by traveling to any other location. This is intuitive in that if it were not the case, no driver would be willing to stay at or travel to that location. In turn, one can effectively replace $U(x, p(x), s^\tau(x))$ with $V(x|p, \tau)$ in the objective, which yields the alternative problem. The main advantage of this new formulation is that the equilibrium utility $V(x|\, p, \tau)$ admits significant structure, as we show next.

4.2 Indifference and Attraction Regions

A key feature of the problem at hand is that, in equilibrium, conditions at different locations are inherently linked as drivers select their destination among all locations. An important object that will help capture the link across various locations is the *indifference region* of a driver departing location x. The indifference region of x represents all the destinations to which drivers from x are willing to travel to. Formally, the indifference region for a driver departing from $x \in \mathcal{C}$ under prices p and flow τ is given by

$$\mathcal{IR}(x|\, p, \tau) \triangleq \Big\{ y \in \mathcal{C} : \lim_{\delta \downarrow 0} V_{B(y,\delta)}(x|p, \tau) = V(x|p,\tau) \Big\},$$

where $B(y, \delta)$ is the open ball in $\mathcal{C}$ of center y and radius δ. Intuitively, the definition above says that if $y \in \mathcal{IR}(x|\, p, \tau)$, then drivers departing from x maximize their utility by relocating to y.

Indifference regions describe the set of best possible destination for a given location. The converse concept which will turn out to be fundamental in our analysis is the *attraction region* of a location z. The attraction region of z represents the set of all possible sources for which location z is their best option. In addition, location z is called a *sink* if it is not willing to travel to any other location. These regions are rich in the sense that enjoy of several appealing properties and, as we will see in Section 5, we can solve for the platform's optimal solution within them. Below we provide a formal definition for an attraction region and a sink location.

Definition 2 (Attraction Region)**.** *Let (p, τ) be a feasible solution of ($\mathcal{P}_2$). For any location $z \in \mathcal{C}$, its* attraction region *$A(z|\, p, \tau)$ is the set of locations from which drivers are willing to relocate to z, i.e.,*

$$A(z|\, p, \tau) \triangleq \{ x \in \mathcal{C} : z \in \mathcal{IR}(x|\, p, \tau) \}.$$

We call a location $z \in \mathcal{C}$ a sink *if its attraction region $A(z|\, p, \tau)$ is non-empty and $z \notin A(z'|\, p, \tau)$ for all $z' \neq z$. When z is a sink, we represent the endpoints of its attraction region by*

$$X_l(z|\, p, \tau) \triangleq \inf\{ x \in A(z|\, p, \tau) \}, \qquad X_r(z|\, p, \tau) \triangleq \sup A(z|\, p, \tau).$$

Furthermore, we say that z is an in-demand sink whenever $\Gamma([z, z+\delta) \cap \mathcal{C}_\lambda)$ and $\Gamma((z-\delta, z]) \cap \mathcal{C}_\lambda)$ are strictly positive for all $\delta > 0$.

The next result characterizes the shape of attraction regions.

Lemma 2 (Attraction Region)**.** *Let (p, τ) be a feasible solution of ($\mathcal{P}_2$). For any sink $z \in \mathcal{C}$, its attraction region $A(z|\, p, \tau)$ is a closed interval containing z.*

The lemma above establishes an intuitive but important transitivity result. Let $x < y < z$ be such that x is in the attraction region of z. Then, y must also be in the attraction region of z.

The structure of the utility function V at a supply equilibrium will play a central role in our analysis. The following lemma establishes the shape of V within attraction regions.

Lemma 3 (Utility Within an Attraction Region)**.** *Let (p,τ) be a feasible solution of ($\mathcal{P}_2$), and let $z \in \mathcal{C}$ be a sink. Then, the equilibrium utility satisfies*

$$V(x|\, p,\tau) = V(z|\, p,\tau) - |z-x|, \quad \textit{for all} \quad x \in A(z|\, p,\tau).$$

This result is closely related to the Envelope Theorem, which is widely used in mechanism design (see Milgrom & Segal (2002)). If a driver originating from x is indifferent to relocating to z, then $V(z|\, p,\tau) - V(x|\, p,\tau)$ must be equal to the relocation cost $|z-x|$.

Importantly, attraction regions occur as frequently as drivers move in the city. The movement of drivers to some final location implies that that location has a non-empty attraction region. At least drivers from locations traveling to it belong to its attraction region. The next proposition formalizes this.

Proposition 2 (Existence of attraction regions)**.** *Let (p,τ) be a feasible solution of ($\mathcal{P}_2$) and suppose that $y \in \mathcal{IR}(x|\, p,\tau)$ for some $x \neq y$. Then, there exists a sink location $z \in \mathcal{C}$ such that $x, y \in A(z|\, p,\tau)$.*

Consider two set $\mathcal{B}$ and $\mathcal{B}'$ such that $\tau(\mathcal{B}\times\mathcal{B}') > 0$. Intuitively, there must exist two locations $x \in \mathcal{B}$ and $y \in \mathcal{B}'$ for which x is willing to travel to y. It follows that location x belongs to the attraction region of y, although y is not necessarily a sink location. The sink location in the proposition is constructed as the last location to which drivers departing from x are willing to travel to. By construction this location must be a sink.

4.3 Spatial Decomposition

Next, we show that attraction regions lead to a natural decoupling of the platform's problem, as they provide a natural way of segmenting the city. The next result establishes a flow separation property induced by attraction regions.

Proposition 3 (Flow Separation)**.** *Let (p,τ) be a feasible solution of ($\mathcal{P}_2$), and let $z \in \mathcal{C}$ be a sink. Then, there is no flow crossing the endpoints of the attraction region, $X_l(z|\, p,\tau)$ and $X_r(z|\, p,\tau)$. Formally,*

- *(i)* $\tau([-H, X_l(z|\, p,\tau)) \times [X_l(z|\, p,\tau), H]) = 0$ *and* $\tau((\{z\} \cup (X_l(z|\, p,\tau), H]) \times ([-H, X_l(z|\, p,\tau)] \setminus \{z\})) = 0$*;*

(ii) $\tau((X_r(z|\,p,\tau),H]\times[-H,X_r(z|\,p,\tau)])=0$ *and* $\tau(([-H,X_r(z|\,p,\tau))\cup\{z\})\times([X_r(z|\,p,\tau),H]\setminus\{z\}))=0$.

Figure 2 illustrates this proposition. In equilibrium, there cannot be any flow crossing the end points of $A(z|\,p,\tau)$ in either direction. Suppose there was positive flow leaving some point $y<X_l(z|\,p,\tau)$ and crossing into $A(z|\,p,\tau)$. This would imply that y should also be an element of $A(z|\,p,\tau)$, contradicting the fact that $X_l(z|\,p,\tau)$ is an end point of the attraction region $A(z|\,p,\tau)$.

We clarify here that this proposition does not impose anything on the direction of flow emerging from the end points $X_l(z|\,p,\tau)$ and $X_r(z|\,p,\tau)$. That is, if there is a mass of drivers starting from the boundary $\partial A(z|\,p,\tau)$, these drivers could move either into or out of the attraction region.

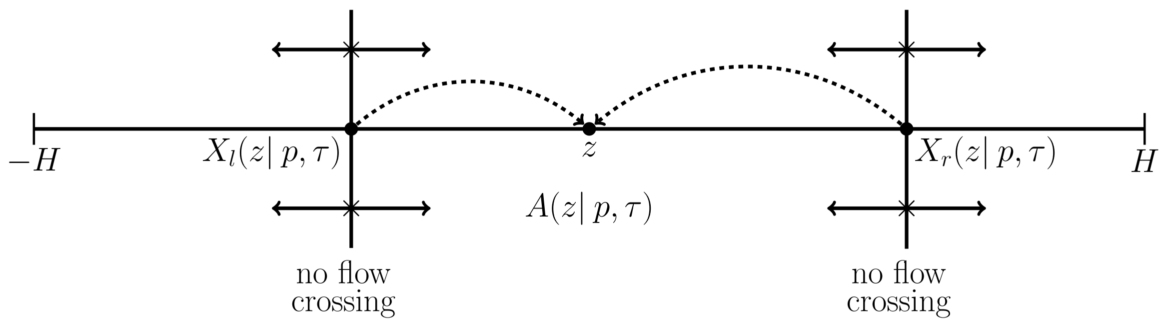

Figure 2: **Flow separation.** Illustration of the result in Proposition 3. No flow crosses the boundaries of $A(z|\,p,\tau)$.

This flow separation result will enable us to geographically decompose the platform's problem into multiple weakly coupled local problems. To that end, we introduce some additional notation that will allow us to "localize the analysis". Formally, for any measurable $\mathcal{B}\subset\mathcal{C}$ and measure $\tilde{\mu}\in\mathcal{M}(\mathcal{B})$, we define the set of feasible flows restricted to $\mathcal{B}$ to be

$$\mathcal{F}_{\mathcal{B}}(\tilde{\mu})=\{\tau\in\mathcal{M}(\mathcal{B}\times\mathcal{B}):\tau_1=\tilde{\mu},\quad \tau_2\ll\Gamma|_{\mathcal{B}}\}.$$

In addition, we define local equilibria as follows.

Definition 3 (Local Equilibrium)**.** *For any $\mathcal{B}\subset\mathcal{C}$ such that $\Gamma(\mathcal{B})>0$ and $\tilde{\mu}\in\mathcal{M}(\mathcal{B})$, a flow $\tau\in\mathcal{F}_{\mathcal{B}}(\tilde{\mu})$ is a local equilibrium in $\mathcal{B}$ if it satisfies*

$$\tau\left(\left\{(x,y)\in\mathcal{B}\times\mathcal{B}:\Pi(x,y,p(y),s^{\tau}_{|\mathcal{B}}(y))=\operatorname*{ess\,sup}_{\mathcal{B}}\Pi\Big(x,\cdot,p(\cdot),s^{\tau}_{|\mathcal{B}}(\cdot)\Big)\right\}\right)=\tilde{\mu}(\mathcal{B}).$$

That is, a local equilibrium in $\mathcal{B}$ is a feasible flow such that no driver wishes to unilaterally change his destination when restricting attention to the set $\mathcal{B}$. With this definition in hand, we may now state our next result. Informally, this result states the following "pasting" property. Suppose we start from a price-equilibrium pair (p,τ) and a sink z and its attraction region $A(z|\,p,\tau)$. Then, we can replace the flow that occurs within $A(z|\,p,\tau)$ with any other local equilibrium within that attraction region as long as we maintain the same conditions at the boundary $\partial A(z|\,p,\tau)$.

Proposition 4. *(Pasting) Let (p,τ) be a feasible solution of ($\mathcal{P}_2$), and let $z\in\mathcal{C}$ be a sink. Denote $\mathcal{A}=A(z|\,p,\tau)$. Let $\tilde{\mu}\in\mathcal{M}(\mathcal{A})$ be the measure representing drivers that stay within $\mathcal{A}$ according to flow τ, i.e., $\tilde{\mu}(\mathcal{B})\triangleq\tau(\mathcal{B}\times\mathcal{A})$ for any measurable set $\mathcal{B}\subseteq\mathcal{A}$. Suppose there exists a measurable price mapping $\tilde{p}:\mathcal{A}\to[0,\overline{V}]$ and a flow $\tilde{\tau}\in\mathcal{F}_{\mathcal{A}}(\tilde{\mu})$ such that $\tilde{\tau}$ is a local equilibrium in $\mathcal{A}$ under pricing $\tilde{p}$. Furthermore, suppose $V_{\mathcal{A}}(\cdot|\,\tilde{p},\tilde{\tau})$ equals $V(\cdot|\,p,\tau)$ in $\partial\mathcal{A}$. Define the pasted pricing function $\hat{p}:\mathcal{C}\to[0,\overline{V}]$,*

$$\hat{p}(x)\triangleq\begin{cases}\tilde{p}(x) & \text{if } x\in\mathcal{A};\\ p(x) & \text{if } x\in\mathcal{A}^c,\end{cases}$$

and the pasted flow $\hat{\tau}\in\mathcal{F}(\mu)$, where for any measurable $\mathcal{L}\subseteq\mathcal{C}\times\mathcal{C}$

$$\hat{\tau}(\mathcal{L})\triangleq\tau(\mathcal{L}\cap(\overline{\mathcal{A}^c}\times\mathcal{A}^c))+\tilde{\tau}(\mathcal{L}\cap(\mathcal{A}\times\mathcal{A})).$$

Then, the pasted solution $(\hat{p},\hat{\tau})$ is a feasible solution of problem ($\mathcal{P}_2$) such that

$$s^{\hat{\tau}}=\begin{cases}s^{\tilde{\tau}}(x) & \text{if } x\in\mathcal{A};\\ s^{\tau}(x) & \text{if } x\in\mathcal{A}^c,\end{cases}\quad \text{and}\quad V(x|\,\hat{p},\hat{\tau})=\begin{cases}V_{\mathcal{A}}(x|\,\tilde{p},\tilde{\tau}) & \text{if } x\in\mathcal{A};\\ V(x|\,p,\tau) & \text{if } x\in\mathcal{A}^c.\end{cases}$$

Propositions 3 and 4 suggest a natural structure for the induced flows by any pricing policy. For a given sink z, Proposition 3 establishes that the attraction region of z and its complement are flow separated. Now Proposition 4 applies this flow separation result and shows how to make local deviations to a feasible solution while maintaining feasibility. More precisely, an equilibrium in $\mathcal{C}$ can be locally modified in the attraction region of z, without losing feasibility, as long the equilibrium utilities of drivers in the boundaries of the attraction region are not modified. The new solution $(\hat{p},\hat{\tau})$ in $\mathcal{C}$ merges the old solution (p,τ) in $A(z|p,\tau)^c$ with the modified solution $(\tilde{p},\tilde{\tau})$ in the attraction region $A(z|p,\tau)$.

5 Congestion Bound and Optimal Flows

In the prior section, we showed that the platform's optimization problem can be reformulated as a problem over equilibrium utilities V and post-relocation supply s^{τ}. We also showed that V is a well-behaved function: it is 1-Lipschitz continuous and it has derivative equal to +1 or -1 over attraction regions. Furthermore, we demonstrated how to use attraction regions to decompose the platform's global problem into localized problems.

In this section, we focus on the optimal relocation of drivers within attraction regions. That is, we will prove that, without loss of optimality, we can restrict attention to flows within attraction regions that take a very specific form. In order to do so, we first need to formalize the notion of congestion level of a given location.

5.1 Congestion Bound

We first introduce some quantities that will be useful throughout our analysis. These quantities emerge from a classical capacitated monopoly pricing problem. Let us consider any location $x \in \mathcal{C}$ and ignore all other locations in the city. The problem that a monopolist faces when supply at x is s and demand is λ_x can be cast as

$$R_x^{loc}(s) \triangleq \max_{q \in [0,\overline{V}]} \; q \cdot \min\{s, \lambda_x \cdot \overline{F}_x(q)\}, \tag{4}$$

with the price $\rho_x^{loc}(s)$ being defined as the argument that maximizes the equation above. Since $q \cdot \overline{F}_x(q)$ is assumed to be unimodular in q, the optimal price $\rho_x^{loc}(s)$ is uniquely determined and is characterized as follows

$$\rho_x^{loc}(s) = \max\{\rho_x^{bal}(s), \rho_x^u\}, \qquad \text{where } s = \lambda_x \cdot \overline{F}_x(\rho_x^{bal}(s)), \quad \rho_x^u \in \arg\max_{\rho \in [0,\overline{V}]} \{\rho \cdot \overline{F}_x(\rho)\}. \tag{5}$$

That is, the optimal local price either balances supply and demand or maximizes the unconstrained local revenue.

For a given local supply s, the maximum revenue that can be generated at location x is $R_x^{loc}(s)$, with a fraction α of that revenue being payed to the drivers. Therefore, $\alpha \cdot R_x^{loc}(s)/s$ is the maximum revenue a driver staying at this location can earn. To capture this notion, we introduce for every location x the supply *congestion* function $\psi_x : \mathbb{R}_+ \to [0, \alpha \cdot \overline{V}]$, which is defined as:

$$\psi_x(s) \triangleq \begin{cases} \alpha \cdot R_x^{loc}(s)/s & \text{if } s > 0; \\ \alpha \cdot \overline{V} & \text{if } s = 0,\ \lambda(x) > 0; \\ 0 & \text{if } s = 0,\ \lambda(x) = 0. \end{cases}$$

The congestion function ψ_x must be decreasing since more drivers (in a single location problem) imply lower revenues per driver.

Lemma 4. *For any $x \in \mathcal{C}_\lambda$ the congestion function $\psi_x(\cdot)$ is a strictly decreasing function.*

More importantly, the congestion function ψ_x yields an upper bound for the utility of drivers at almost any location with respect to the city measure.

Proposition 5 (Congestion Bound)**.** *Let (p, τ) be a feasible solution of $(\mathcal{P}_2)$. Then the equilibrium driver utility function is bounded as follows:*

$$V(x|\, p, \tau) \;\leq\; \psi_x\left(s^\tau(x)\right) \qquad \Gamma - a.e.\ x \;\; in\ \mathcal{C}_\lambda.$$

When there is a single location, the inequality above is an equality by the definition of ψ_x. For multiple locations, drivers may travel to any location and there is no a priori connection between the utility that drivers originating from x can garner, $V(x|\, p, \tau)$, and $\psi_x(s^\tau(x))$. The result above establishes that the latter upper bounds the former. The bound captures the structural property that as equilibrium supply increases at a location, and hence driver congestion increases, the drivers originating from that location will earn less utility.

5.2 Optimal Supply Reallocation in Attraction Regions

We now consider the problem of how to optimize flows within an attraction region. The key idea is to use the structural properties about the equilibrium utility function as well as the pasting result developed in Section 4, in conjunction with a relaxation to the platform's problem within an attraction region that leverages the congestion bound established in Proposition 5.

Consider a feasible solution (p, τ) of $(\mathcal{P}_2)$. Let $z \in \mathcal{C}$ be a sink and $A(z|\, p, \tau)$ its corresponding attraction region. We will now show how to construct a second feasible solution of $(\mathcal{P}_2)$ for which the revenue is weakly larger and we can fully characterize its prices and flows within the attraction region $A(z|\, p, \tau)$ as defined by the original solution (p, τ).

Theorem 1. *(Optimal Supply Within an Attraction Region) Consider a feasible solution (p, τ) of $(\mathcal{P}_2)$, and let $z \in \mathcal{C}$ be an in-demand sink. Then, there exists another feasible solution $(\hat{p}, \hat{\tau})$ that weakly revenue dominates (p, τ), and is such that $V(\cdot|\, \hat{p}, \hat{\tau})$ coincides with $V(\cdot|\, p, \tau)$ in $A(z|\, p, \tau)$ and its supply $s^{\hat{\tau}}$ in $A(z|\, p, \tau)$ is given by:*

$$s^{\hat{\tau}}(x) = \begin{cases} \psi_x^{-1}(V(z|\, p, \tau) - |x - z|) \cdot \mathbf{1}_{\{\lambda(x)>0\}} & \text{if } x \in (z_l, z_r); \\ s_i & \text{if } x = z_i,\ i \in \{l, r\}; \\ 0 & \text{otherwise,} \end{cases}$$

for a set of values $z_l \leq z, z_r \geq z$, and $s_i \geq 0, i \in \{l, r\}$. Furthermore,

$$\hat{p}(x) = \begin{cases} \rho_x^{loc}(s^{\hat{\tau}}(x)) & \text{if } x \in A(z|\, p, \tau) \setminus \{z_l, z_r\}; \\ p_i & \text{if } x = z_i,\ i \in \{l, r\}, \end{cases}$$

where p_i is such that $U(z_i, p_i, s_i) = V(z_i|\, p, \tau) \cdot \mathbf{1}_{\{\lambda(z_i)>0\}}$ for $i \in \{l, r\}$.

The theorem above characterizes an optimal solution, including both prices and flows, within an attraction region. In particular, the optimality of a pricing policy implies that it is sufficient to focus on solutions that have post-movement equilibrium supply around the sink z in $[z_l, z_r]$ while potentially creating regions with zero equilibrium supply away from the sink, in $[X_l, z_l)$ and $(z_r, X_r]$. These regions "feed" the region around the sink z with drivers. Furthermore, the optimal prices are fully characterized in any attraction region through the post-relocation supply. We will highlight the main implications of Theorem 1 through a prototypical family of instances in Section 6, where we will characterize the optimal solution in quasi-closed form.

Key ideas for Theorem 1. The key idea underlying the proof of the result is based on optimizing the contribution of the attraction region $A(z|p, \tau)$ to the overall objective by reallocation the supply around the sink, and then showing that this reallocation of supply constitutes an equilibrium flow in the original problem.

In order to optimize the supply around the sink we consider the following optimization problem which, as explained below, is a relaxation of $(\mathcal{P}_2)$ within $A(z|\, p, \tau)$:

$$\begin{aligned}
\max_{\tilde{s}(\cdot)\geq 0} \quad & \int_{[X_l, X_r]} V(x|\, p, \tau) \cdot \tilde{s}(x)\, d\Gamma(x) && (\mathcal{P}_{KP}(z)) \\
\text{s.t} \quad & \tilde{s}(x) \leq \psi_x^{-1}(V(x)) \quad \Gamma - a.e.\ x \ \text{ in } \mathcal{C}_\lambda, && \text{(Congestion Bound)} \\
& \int_{[X_l, X_r]} \tilde{s}(x) d\Gamma(x) = \tau_c, && \text{(Flow Conservation)} \\
& \int_{(z, X_r]} \tilde{s}(x) d\Gamma(x) \leq \tau_r, && \text{(No Flow Crossing Left to Right)} \\
& \int_{[X_l, z)} \tilde{s}(x) d\Gamma(x) \leq \tau_l, && \text{(No Flow Crossing Right to Left)}
\end{aligned}$$

where τ_c corresponds to the total flow that τ transports from $A(z|\, p, \tau)$ to $A(z|\, p, \tau)$, and τ_l (τ_r) correspond to the total flow in $A(z|\, p, \tau)$ that is transported to the left (right) of z, excluding z. Recall that given the post-relocation supply, $\tilde{s}$, the quantity

$$\int_{\mathcal{B}} \tilde{s}(x)\, d\Gamma(x),$$

represents the post-relocation supply induced by $\tilde{s}$ in $\mathcal{B}$. Thus, the last three constraints in $(\mathcal{P}_{KP}(z))$ stand for consistency of the total post-relocation supply in each one the relevant subregions of $A(z|\, p, \tau)$.

In $(\mathcal{P}_{KP}(z))$, we fix the driver utilities and ask what should be the optimal allocation of drivers while satisfying flow balance in the regions $[X_l, z]$ and $[z, X_r]$ and imposing the congestion bound. Clearly selecting $\tilde{s} = s^\tau$ is feasible for the problem above and hence the optimal value upper bounds the value generated by the initial price-equilibrium pair (p, τ) in the region $A(z|\, p, \tau)$. In the proof, we show that this relaxation is tight. Namely, it is possible to construct prices and equilibrium flows achieving the value of Problem ($\mathcal{P}_{KP}(z)$). The proof consists of two main steps: 1) solving problem ($\mathcal{P}_{KP}(z)$) and 2) showing that the post-relocation supply that solves the relaxation can actually be obtained from appropriate prices and flows. For step 1), the main idea relies on recognizing that Problem ($\mathcal{P}_{KP}(z)$) is a measure-theoretical instance of two coupled *Continuous Bounded Knapsack Problems*. In particular, the congestion constraint corresponds to the availability constraint in the classical knapsack problem. The solution to ($\mathcal{P}_{KP}(z)$) is obtained by allocating as much as possible at locations where we can make the most revenue per unit of volume, i.e., we would like to make $\tilde{s}(x)$ as large as possible at locations where $V(x|\, p, \tau)$ is the largest. Hence the solution starts by allocating as much supply as possible at location z. The challenge here is that flow-crossing conditions need also to be satisfied and hence whether flow is sent to z from the left or the right is key and needs to be tracked. For step 2), we explicitly construct prices and flows that generate the post-relocation supply $\tilde{s}$ we obtained earlier, and then we apply the pasting result (Proposition 4) to obtain a feasible price-equilibrium in the whole city $\mathcal{C}$.

6 Response to Demand Shock: Optimal Solution and Insights

The results derived in the previous sections characterize the structure of an optimal pricing policy and the corresponding supply response in attraction regions for general demand and supply conditions. In this section, to crisply isolate the interplay of spatial supply incentives and spatial pricing, we focus on a special family of instances that will be rich enough to capture spatial supply-demand imbalances while isolating the interplay above.

In particular, we focus on a family of models that captures a potential local surge in demand. Namely, we specialize the model to the case where the city measure is given by

$$\mathit{\Gamma}(B) = \mathbf{1}_{\{0 \in B\}} + \int_B dx,$$

that is, the origin may admit point masses of supply and demand while the rest of the locations only admit infinitesimal amounts of supply and demand. In what follows, we fix the city measure throughout, but we parametrize the supply and demand measures.

Supply is initially evenly distributed throughout the city, with a density of drivers equal to μ_1 everywhere. Potential demand will be also be assumed to be have a uniform density on the line interval, except potentially at the origin.

We analyze what happens when a potential demand shock at the origin (the potential high demand location) materializes and, in particular, we investigate the optimal pricing policy in response to such a shock. We represent the demand shock by a Dirac delta at this location. Therefore, for any measurable set $B \subseteq \mathcal{C}$, the potential demand measure (after the shock) is given by

$$\Lambda(B) = \lambda_0 \cdot \mathbf{1}_{\{0 \in B\}} + \int_B \lambda_1 dx,$$

where $\lambda_0 \geq 0$ and $\lambda_1 > 0$. In particular, we refer to the case $\lambda_0 = 0$ as the *pre-demand shock environment* and the case $\lambda_0 > 0$ as the *demand shock environment*.

For this family of models, we assume that customer willingness to pay is drawn from the same distribution $F(\cdot)$ for all locations in the city (and this function is assumed to satisfy the regularity conditions of Section 3). Figure 3 provides a visual representation of this family of cases.

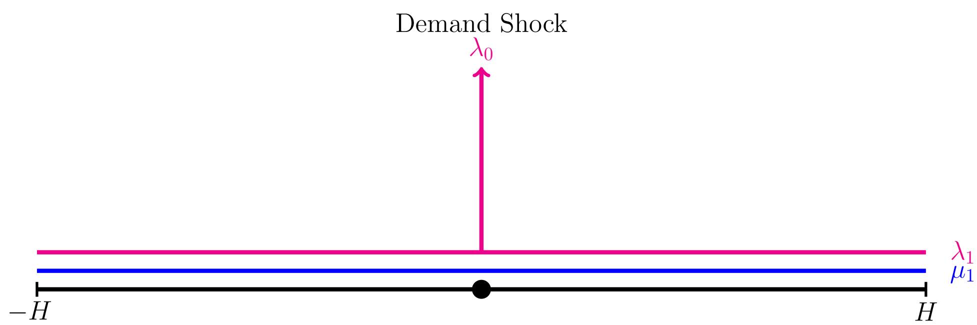

Figure 3: **Prototypical family of models with demand surge.** The supply is initially uniformly distributed in the city with density μ_1, and potential demand is uniformly distributed in the city with density λ_1, with a sudden demand surge at location 0.

This special structure will enable to elucidate the spatial supply response induced by surge pricing and the structural insights on the optimal policies that emerge.

Throughout this section we will use short-hand notation to present the optimal solution in a streamlined fashion. Let (p, τ) be a price equilibrium pair we use $A(0), X_l$ and X_r to denote $A(0|\, p, \tau)$, $X_l(0|\, p, \tau)$ and $X_r(0|\, p, \tau)$, respectively. Moreover, when clear from context, we write $V(\cdot)$ instead of $V(\cdot|\, p, \tau)$.

6.1 The Pre-demand Shock Environment

We start by analyzing the pre-shock environment. In this environment, there is no demand shock, $\lambda_0 = 0$, and both demand and supply are uniformly distributed along the city, with respective densities λ_1 and μ_1.

If one were to look at each location in isolation, the optimal local price at a location x with demand density λ_1 and supply density μ_1 is $\rho_x^{loc}(\mu_1)$, as defined in Eq. (5). Note that in the current environment $\rho_x^{loc}(\mu_1)$ is not location dependent and we denote it by ρ_1 throughout.

The next result characterizes the optimal solution in this environment.

Proposition 6 (Pre-demand Shock Environment)**.** *Suppose $\lambda_0 = 0$. Then, the optimal policy and corresponding supply equilibrium and flows can be characterized as follows.*

(i) (Prices) The optimal pricing policy is given by $p(x) = \rho_1$, for all x in $\mathcal{C}$.

(ii) (Flow) All supply units stay at their original locations.

Furthermore, the optimal revenue equals $\gamma \cdot \psi_1 \cdot \mu_1 \cdot 2H$.

This result is simply says that if the initial demand-supply conditions are identical across the city, then the optimal price policy does not induce any movement for supply, and the optimal price at each location is simply that of a single location capacitated pricing problem. In such a solution, the expected utilization of all drivers is equal to 1 if $\mu_1 \leq \lambda_1 \cdot \overline{F}(\rho^u)$, and otherwise is strictly below 1. In the latter case, there is oversupply and driver congestion at all locations. The optimal revenue, recalling the reformulation in Proposition 1, is given by the equilibrium utility of drivers ψ_1, times the density of equilibrium supply, integrated across all locations (times a scaling factor).

6.2 Benchmark: Local Price Response to a Demand Shock

We next start our analysis of the demand shock environment. Before turning our attention to an optimal policy in Section 6.3, we first focus on a simple type of pricing heuristic which responds to changes in demand conditions through changes in prices *only* where these changes occur. In particular, in the context of the demand shock model, this corresponds to responding to a shock in demand at the origin by only adjusting the price at the origin; we call this policy the *local price response*. This provides a benchmark to better understand the structure and performance of an optimal policy. We next characterize an optimal local price response, when prices are fixed everywhere at the pre-demand shock environment solution, except at the origin.

Proposition 7 (Local Price Response to a Demand Shock)**.** *Fix $\lambda_0 > 0$. Suppose that $p(x) = \rho_1$ for all x in $\mathcal{C} \setminus \{0\}$ and that the firm optimizes for the price $p(0)$. Then,*

(i) (Prices) The optimal price at the origin is given by $p(0) = \rho_0^{loc}(s^\tau(0))$, and $p(0) \geq \rho_1$.

(ii) (Movement) There exists two thresholds $X_r \geq X_r^0 \geq 0$, such that $X_r > 0$ and:

- *for all x in $[-X_r^0, X_r^0]$, all of the supply units move to the origin,*

- *for all x in $[-X_r, -X_r^0]$ and all x in $[X_r^0, X_r]$, a fraction of the supply units move to the origin and the other fraction does not move,*
- *for all x in $\mathcal{C} \setminus [-X_r, X_r]$, no supply unit moves.*

Furthermore, the platform's revenue is strictly larger than in the pre-demand shock environment.

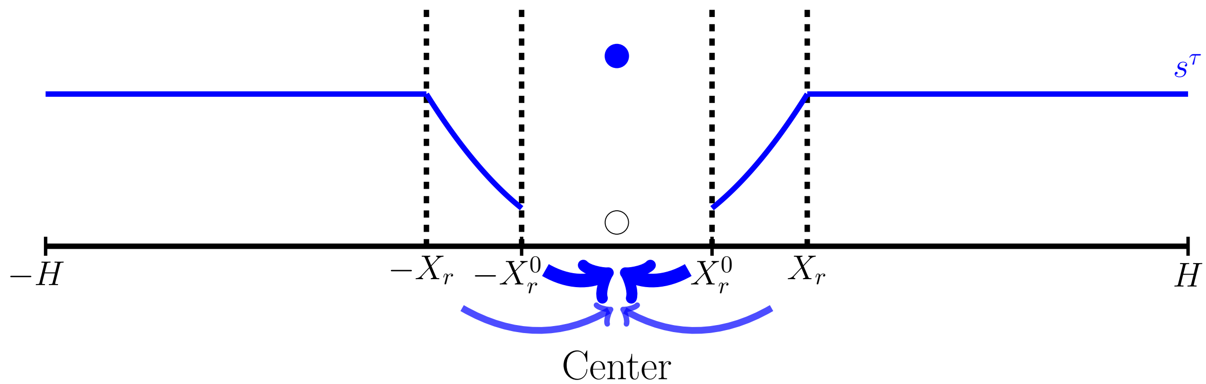

Figure 4: **Optimal local price response: induced supply response for a case with $\mu_1 > \lambda_1 \cdot \overline{F}(\rho^u)$.**

The result above characterizes the structure of an optimal local price response as well as the structure of the supply movement it induces. Figure 4 depicts the structure of the supply response. In particular, the optimal local price response leads to a higher price at the origin to respond to the surge of demand at that location. In turn, this higher price attracts drivers from a symmetric region around the origin. In that region, for locations close to the origin, all supply units move to the origin. After a given threshold X_r^0, only a fraction of the drivers will move to the origin. Intuitively, as one gets further from the origin, traveling to the origin becomes a less attractive option, compared to staying put or traveling elsewhere. As that becomes the case, a smaller and smaller fraction of units travels to the origin. Furthermore, we establish that supply units have no incentive to travel anywhere else in the city and, as a result, units that do not travel to the origin stay put and serve local demand. Beyond the threshold X_r, no supply units move in the equilibrium induced by the optimal local price response.

The threshold X_r corresponds to the location of the last drivers willing to travel to the origin. In the current environment, prices are not flexible and, therefore, X_r must equals $V(0) - \psi_1$ since drivers who are further than that will prefer to earn ψ_1 by staying put compared to driving to the origin to earn $V(0)$ minus driving costs. If we are in a supply constrained regime, $\mu_1 \leq \lambda_1 \cdot \overline{F}(\rho^u)$, then all drivers within $[-X_r, X_r]$ drive to the origin, i.e., $X_r^0 = X_r$. However, in a supply unconstrained regime, $\mu_1 > \lambda_1 \cdot \overline{F}(\rho^u)$, the two thresholds are different, $X_r^0 < X_r$, as depicted in Figure 4. This occurs because in locations further from the origin but still within $[-X_r, X_r]$, as underutilized drivers drive toward the origin, conditions at the departing point improve and in equilibrium, staying put becomes competitive with driving to the origin.

6.3 Optimal Solution

The previous subsection provided an optimal local price response to a demand shock and the supply movement it induces. In this subsection, we focus on the optimal *global* price response across all locations in the city. To that end, we will leverage the results developed for the general model to obtain a quasi-closed form solution to the platform's problem in this specialized setting.

We begin by showing that the origin is an in-demand sink location and, therefore, the results from Sections 4 and 5 apply to the attraction region of the origin.

By leveraging structural properties of the equilibrium utility function, the congestion bound, and a novel flow-mimicking technique, we next fully characterize in Theorem 2 the optimal equilibrium utility of supply units $V(\cdot)$, not only in the attraction region of the origin, but across the entire city. In particular, this characterization yields a spatial separation of the city into three attractions regions and regions of no-movement. Leveraging Theorem 1 and a symmetry argument, we solve for the optimal s^τ and the corresponding prices in each attraction region. The solution for the no-movement regions reduces to the pre-shock environment. Leveraging the pasting result (cf. Proposition 4) yields the optimal solution to the platform's problem as presented in Theorem 3.

Our first result in this section demonstrates that we can focus on price-equilibrium pairs such that the high demand location is a sink that has drivers coming towards it from left and right.

Lemma 5 (Origin is in-demand sink)**.** *Without loss of optimality, one can restrict attention to price-equilibrium pairs (p, τ) such that the origin is an in-demand sink such that $X_l < 0 < X_r$.*

The intuition behind this proposition harks back to the fact that the performance of the pre-shock environment is dominated by that of the local price response solution. Solutions for which the origin is not an in-demand sink have revenues capped by that of the pre-demand shock environment. At a high-level, in those solutions, there is no positive mass of drivers willing to travel to the demand shock location and, thus, the city resembles a city without a demand shock. However, the local price response solution incentivizes drivers from both sides to travel to the demand shock and has a strictly larger revenue. This implies that at optimality we must have drivers coming from both sides to the origin, that is, $X_l < 0 < X_r$.

In what follows we solve for the key objects of the platform's optimization problem $(\mathcal{P}_2)$. To make

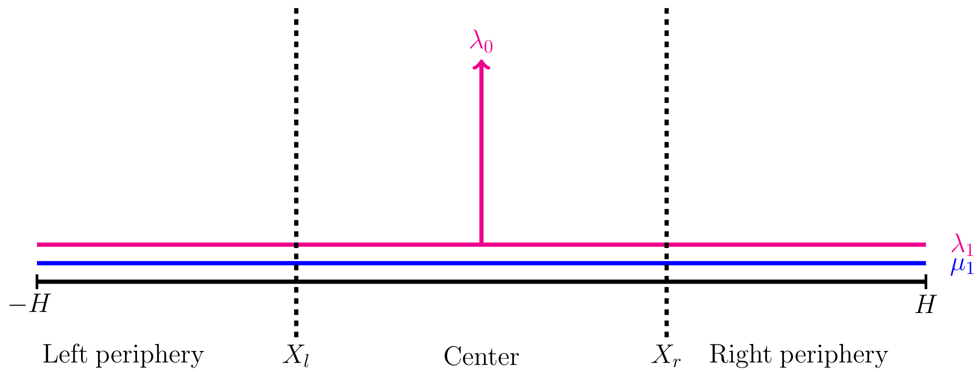

Figure 5: Three region-decomposition.

our exposition clear and highlight the spatial aspects of our solution we call the interval $[X_l, X_r]$ the *center* region, and the region outside of it will be referred to as the *periphery* (see Figure 5).

6.3.1 Equilibrium Utilities

In this subsection we characterize $V(\cdot)$ throughout $\mathcal{C}$. We begin by stating the main result of this subsection. We then we discuss some of the implications and associated intuition.

Theorem 2. *(Equilibrium utilities) Under an optimal price-equilibrium pair(p, τ), the equilibrium utility function $V(\cdot)$ is fully parametrized by the three values $V(0)$ and X_l, X_r as follows:*

$$V(x) = \begin{cases} V(0) - |x| & \text{if } x \in [X_l, X_r], \\ \min\{V(0) - 2X_r + x, \psi_1\} & \text{if } x > X_r, \\ \min\{V(0) - 2|X_l| + |x|, \psi_1\} & \text{if } x < X_l. \end{cases}$$

Moreover, $V(0) > \psi_1$ and $V(X_l), V(X_r) \leq \psi_1$.

The first main implication of this result is that we know exactly how much utility each supply unit garners under optimal prices throughout the entire city. Quite strikingly, the characterization of $V(\cdot)$ is "independent" of the flows. That is, in order to characterize the equilibrium utility we did not need to pin down the distribution of after-movement supply.

The second implication is that the city has at most three types of regions. Figure 6 depicts the equilibrium utility function. The center $[X_l, X_r]$ is by definition an attraction region. Let W_r and Y_r be defined as the points to the left and to the right of X_r where the driver's equilibrium utility function equals the pre-shock utility level ψ_1. To the right of the origin (and similarly to the left), we can observe three main regions. We first have the interval $[0, W_r]$, where drivers' utilities are

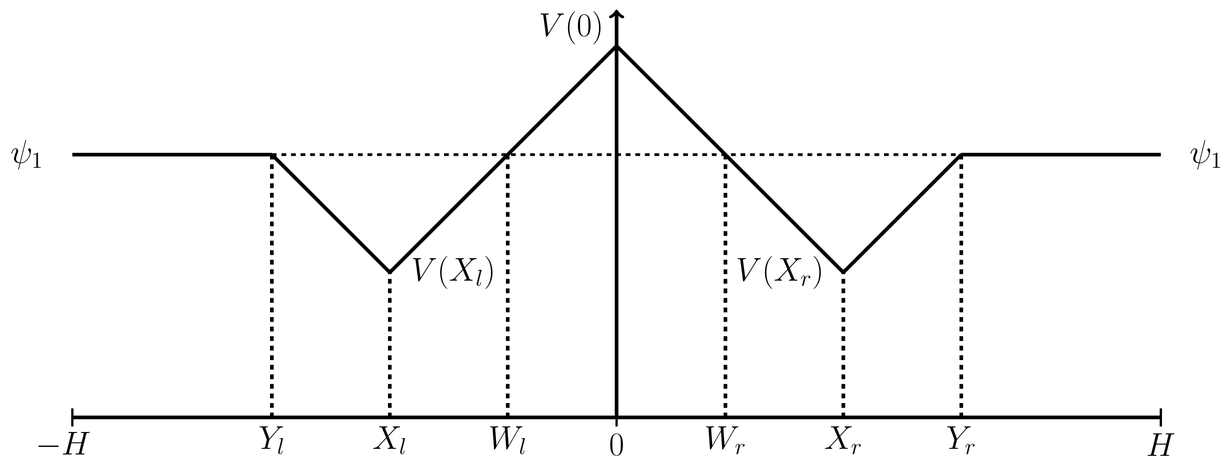

Figure 6: **Drivers' equilibrium utility under an optimal pricing policy.** The equilibrium utility is fully characterized up to $V(0)$, X_l and X_r.

above the pre-shock utility level. Drivers in this region are positively impacted by the shock of demand at the origin (and the global optimal prices). The second region $[W_r, Y_r]$ is notable. Here, drivers garner strictly less utility compared to the pre-shock environment. In $[W_r, X_r]$ drivers are "too far" from the origin so their utilities are negatively affected by the cost of driving to the origin. Drivers in $[X_r, Y_r]$ are outside the origin's attraction region and, thus, do not relocate to the origin. Interestingly, drivers in $[X_r, Y_r]$ suffer because the platform has to make sure that drivers in $[0, X_r]$ stay within the attraction region of the origin. For the marginal drivers at X_r to be willing to travel to the origin, the conditions to the right of X_r should not be too attractive. The final region corresponds to $[Y_r, H]$; this region is not affected by the shock of demand as it is effectively too far from the origin.

Key ideas for the proof of Theorem 2. We now present the main arguments that enable us to establish Theorem 2. At a high level, we focus on each region separately, center and periphery, and solve for $V(\cdot)$ in each of these regions. We first present the arguments for the center, and then for the periphery.

Center. The center region is easy to analyze. Lemma 5 establishes that we can focus on solutions such that $A(0) = [X_l, X_r]$ is a non-empty interval that strictly contains the origin. Our envelope result (Lemma 3) characterizes the equilibrium utility function in any attraction region. In turn, this implies that

$$V(x) = V(0) - |x|, \quad \text{for all } x \in [X_l, X_r].$$

Importantly, the characterization of $V(\cdot)$ in this region only depends on three parameters, namely,

$V(0)$, X_l and X_r. In Section 7, we will leverage this fact to numerically compute the optimal value for these parameters.

Periphery. Consider the right periphery $(X_r, H]$. We first argue that, in this region, the drivers' equilibrium utility has a non-trivial upper bound, and then establish that this upper bound is achieved. The treatment for the left periphery is analogous.

Lemma 6. *(Upper bound) An optimal price-equilibrium pair (p, τ) satisfies*

$$V(x) \le \min\{V(X_r) + x - X_r, \psi_1\}, \quad \textit{for all } x \in (X_r, H]. \tag{6}$$

The upper bound above follows from two bounds. A first upper bound can be derived using the 1-Lipschitz property of V (Lemma 1), which ensures that V can grow at a rate of at most 1. Thus, $V(x)$ is bounded by $V(X_r)+x-X_r$. A second bound may be obtained by leveraging the congestion bound (Proposition 5). One may show that that drivers from almost any location that do not have an incentive to travel to the origin have their utilities capped by the pre-demand shock utility level ψ_1. Locations different than the origin that receive supply increase their driver congestion with respect to the initial congestion level which, in turn, reduces the driver utility at that location. In addition, drivers traveling to these locations have to incur a transportation cost further decreasing their utilities. Thus $V(\cdot)$ has to be bounded by ψ_1 in $(X_r, H]$.

The core of the argument toward characterizing the equilibrium utilities in the periphery resides in establishing that the upper bound in Eq. (6) is always binding for any x in $(X_r, H]$, a result we will present in Proposition 9. We show this result in two steps: we first establish that the value function has to be non-decreasing in $[X_r, H]$ and then leverage this to establish that the upper bound is achieved under an optimal pricing policy.

By our characterization of a driver's utility in an attraction region (see Lemma 3), the upper bound would not be binding if there were drivers willing to move left in $(X_r, H]$. That would imply the existence of an attraction region (see Proposition 2) inside of which $V(\cdot)$ is decreasing. Our first proposition proves this cannot happen by establishing that, in an optimal solution, $V(\cdot)$ is a non-decreasing function in the right periphery.

Proposition 8. *(Monotonicity in the periphery) Without loss of optimality, we can focus on price-equilibrium pairs (p, τ) such that $V(\cdot)$ is non-decreasing in $(X_r, H]$. Furthermore, if $V(X_r) = \psi_1$, then $V(x) = \psi_1$ for all $x \ge X_r$.*

We first observe that the attraction region around the origin of the demand shock location is always wider under the optimal solution than under the local best response. That is, $A^{\mathrm{lr}}(0) \subset A^{\mathrm{opt}}(0)$. In particular, this means that more locations are affected by a demand shock in the optimal solution than under the local price response. Hence, the largest interval in which both solutions differ corresponds to $[-Y_r^{\mathrm{opt}}, Y_r^{\mathrm{opt}}]$. We denote this interval by $\mathcal{C}_{\mathrm{diff}}$.

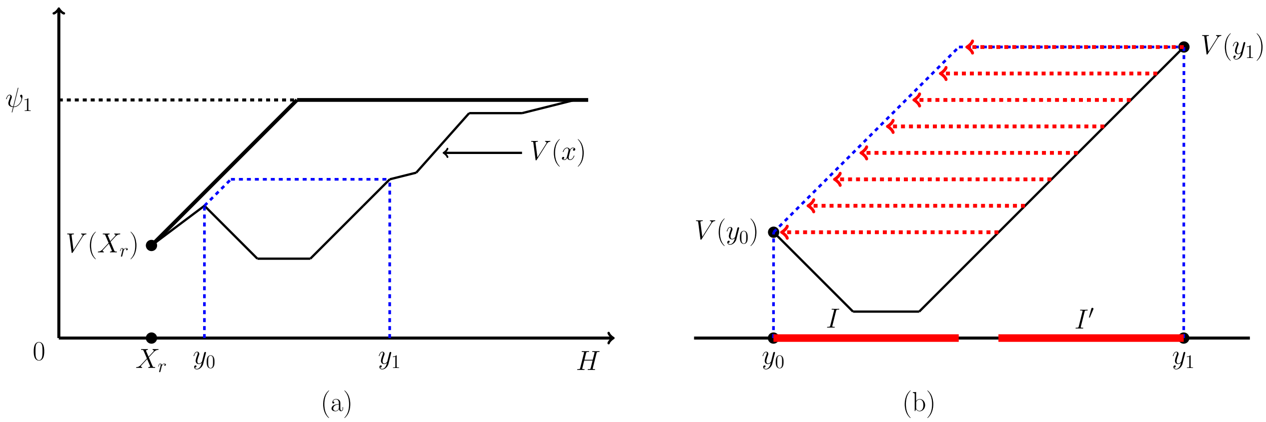

Figure 7: Illustration of the main argument in the proof of Proposition 8.

The key argument behind the proof of Proposition 8 is to construct a (strictly) profitable deviation whenever $V(\cdot)$ is decreasing in some region. We illustrate the main idea of the argument in Figure 7. Suppose the value function is decreasing in some interval as illustrated in Figure 7(a). We will construct a deviation over a superset of that interval, denoted by $[y_0, y_1]$ in the figure. The construction of a deviation contains three main ideas.

First, the interval $[y_0, y_1]$ is constructed in such a way that it is flow separated. That is, there is no flow of drivers leaving this interval and no drivers coming in ($\tau - a.e$). This separation permits us to analyze this region as an individual sub-problem, where the behavior of drivers is relatively "controlled". In particular, we construct the interval $[y_0, y_1]$ in such a way there is at most one maximal subinterval where $V(\cdot)$ decreases at rate -1, and at most one maximal subinterval where where V increases at rate 1. Where $V(\cdot)$ decreases at rate -1 drivers can only move left, and where $V(\cdot)$ increases at rate 1 drivers can only move right.

Second, the best incentive compatible deviation that ensures a non-decreasing value function coincides with the dashed blue line. Because V can increase at most at a rate of 1, after y_0 the best deviation equals $V(y_0) + (x - y_0)$ (recall Eq. (6)). Moreover, since the interval ends at y_1 and we want the deviation to be a non-decreasing function, it has to be bounded by $V(y_1)$.

The final idea is a subtle, but critical one. We know from Proposition 1 that the platform earns revenues from a location x proportionally to $V(x) \cdot s^\tau(x)$. As a result, one needs to focus on *both* $V(\cdot)$ and the post-movement supply s^τ to establish a profitable deviation. We need to argue that overall the platform will earn higher revenues after the drivers move. Our argument, which relies on judicious price setting as well as a proper mapping of revenue contributions in different space regions between the old and new flows, is illustrated in Figure 7(b). We set prices in such a way that it is incentive compatible for drivers not to move within the interval $[y_0, y_1]$, except for the region we denote by I near y_0. In this region, we set prices to incentivize the drivers to behave as they did in region I' in the old (non-monotone) solution. This enables us not only to achieve the upper bound constructed, but also to obtain a strict revenue improvement for the platform.

In brief, at the optimal solution, $V(\cdot)$ must be a non-decreasing function in $(X_r, H]$. This implies that drivers only move right (or do not move) in the right peripheral region. Our next result shows that Eq. (6) is indeed binding.

Proposition 9. *(Tight upper bound) Without loss of optimality, we can focus on price-equilibrium pairs (p, τ) such that the upper bound in Eq.* (6) *is tight.*

The proof of Proposition 9 relies on the monotonicity in the periphery of $V(\cdot)$ to construct a strict improvement whenever we have a solution (p, τ) for which the upper bound in Eq. (6) is not tight. We start by separating intervals that form maximal attraction regions, that is, attraction regions with a sink at an end point. In these regions, $V(\cdot)$ is differentiable and has slope equal to 1. Such intervals can be mapped onto the interval where the upper bound in Eq. (6) also has slope 1. This mapping in represented by dashed lines and arrows in Figure 8.

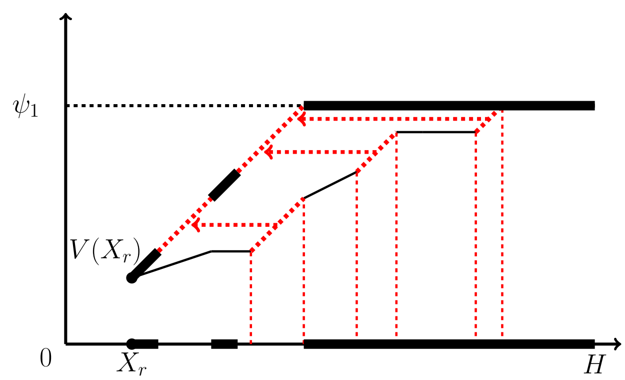

Figure 8: Illustration of the main idea underlying the proof of Proposition 9. The dashed lines in $V(x)$ correspond with interval where $dV(x)/dx = 1$. These intervals are mapped onto the intervals in $[X_r, H]$ where the upper bound in Eq. (6) has slope 1. The thick black lines correspond to both the intervals and parts of the upper bound that are left after the mapping.

We can then use a flow mimicking argument similar to the one used in Figure 7(b). The solutions in the initial intervals in the mapping can be replicated in the new intervals, which we illustrate in Figure 8. Thus, this mapping preserves the platform's revenue in the intervals being mapped.

The regions that are left after the mapping (thick black lines in the figure) are given prices such that drivers in them prefer not to relocate, and V coincides with the upper bound. By pasting the solutions in the intervals we obtain then a solution for which the upper bound is tight and whose revenue is strictly larger than that of (p, τ).

6.3.2 From Equilibrium Utilities to Supply Distribution and Optimal Prices

Given that we pinned down the equilibrium utility function across the city, the natural next step as prescribed by the problem reformulation in Proposition 1 is to solve for prices and post-relocation supply.

Theorem 3. *(Optimal prices and flows) An optimal price-equilibrium pair (p, τ) is such that $V(\cdot)$ is as in Theorem 2, $X_r = -X_l$, and prices and flows are characterized as follows.*

1. *(Post-relocation supply) There exists unique $\beta_c \in [0, W_r]$ and $\beta_p \in [X_r, Y_r]$ such that*

$$\int_{-\beta_c}^{\beta_c} \psi_x^{-1}(V(x)) d\Gamma(x) = \mu_1 \cdot 2 \cdot X_r \quad \text{and} \quad \int_{\beta_p}^{Y_r} \psi_x^{-1}(V(x)) d\Gamma(x) = \mu_1 \cdot (Y_r - X_r),$$

   and the optimal post-relocation supply is given by

$$s^\tau(x) = \begin{cases} 0 & \text{if } x \in (\beta_c, \beta_p) \cup (-\beta_p, \beta_c), \\ \psi_x^{-1}(V(x)) & \text{otherwise.} \end{cases}$$

2. *(Prices) The optimal prices are given by $p(x) = \rho_x^{loc}(s^\tau(x))$, where $s^\tau(x)$ is as above.*
3. *(Movement)*
   - *for all x in $[-\beta_c, \beta_c]$, drivers move in the direction of the origin,*
   - *for all x in $[-X_r, -\beta_c) \cup (\beta_c, X_r]$, all drivers move to $[-\beta_c, \beta_c]$,*
   - *for all x in $[X_r, \beta_p)$, all drivers move to $[\beta_p, Y_r]$.*
   - *for all x in $(-\beta_p, -X_r]$, all drivers move to $[-Y_r, -\beta_p]$.*
   - *for all x in $[\beta_p, Y_r]$, drivers move in the direction of Y_r,*
   - *for all x in $[-Y_r, -\beta_p]$, drivers move in the direction of $-Y_r$,*
   - *for all x in $[-H, -Y_r) \cup (Y_r, H]$, drivers do not relocate.*

The key idea underlying Theorem 3 is to recognize the structure of the regions. The center $[X_l, X_r]$ is by definition an attraction region. The other two attraction regions correspond to the intervals $[Y_l, X_l]$ and $[X_r, Y_r]$ (to recall the definitions of these terms, please revisit Figure 6). Consider the last of these intervals. In it, $V(\cdot)$ increases at a rate of 1 and drivers only move towards Y_r but not beyond it. The shape of $V(\cdot)$ then ensures that all drivers in this region are willing to travel to Y_r and, therefore, this location has to be a sink with its associated attraction region being $[X_r, Y_r]$. We can thus leverage Theorem 1 to characterize the flow structure within attraction regions and then paste solutions appropriately. Finally, we show that the optimal solution has to be symmetric around the origin. In particular, now all the relevant quantities that characterize the optimal solution depend only on two values: $V(0)$ and X_r.

Discussion. We depict in Figure 9 the structure of the solution obtained in Theorem 3. The main

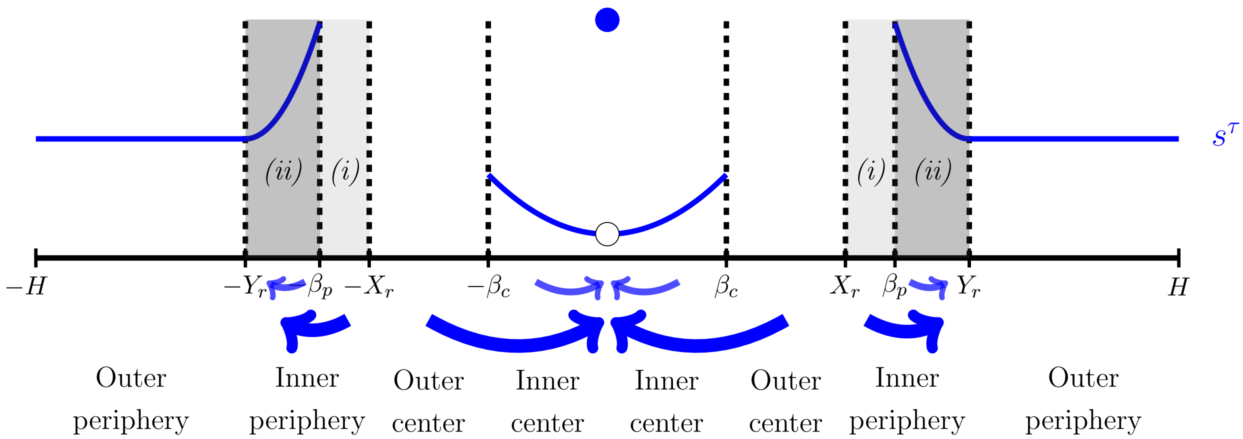

Figure 9: **Supply response induced by optimal prices.**

feature of the optimal solution is that it separates each side of the city, with respect to the origin, into six regions. We focus our discussion on the right side of the city, as the result on the left side is completely symmetric.

The origin receives a mass of supply equal to $\psi_0^{-1}(V(0))$. This mass of drivers comes from two regions, the inner and the outer center, which we now define. The first corresponds to the interval $(0, \beta_c]$. Some drivers in this region choose to stay put while others, attracted by the favorable conditions at the center of the city, choose to drive to the origin. In equilibrium, drivers staying or traveling to the origin garner the same utility. The outer center is the interval $(\beta_c, X_r]$. Here, the platform sets prices to $\overline{V}$ (or 0) and therefore supply is equal to zero. That is, the platform chooses prices to shut down demand, giving no incentive for drivers to stay there (or alternatively sets prices at zero to again give no incentive for drivers to stay there). In turn, this incentivizes all drivers in this region to move somewhere else. In order to incentivize these drivers to move towards the origin, the platform creates one more region: the inner periphery.

The inner periphery corresponds to the interval $(X_r, Y_r]$. The platforms "artificially" degrades the conditions for drivers in this interval in two different ways, leading to the two sub regions, *(i)* and *(ii)* in Figure 9. In region (i), the platform sets prices equal to $\overline{V}$ in $(X_r, \beta_p]$, shutting down demand, so no drivers want to either travel to or stay in this region. As a result the interval $(\beta_p, Y_r]$ receives all drivers from $(X_r, \beta_p]$. This creates driver congestion and, thus, endogenously worsens driver conditions in the interval $(\beta_p, Y_r]$. The reason the platform to choose these inner periphery prices is to discourage drivers in the outer center from driving towards the periphery. Quite strikingly, the optimal global price response to a demand shock at the origin induces supply movement *away* from the origin in the inner periphery.

The final region is the outer periphery. All drivers in this region stay put, leading to $s^\tau(x) = \mu_1$. Here, drivers collect the same utility they would make if there was no demand shock at the origin.

In sum, the optimal global price response to a demand shock, while correcting the supply-demand imbalance at the origin, also creates significant imbalances across the city. This is driven by the self-interested nature of capacity units and the need to incentivize them through spatial pricing. In particular, we observe that the structure of the optimal pricing policy is very different from that of the local price response (cf. Proposition 7). In Section 7, we explore the structure of the policies numerically, and compare their structure as well as their performance.

7 Local Price Response versus Optimal (Global) Prices: Structure and Numerics

In this section, we will use the optimal local price response solution as a benchmark for comparison to put the optimal solution in perspective. The objective is to illustrate through several metrics the different features of the optimal solution as well as its performance in terms of revenue maximization and welfare. Throughout this section, we use superscripts *lr* and *opt* to label relevant quantities associated with the local price response and optimal solution, respectively (except when obvious from the context).

We first observe that the attraction region around the origin of the demand shock location is always wider under the optimal solution than under the local best response. That is, $A^{\text{lr}}(0) \subset A^{\text{opt}}(0)$. In particular, this means that more locations are affected by a demand shock in the optimal solution than under the local price response. Hence, the largest interval in which both solutions differ corresponds to $[-Y_r^{\text{opt}}, Y_r^{\text{opt}}]$. We denote this interval by $\mathcal{C}_{\text{diff}}$.

Next, we illustrate and discuss through a set of numerics the differences between the two policies. We consider a range of instances that includes various levels of supply availability. We fix the city to be characterized by $H = 1$ and assume that the demand is uniformly distributed across locations with $\lambda_1 = 4$. The origin experiences a shock of demand ranging from low to high: $\lambda_0 \in \{3, 6, 9\}$. We vary the initial supply $\mu_1 \in \{1, 1.5, 2, \ldots, 4.5, 5\}$ so that when low, the city (excluding the origin) is supply constrained, and when high, the city is supply unconstrained. Consumer valuation is uniformly distributed in the unit interval. Note that the city (excluding the origin) is supply constrained whenever $\mu_1 < \lambda_1 \cdot \overline{F}(p^u) = 2$. To eliminate any strong dependence on the choice of H, for each instance, we compare the local price response performance and optimal solution performances within the sub-region of the city corresponding to the largest interval in which both solutions differ, $\mathcal{C}_{\text{diff}}$. Given the symmetry of the solutions, in all that follows we focus on the right side of the city $[0, H]$.

Policy structure. Figure 10 depicts the core spatial thresholds characterizing the optimal pricing policy and the local price response as the supply conditions μ_1 changes (on the y-axis). In particular, we track the changes in X_r, β_p, β_c and Y_r for the optimal solution (cf. Theorem 3) and the changes in X_r and X_r^0 for the local price response (cf. Proposition 7).

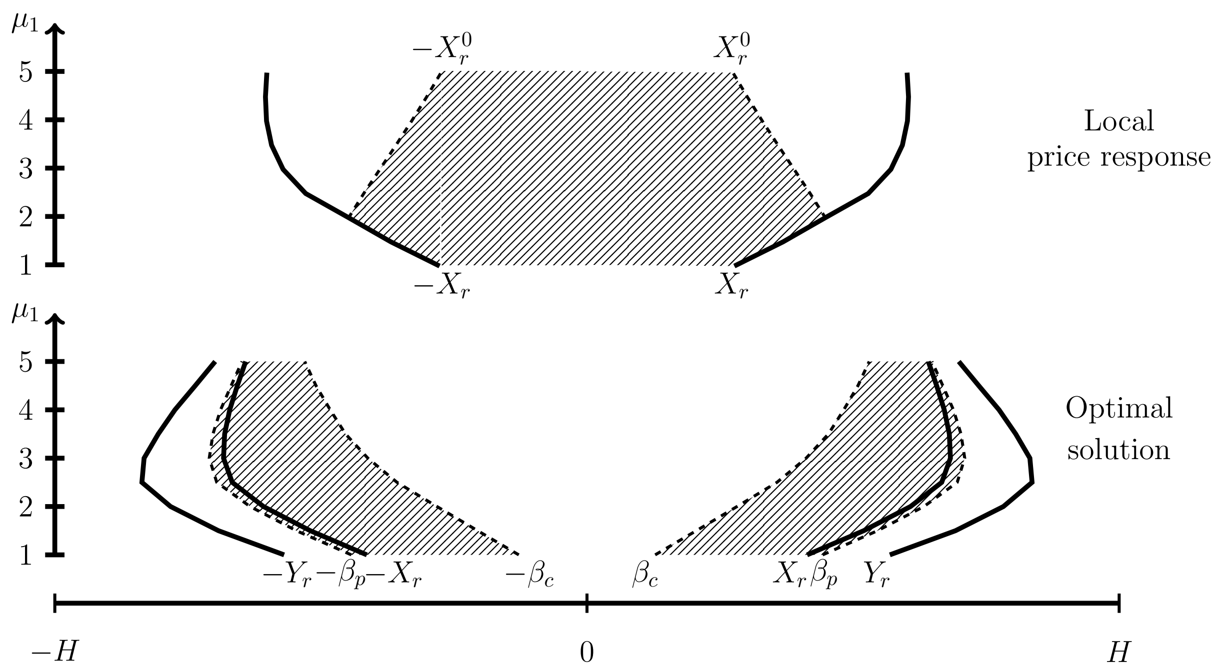

Figure 10: **Policy structure.** Spatial thresholds characterizing the optimal pricing policy and the local price response as the the supply conditions changes. The shaded regions have no supply in equilibrium. The figure assumes $\lambda_0 = 9$ and $\lambda_1 = 4$.

The first thing to note is that the structure of supply in the attraction region of 0 differs significantly between the local price response and the optimal policy. In the local price response, there are no drivers who stay put around the origin; and post-relocation, drivers are either at the origin or in $[X_r^{0,\mathrm{lr}}, X_r^{\mathrm{lr}}]$. In contrast for the the optimal policy, there are no drivers in a region separated from the origin $[\beta_c, X_r^{\mathrm{opt}}]$ but there are drivers in $[0, \beta_c]$. This contrast can be better understood through the reformulation of the objective in Proposition 1, in conjunction with the shape of the equilibrium utility function in the attraction region of 0 (cf. Lemma 3). Given the objective, the platform would ideally like to have supply as close to the origin as possible (subject to the congestion bound constraint) as it maximizes the integral of $V(x) \cdot s^\tau(x)$. With a local price response, as a result of the lack of flexibility in setting prices throughout the city, the platform is unable to "optimize" the supply in the attraction region and ends up with drivers at locations with low V in $[X_r^{0,\mathrm{lr}}, X_r^{\mathrm{lr}}]$ while locations with higher V's have no drivers in $(0, X_r^{\mathrm{lr}}]$. Meanwhile, the optimal policy is able to set prices so as to induces the best possible distribution of supply in the attraction region.

In the periphery of the optimal solution, which is outside the origin's attraction region under pricing policy, the local price response behaves exactly as in the pre-demand shock environment. In stark contrast, the optimal solution incentivizes movement of drivers from the periphery away from the demand shock. In particular, the region $[X_r, Y_r]$, which has a non-trivial size, is artificially damaged. This region is needed for the optimal solution to steer more drivers towards the origin, an issue we address in more detail in the revenue improvement discussion below.

Revenue Improvement. The revenue performance of the optimal solution with respect to our benchmark in $\mathcal{C}_{\mathrm{diff}}$ is shown in Table 1.

μ_1	1	1.5	2	2.5	3	3.5	4	4.5	5
$\lambda_0 = 3$	2.05	4.64	9.59	13.02	13.87	12.92	11.00	8.60	5.91
$\lambda_0 = 6$	2.17	3.11	4.99	8.73	9.96	10.01	9.56	8.92	8.21
$\lambda_0 = 9$	2.69	3.51	4.69	8.75	10.16	10.30	9.81	9.10	8.29

Table 1: Revenue improvement (in %) of optimal solution over optimal local prices response solution in $\mathcal{C}_{\mathrm{diff}}$.

For any level of demand shock, we observe that the revenue improvement reaches its maximum value for medium to high levels of supply, and can be significant, above 10%.

In order to appreciate where the revenue gains stem from, consider Figure 10 and Table 2 below, which summarizes some key quantities for the case $\mu_1 = 3$, $\lambda_0 = 9$ (so that ψ_1 equals 0.27). Let us analyze the various contributions to revenues under both policies. We start by noticing that the drivers' equilibrium utility at the shock location is lower under the optimal solution than under the local price response, $V^{\mathrm{opt}}(0) = 0.62$ and $V^{\mathrm{lr}}(0) = 0.65$. However, since $X_r^{\mathrm{opt}} = 0.46$ and $X_r^{\mathrm{lr}} = 0.38$,

$V^{\text{opt}}(0)$	$s^{\text{opt}}(0)$	$p^{\text{opt}}(0)$	X_r^{opt}	Y_r^{opt}	$V^{\text{lr}}(0)$	$s^{\text{lr}}(0)$	$p^{\text{lr}}(0)$	X_r^{lr}	$X_r^{0,\text{lr}}$
0.62	1.97	0.78	0.46	0.57	0.65	1.66	0.81	0.38	0.25

Table 2: Metrics for the local response and optimal solution for the case $\mu_1 = 3$, $\lambda_0 = 9$.

the optimal solution is able to incentivize the movement of a larger mass of drivers towards the demand shock, leading to a mass $s^{\text{opt}}(0) = 1.97$ and $s^{\text{lr}}(0) = 1.66$. Focusing on the objective reformulation in Proposition 1, this extra mass of drivers delivers 0.14 units ($0.62 \times 1.97 - 0.65 \times 1.66$) of extra revenue to the platform. The revenue difference is further increased by the fact that the remainder 0.79 units of drivers in the attraction region of zero ($2 \times 3 \times 0.46 - 1.97$) in the optimal solution travel to locations nearby the demand shock, where $V(\cdot)$ is close to 0.62. In contrast, the benchmark solution has the remainder 0.62 drivers ($2 \times 3 \times 0.38 - 1.66$) staying within $[X_r^{0,\text{lr}}, X_r^{\text{lr}}]$ where $V(\cdot)$ is below 0.37 ($V^{\text{lr}}(0) - X_r^{0,\text{lr}}$). Through these two mechanisms, the optimal policy garners more revenue than the benchmark solution in the region $[-X_r^{\text{opt}}, X_r^{\text{opt}}]$.

However, the benefits come at a cost. In particular, to induce the "right" incentives in the shock's attraction region, the platform has to alter conditions to the right of the attraction region. In order to incentivize the movement of drivers in $[-X_r^{\text{opt}}, X_r^{\text{opt}}]$ towards the demand shock, the region $[X_r^{\text{opt}}, Y_r^{\text{opt}}]$ is damaged by having the 0.22 units of drivers in it ($2 \times (0.57 - 0.46)$) contributing values strictly below $\psi_1 = 0.27$ to the platform's objective. The same units of drivers in the benchmark solution contribute exactly 0.27 per unit to the platform's revenue. This cost is offset by the proceeds that incentivizing the movement of a larger amount of drivers toward the demand shock generates.

Welfare Implications. The revenue improvement of the optimal solutions relies on creating a special region in which drivers' utilities are below of what they could earn if the platform responded only locally to the demand shock. This raises the question of whether revenue-optimal pricing leads to lower or higher surpluses for drivers and consumers compared to the benchmark solution.

The social welfare (SW) equals the sum of the platform's revenue, and the driver (DS) and consumer surpluses (CS), as given by

$$DS = \int_{\mathcal{C}_{\text{diff}}} V(x)\, d\mu(x), \quad CS = \int_{\mathcal{C}_{\text{diff}}} \mathbb{E}[(v - p(x)) | v \geq p(x)] \cdot \min\left\{ s^{\tau}(x), \lambda_x \cdot \overline{F}(p(x)) \right\} d\Gamma(x).$$

Driver surplus corresponds to nothing more than the integral of driver equilibrium utilities across all locations in $\mathcal{C}_{\text{diff}}$. Similarly, consumer surplus corresponds to the gains enjoyed across $\mathcal{C}_{\text{diff}}$ by all those consumers who are willing to pay and are matched to some driver.

In Table 3 we display the percentage differences of driver and consumer surpluses, as well as social welfare between the optimal and benchmark solutions. We note that there are instances where the optimal solution is a Pareto improvement over the local price response, in the sense that it is better for the platform, drivers and consumers. There are also instances where the platform's revenue gain is at the expense of both drivers and consumers.

μ_1		1	1.5	2	2.5	3	3.5	4	4.5	5
	$\lambda_0 = 3$	-0.67	3.09	11.3	13.64	14.6	12.44	10.00	7.53	4.92
DS	$\lambda_0 = 6$	-4.15	-3.99	-1.62	-2.01	-0.82	0.74	3.00	5.35	7.80
	$\lambda_0 = 9$	-6.22	-7.35	-7.48	-9.45	-9.72	-9.02	-8.14	-6.36	-4.32
	$\lambda_0 = 3$	-10.96	-14.1	-18.48	-7.24	-3.15	-0.44	1.01	1.57	1.58
CS	$\lambda_0 = 6$	-12.03	-10.58	-17.15	-6.32	1.18	4.18	4.24	2.85	0.69
	$\lambda_0 = 9$	-14.33	-11.94	-22.43	-12.58	-1.39	5.77	9.73	10.98	10.44
	$\lambda_0 = 3$	-1.04	0.81	4.26	8.28	9.70	8.83	7.44	5.8	3.96
SW	$\lambda_0 = 6$	-3.60	-3.56	-3.49	-1.05	1.50	3.16	4.43	5.29	5.87
	$\lambda_0 = 9$	-5.24	-5.95	-8.16	-6.84	-4.40	-2.32	-0.86	0.51	1.58

Table 3: Driver surplus, consumer surplus and social welfare difference (in %) of optimal solution over optimal local prices response solution in $\mathcal{C}_{\text{diff}}$.

For a given level of supply, the driver surplus degrades with respect to the benchmark as the demand shock becomes more intense. We also find that, independently of the size of the demand shock, the optimal solution performs better than the benchmark in terms of consumer surplus when the supply level is high. More drivers in the city imply more matches and lower prices and, thus, higher consumer surplus.

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Appendix for:
Surge Pricing and its Spatial Supply Response

A Proofs for Section 4

Proof of Lemma 1. Consider any $z, y \in \mathcal{C}$. Then, for essentially any $w \in \mathcal{B}$, we have

$$\begin{aligned} V_{\mathcal{B}}(y) &\geq U(w) - |w-y| \\ &= U(w) - |z-w| + |z-w| - |w-y| \\ &\geq U(w) - |z-w| - |z-y|, \end{aligned}$$

where the second inequality comes from the triangular inequality. This implies that

$$V_{\mathcal{B}}(y) + |z-y| \geq V_{\mathcal{B}}(z).$$

Since by Lemma A-1 (stated and proved right after this proof), $V_{\mathcal{B}}(y)$ is finite for any y in $\mathcal{C}$, we have

$$V_{\mathcal{B}}(z) - V_{\mathcal{B}}(y) \leq |z-y|.$$

Since we can interchange the roles of z and y, we have proved that

$$|V_{\mathcal{B}}(z) - V_{\mathcal{B}}(y)| \leq |z-y|, \quad \text{for all } z, y \in \mathcal{C}.$$

□

Lemma A-1. *Consider a measurable set $\mathcal{B} \subseteq \mathcal{C}$ such that $\mathit{\Gamma}(\mathcal{B}) > 0$, let p be a measurable mapping $p : \mathcal{B} \to \mathbb{R}_+$, and let $\tau \in \mathcal{F}(\mu)$. Then, $V_{\mathcal{B}}(x|\, p, \tau) \in \left(-\infty, \alpha \cdot \overline{V}\right]$ for all $x \in \mathcal{C}$. Furthermore, $V(x|\, p, \tau) \geq 0$ for all $x \in \operatorname{supp}(\mathit{\Gamma})$.*

Proof. Fix $x \in \mathcal{C}$. Note that for any $y \in \mathcal{B}$, we have $U(y) - |y-x| \geq -2 \cdot H$. Since $\mathit{\Gamma}(\mathcal{B}) > 0$, this implies that $V_{\mathcal{B}}(x|\, p, \tau) \geq -2 \cdot H$.

Similarly, note that for any $y \in \mathcal{B}$, $U(y) \leq \alpha \cdot \overline{V}$ and hence

$$\mathit{\Gamma}(y \in \mathcal{B} : U(y) - |y-x| > \alpha \cdot \overline{V}) \ \leq\ \mathit{\Gamma}(y \in \mathcal{B} : -|y-x| > 0) \ =\ 0.$$

This implies that $V_{\mathcal{B}}(x|\, p, \tau) \leq \alpha \cdot \overline{V}$.

Finally, we show that $V(x|\, p, \tau) \geq 0$ for all $x \in \operatorname{supp}(\mathit{\Gamma})$. Since $x \in \operatorname{supp}(\mathit{\Gamma})$ we have that $\mathit{\Gamma}(B(x, \delta)) > 0$ for all $\delta > 0$ and, therefore, because $U(y) \geq 0$

$$\mathit{\Gamma}(y \in B(x, \delta) : U(y) - |y-x| > -\delta) > 0, \quad \forall \delta > 0.$$

This implies that $V(x|\, p, \tau) > -\delta$ for all $\delta > 0$, implying that $V(x|\, p, \tau) \geq 0$. $\square$

Proof of Proposition 1. The platform's objective function satisfies the following series of equalities:

$$\begin{aligned}
\alpha \int_{\mathcal{C}} p(y) \cdot \min\left\{s^\tau(y), \overline{F}_y(p(y))\lambda(y)\right\} d\Gamma(y) &= \alpha \int_{\mathcal{C}} p(y) \cdot \min\left\{s^\tau(y), \overline{F}_y(p(y))\lambda(y)\right\} \mathbf{1}_{\{s^\tau(y)>0\}} d\Gamma(y) \\
&= \alpha \int_{\mathcal{C}} p(y) \cdot \min\left\{1, \frac{\overline{F}_y(p(y))\lambda(y)}{s^\tau(y)}\right\} \mathbf{1}_{\{s^\tau(y)>0\}} s^\tau(y) d\Gamma(y) \\
&= \int_{\mathcal{C}} U(y, p(y), s^\tau(y)) \mathbf{1}_{\{s^\tau(y)>0\}} s^\tau(y) d\Gamma(y) \\
&\stackrel{(a)}{=} \int_{\mathcal{C}} U(y, p(y), s^\tau(y)) \mathbf{1}_{\{s^\tau(y)>0\}} d\tau_2(y) \\
&\stackrel{(b)}{=} \int_{\mathcal{C}_\lambda} U(y, p(y), s^\tau(y)) \mathbf{1}_{\{s^\tau(y)>0\}} d\tau_2(y),
\end{aligned}$$

where (a) follows from the fact that $U(y, p(y), s^\tau(y))\mathbf{1}_{\{s^\tau(y)>0\}}$ is a measurable function with values in $[0, +\infty)$, and (b) holds because whenever $\lambda(y) = 0$ we have $U(y, p(y), s^\tau(y)) = 0$. The key step is then to establish that

$$U\left(x, p(x), s^\tau(x)\right) \;=\; V(x|\, p, \tau) \qquad \tau_2 - a.e.\ x \in \mathcal{C},$$

namely, whenever there is post-relocation supply at a given location, the drivers originating at such a location can achieve maximum utility by staying at that location. We state and prove this result in Lemma A-2 (stated and proved following this proof). In turn, one obtains that

$$\begin{aligned}
\gamma \cdot \alpha \int_{\mathcal{C}} p(y) \cdot \min\left\{s^\tau(y), \overline{F}_y(p(y))\lambda(y)\right\} d\Gamma(y) &= \int_{\mathcal{C}_\lambda} V(y) \mathbf{1}_{\{s^\tau(y)>0\}} d\tau_2(y) \\
&\stackrel{(a)}{=} \int_{\mathcal{C}_\lambda} V(y) \mathbf{1}_{\{s^\tau(y)>0\}} s^\tau(y) d\Gamma(y) \\
&= \int_{\mathcal{C}_\lambda} V(y) s^\tau(y) d\Gamma(y),
\end{aligned}$$

where (a) follows since $V(y)\mathbf{1}_{\{s^\tau(y)>0\}}$ is measurable with values in $[0, +\infty)$. This concludes the proof. $\square$

Lemma A-2 (Equilibrium Utilities)**.** *For any price mapping p and corresponding equilibrium τ, let $\mathcal{B} \subseteq \mathcal{C}$ such that $\Gamma(\mathcal{B}) > 0$, then*

$$U\left(x, p(x), s^\tau(x)\right) \;=\; V_{\mathcal{B}}(x|\, p, \tau) = V(x|\, p, \tau) \qquad \tau_2 - a.e.\ x \in \mathcal{B}.$$

Proof. We prove that

$$U\left(x, p(x), s^\tau(x)\right) \;=\; V_{\mathcal{B}}(x|\, p, \tau) \qquad \tau_2 - a.e.\ x \in \mathcal{B}.$$

The proof for $V(x|\, p, \tau)$ instead of $V_{\mathcal{B}}(x|\, p, \tau)$ follows the same steps and is, thus, omitted.

Let $A \subset \mathcal{B}$ be a set defined by

$$A \triangleq \{x \in \mathcal{B} : U(x) = V_{\mathcal{B}}(x)\}. \tag{A-1}$$

We want to prove that $\tau_2(A^c) = 0$, where the complement is taken with respect to $\mathcal{B}$. Consider the sets

$$\begin{aligned} A^- &\triangleq \{x \in \mathcal{B} : U(x) < V_{\mathcal{B}}(x)\} \\ A^+ &\triangleq \{x \in \mathcal{B} : U(x) > V_{\mathcal{B}}(x)\}. \end{aligned}$$

We show that $\tau_2(A^-) = 0$ and $\tau_2(A^+) = 0$. We begin with A^-:

$$\begin{aligned} \tau_2(A^-) &= \tau(\mathcal{C} \times A^-) \\ &= \tau(\{(x,y) \in \mathcal{C} \times A^- : \mathit{\Pi}(x,y) = V(x)) \\ &\leq \tau(\{(x,y) \in \mathcal{C} \times A^- : U(y) \geq V(y)) \\ &\leq \tau(\{(x,y) \in \mathcal{C} \times A^- : U(y) \geq V_{\mathcal{B}}(y)) \\ &\leq \tau(\{(x,y) \in \mathcal{C} \times \mathcal{B} : V_{\mathcal{B}}(y) > U(y) \geq V_{\mathcal{B}}(y)) \\ &= 0, \end{aligned}$$

where the second equality comes from the equilibrium definition. The first inequality follows from the fact that $V(x) + |x - y| \geq V(y)$ (see Lemma 1), the second from $V(y) \geq V_{\mathcal{B}}(y)$, while the third from $y \in A^-$.

To show that $\tau_2(A^+) = 0$, for any $n \in \mathbb{N}$ define the set

$$A_n^+ = \{y \in \mathcal{B} : U(y) \geq V_{\mathcal{B}}(y) + \frac{1}{n}\},$$

and note that $A^+ = \bigcup_{n \in \mathbb{N}} A_n^+$. It is enough to show that $\tau_2(A_n^+) = 0$ for all $n \in \mathbb{N}$. We proceed by contradiction, suppose there exists $n \in \mathbb{N}$ such that $\tau_2(A_n^+) > 0$. Let $\epsilon > 0$ be such that $\epsilon < \frac{1}{2n}$, and consider the finite partition $\{I_i^\epsilon\}_{i=1}^{K(\epsilon)}$ of $\mathcal{C}$, where for any $x, y \in I_i^\epsilon$ we have $|x - y| \leq \epsilon$. Observe that

$$0 < \tau_2(A_n^+) = \tau_2(A_n^+ \cap \bigcup_{i=1}^{K(\epsilon)} I_i^\epsilon) = \sum_{i=1}^{K(\epsilon)} \tau_2(A_n^+ \cap I_i^\epsilon),$$

therefore, there exists $i \in \{1, \ldots, K(\epsilon)\}$ such that $\tau_2(A_n^+ \cap I_i^\epsilon) > 0$. Since, $\tau_2 \ll \mathit{\Gamma}$, we also have

$\mathit{\Gamma}(A_n^+ \cap I_i^\epsilon) > 0$. Take $x \in I_i^\epsilon$, then for any $y \in A_n^+ \cap I_i^\epsilon$

$$\begin{aligned} U(y) &\geq V_{\mathcal{B}}(y) + \frac{1}{n} \\ &\geq V_{\mathcal{B}}(x) - |y-x| + \frac{1}{n} \\ &> V_{\mathcal{B}}(x) - |y-x| + 2\epsilon \\ &\geq V_{\mathcal{B}}(x) + |y-x|, \end{aligned}$$

where the second inequality comes from the Lipschitz property (see Lemma 1). The last two inequalities hold because of our choice of ϵ and $x, y \in I_i^\epsilon$. We conclude that

$$A_n^+ \cap I_i^\epsilon \subseteq \{y \in \mathcal{B} : \mathit{\Pi}(x,y) > V_{\mathcal{B}}(x)\},$$

and, therefore, $\mathit{\Gamma}(\{y \in \mathcal{B} : \mathit{\Pi}(x,y) > V_{\mathcal{B}}(x)\}) > 0$. This contradicts the definition of essential supremum. □

Proof of Lemma 2. For ease of notation let us denote $X_l(z|\, p,\tau)$ and $X_r(z|\, p,\tau)$ by x_l and x_r, respectively. We also denote $A(z|p,\tau)$ by $A(z)$. We show that $A(z) = [x_l, x_r]$. The definitions of x_l and x_r immediately imply that $A(z) \subseteq [x_l, x_r]$. Next, we prove the reverse inclusion also holds. First, we show that $x_l, x_r \in A(z)$, that is,

$$z \in \mathcal{IR}(x_l) \quad \text{and} \quad z \in \mathcal{IR}(x_r).$$

It is enough to show this for x_r, as the proof for x_l is analogous. Since $A(z) \neq \emptyset$ and $\mathcal{C}$ is a bounded set, x_r is well defined. Also, $A(z) \neq \emptyset$ implies the existence of x such that $z \in \mathcal{IR}(x|\, p,\tau)$. This together with Lemma A-3 (stated and proved right after this proof) imply that $z \in \mathcal{IR}(z|\, p,\tau)$, leading to the conclusion that $z \leq x_r$.

If $z = x_r$ we are done. Assume that $z < x_r$. We want to prove that

$$\lim_{\delta \downarrow 0} V_{B(z,\delta)}(x_r) = V(x_r). \tag{A-2}$$

Let us first construct a sequence $\{x^n\}_{n\in\mathbb{N}} \subset A(z)$ such that $x^n \to x_r$. Then, $z \in \mathcal{IR}(x^n)$ for all $n \in \mathbb{N}$, that is,

$$\lim_{\delta \downarrow 0} V_{B(z,\delta)}(x^n) = V(x^n), \quad \forall n \in \mathbb{N}. \tag{A-3}$$

Note that Eq.(A-3) implies that $V_{B(z,\delta)}(\cdot)$ takes finite values. We prove Eq. (A-2) from first principles. Take an arbitrary $\epsilon > 0$, it is enough to show that

$$\exists \delta_0 > 0 \text{ such that } \forall \delta \leq \delta_0,\ \epsilon + V_{B(z,\delta)}(x_r) \geq V(x_r).$$

Since x^n converges to x_r we can find $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $|x^n - x_r| \leq \frac{\epsilon}{3}$. In particular, from Eq. (A-3) applied to n_0 we deduce

$$\exists \delta_0 > 0, \text{ such that } \forall \delta \leq \delta_0, \ \frac{\epsilon}{3} + V_{B(z,\delta)}(x^{n_0}) \geq V(x^{n_0}). \tag{A-4}$$

Using the Lipchitz property of $V_{B(z,\delta)}(\cdot)$ and $V(\cdot)$, and the fact that $|x^{n_0} - x_r| \leq \frac{\epsilon}{3}$ yields

$$\frac{\epsilon}{3} + V_{B(z,\delta)}(x_r) \geq V_{B(z,\delta)}(x^{n_0}) \text{ and } V(x^{n_0}) \geq V(x_r) - \frac{\epsilon}{3}.$$

Plugging this into Eq. (A-4) yields Eq. (A-2) and, therefore, $x_r \in A(z)$. Similarly we have that $x_l \in A(z)$.

To conclude the result we make use of Lemma A-3 (stated and proved right after this proof) one more time. Since $A(z) \neq \emptyset$ we must have $z \in A(z)$ and, therefore, $z \in [x_l, x_r,]$. Consider $x \in [x_l, x_r]$ then the lemma together with the fact that $x_l, x_r \in A(z)$ imply $z \in \mathcal{IR}(x|p,\tau)$. This concludes the proof. $\square$

Lemma A-3. *For any price mapping p and corresponding equilibrium τ, if $y \in \mathcal{IR}(x|\, p,\tau)$ then $y \in \mathcal{IR}(z|\, p,\tau)$ for all $z \in [x \wedge y, x \vee y]$. Furthermore, let $x < y < z$ then if $y \in \mathcal{IR}(x|\, p,\tau)$ and $z \in \mathcal{IR}(y|\, p,\tau)$ then $z \in \mathcal{IR}(x|\, p,\tau)$.*

Proof. Suppose that $y \in \mathcal{IR}(x|\, p,\tau)$. If $x = y$ there is nothing to prove. Without loss of generality, suppose $x < y$. Fix $z \in [x, y]$, we want to prove that $y \in \mathcal{IR}(z|\, p,\tau)$, i.e.,

$$\lim_{\delta \downarrow 0} V_{B(y,\delta)}(z) = V(z). \tag{A-5}$$

First, observe that $\mathit{\Gamma}(B(y,\delta)) > 0$ for all $\delta > 0$. If this is not true then there exists $\hat{\delta} > 0$ such that $\mathit{\Gamma}(B(y,\hat{\delta})) = 0$ and, therefore, $\mathit{\Gamma}(B(y,\delta)) = 0$ for all $\delta \leq \hat{\delta}$. In turn, this implies that $V_{B(y,\delta)}(x) = -\infty$ which contradict that $y \in \mathcal{IR}(x|\, p,\tau)$. Lemma A-1 implies that $V_{B(y,\delta)}(\cdot)$ takes finite values for all $\delta > 0$.

Now we proceed to prove Eq. (A-5). Clearly, $V(z) \geq V_{B(y,\delta)}(z)$ for any $\delta > 0$ and, therefore, it is enough to verify that

$$\forall \epsilon > 0, \, \exists \delta_0 > 0 \, \text{such that } \forall \delta \leq \delta_0, \, V_{B(y,\delta)}(z) + \epsilon \geq V(z).$$

Consider $\epsilon > 0$ and $\delta_1 > 0$ such that $x \notin B(y, \delta_1)$, and note that since $y \in \mathcal{IR}(x|\, p,\tau)$ we can find $\delta_0 > 0$ such that

$$V(x) \leq V_{B(y,\delta)}(x) + \frac{\epsilon}{3}, \quad \forall \delta \leq \delta_0.$$

Consider $\delta \leq \min\{\delta_1, \delta_0, \frac{\epsilon}{6}\}$, then

$$U(w) - |w - x| \leq V_{B(y,\delta)}(x) + \frac{\epsilon}{3}, \qquad \mathit{\Gamma} - a.e. \ w \ \text{ in } \mathcal{C}. \tag{A-6}$$

Note that since $z \in [x, y]$, for any $y' \in B(y, \delta)$ we have

$$|y' - x| - |y' - z| \geq -2\delta + |z - x|,$$

and, therefore,

$$\min_{y' \in B(y,\delta)} \{|y' - x| - |y' - z|\} \geq -2\delta + |z - x|.$$

This and Lemma A-4 (which we state and prove after the present proof) deliver

$$V_{B(y,\delta)}(z) \geq V_{B(y,\delta)}(x) - \frac{\epsilon}{3} - 2\delta + |y - x|.$$

This inequality together with Eq. (A-6) deliver:

$$V_{B(y,\delta)}(z) + \frac{\epsilon}{3} + 2\delta - |y - x| \geq U(w) - |w - x| - \frac{\epsilon}{3}, \qquad \mathit{\Gamma} - a.e. \ w \ \text{ in } \mathcal{C}.$$

Then, $\mathit{\Gamma} - a.e. \ w$ in $\mathcal{C}$ we have

$$\begin{aligned} V_{B(y,\delta)}(z) + \frac{2}{3}\epsilon + 2\delta &\geq U(w) - |w - x| + |y - x| \\ &= U(w) - |w - y| + |w - y| - |w - x| + |y - x| \\ &\geq U(w) - |w - y| - |y - x| + |y - x| \\ &= U(w) - |w - y|, \end{aligned}$$

implying that $V_{B(y,\delta)}(z) + \frac{2}{3}\epsilon + 2\delta \geq V(z)$. Finally, since $2\delta \leq \frac{\epsilon}{3}$ we conclude that $V_{B(y,\delta)}(z) + \epsilon \geq V(z)$. This concludes the proof. □

Lemma A-4. *Let $\epsilon, \delta > 0$ and $x, y, z \in \mathcal{C}$. Then,*

$$V_{B(y,\delta)}(z) \geq V_{B(y,\delta)}(x) - \epsilon + \min_{y' \in B(y,\delta)} \{|y' - x| - |y' - z|\},$$

Proof. Define the following set

$$R \triangleq \Big\{ y' \in B(y, \delta) : \mathit{\Pi}(x, y') \geq V_{B(y,\delta)}(x) - \epsilon \Big\},$$

and observe that $\mathit{\Gamma}(R) > 0$. Otherwise, we could find a lower essential upper bound in $B(y, \delta)$. Let $y' \in R$ then

$$\begin{aligned} \mathit{\Pi}(z, y') &= U(y') - |y' - z| - |y' - x| + |y' - x| \\ &= \mathit{\Pi}(x, y') - |y' - z| + |y' - x| \\ &\geq V_{B(y,\delta)}(x) - \epsilon - |y' - z| + |y' - x| \\ &\geq V_{B(y,\delta)}(x) - \epsilon + \min_{y' \in B(y,\delta)} \{|y' - x| - |y' - z|\}. \end{aligned}$$

Since $\varGamma(R) > 0$ we must have that

$$V_{B(y,\delta)}(z) \geq \varPi(z, y') \qquad \varGamma- \ a.e \ y' \in R.$$

Putting the last two inequalities together yields the desired result. $\square$

Proof of Lemma 3. We make use of a more general result which we prove and state after this proof in Lemma A-5 . Namely, for any $x, y \in \mathcal{C}$

$$\text{If } \ y \in \mathcal{IR}(x|\, p, \tau) \ \text{ then } \ V(z|\, p, \tau) = V(x|\, p, \tau) + |z - x|, \ \text{ for all } z \in [x \wedge y, x \vee y]. \tag{A-7}$$

Since z is sink, by lemma 2, $A(z)$ is a closed interval such that $z \in A(z)$. Denote this interval by $[x_l, x_r]$ and let x be in this interval. If $x_l \leq x \leq z$ then Eq. (A-7) yields both

$$V(x|\, p, \tau) = V(x_l|\, p, \tau) + (x - x_l) \ \text{ and } \ V(z|\, p, \tau) = V(x_l|\, p, \tau) + (z - x_l).$$

Putting these two equalities together delivers $V(x|\, p, \tau)$ equal to $V(z|\, p, \tau) - (z - x)$. A similar argument applies for the case that $z \leq x \leq x_r$. In either case, for any $x \in A(z|\, p, \tau)$ we have that $V(x|\, p, \tau)$ equals $V(z|\, p, \tau) - |z - x|$. $\square$

Lemma A-5. *Let (p, τ) be a price equilibrium pair. Then for any $x, y \in \mathcal{C}$*

$$\text{If } \ y \in \mathcal{IR}(x|\, p, \tau) \ \text{ then } \ V(z|\, p, \tau) = V(x|\, p, \tau) + |z - x|, \ \text{ for all } z \in [x \wedge y, x \vee y].$$

Proof. Without loss of generality let $x < y$, and assume that $y \in \mathcal{IR}(x)$. We prove that $V(z) = V(x) + |z - x|$. By the Lipschitz property (see Lemma 1) of V we have

$$V(z) \leq V(x) + |z - x|.$$

It remains to prove the opposite inequality. Fix $z \in (x, y)$ and $\epsilon > 0$, and choose $\delta_0 > 0$ such that $z \notin B(y, \delta)$ for all $\delta \leq \delta_0$. Since $y \in \mathcal{IR}(x)$ we can find $\delta_1 > 0$ such that

$$\frac{\epsilon}{2} + V_{B(y,\delta)}(x) \geq V(x), \quad \forall \delta \leq \delta_1, \tag{A-8}$$

and $V_{B(y,\delta)}(\cdot)$ takes finite values. Let $\hat{\delta}_0$ to be equal to $\delta_0 \wedge \delta_1$ and define the set

$$R^{x,\delta,\epsilon} \triangleq \left\{ y' \in B(y, \delta) : \varPi(x, y') > V(x) - \epsilon \right\}.$$

Note that for all $\delta \leq \hat{\delta}_0$ we have $\varGamma(R^{x,\delta,\epsilon}) > 0$. Otherwise, we can find $\delta \leq \hat{\delta}_0$ such that $\varGamma(R^{x,\delta,\epsilon}) = 0$, which implies that $V(x) - \epsilon \geq V_{B(y,\delta)}(x)$. This together with Eq. (A-8) yields a contradiction.

For any $y' \in R^{x,\delta,\epsilon}$

$$\begin{aligned} \Pi(z,y') = U(y') - |y'-z| &= U(y') - |y'-x| + |y'-x| - |y'-z| \\ &\geq V(x) - \epsilon + |y'-x| - |y'-z| \\ &= V(x) - \epsilon + |z-x|. \end{aligned}$$

We deduce by the definition of the essential supremum that

$$V_{B(y,\delta)}(z) \geq V(x) + |z-x| - \epsilon.$$

The choice of ϵ was arbitrary. Letting $\epsilon \downarrow 0$, we obtain that

$$V_{B(y,\delta)}(z) \geq V(x) + |z-x|,$$

which implies that $V(z) \geq V(x) + |z-x|$. Thus,

$$V(z|p,\tau) = V(x|p,\tau) + |z-x|, \quad \text{for all} \quad z \in [x \wedge y, x \vee y].$$

Also, $V(\cdot|p,\tau)$ is differentiable over $(x \wedge y, x \vee y)$ with derivative equal to either 1 or -1. $\square$

Proof of Proposition 2. WLOG suppose $x < y$, since $y \in \mathcal{IR}(x|\, p,\tau)$ the following quantity is well defined

$$z \triangleq \sup\{y' \in \mathcal{C} : y' \in \mathcal{IR}(x|\, p,\tau)\}.$$

We prove that z is a sink location such that $x, y \in A(z|\, p,\tau)$. First, we show that $z \in \mathcal{IR}(x|\, p,\tau)$. Consider a sequence $\{z_n\}$ such that $z_n \in \mathcal{IR}(x|p,\tau)$ and $z_n \to z$. Fix $\epsilon > 0$, $\hat{\delta} > 0$ and choose n such that $|z_n - z| < \hat{\delta}/2$. Since $z_n \in \mathcal{IR}(x|p,\tau)$ then there exists $\delta_0(n,\epsilon) > 0$ such that for all $\delta \leq \delta_0(n,\epsilon)$ we have $V_{B(z_n,\delta)}(x) \geq V(x) - \epsilon$. In particular, for any $\delta \leq \min\{\delta_0(n,\epsilon), \hat{\delta}/2\}$ we have $B(z_n,\delta) \subseteq B(z,\hat{\delta})$ and, therefore,

$$V_{B(z,\hat{\delta})}(x) \geq V_{B(z_n,\delta)}(x) \geq V(x) - \epsilon.$$

Since the choice of ϵ and $\hat{\delta}$ was arbitrary we conclude that $\lim_{\hat{\delta}\downarrow 0} V_{B(z,\hat{\delta})}(x) = V(x)$. That is, $z \in \mathcal{IR}(x|\, p,\tau)$ which also shows that $A(z) \neq \emptyset$. To complete the argument that z is a sink location we argue that we cannot have $z \in A(z')$ for some $z' \neq z$. If we did then $z' \in \mathcal{IR}(z|\, p,\tau)$ for some $z' \neq z$. If $z' > z$ this would contradict the definition of z as being maximal. If $z' < z$ then by Lemma A-5 the function $V(\cdot)$ would be decreasing in (z', z), and by the same lemma it would be increasing in (x, z). Since we cannot have both at the same we deduce that z is a sink location. Moreover, by construction $x \in A(z)$, and because $x < y \leq z$ Lemma A-3 guarantees that $y \in A(z)$. $\square$

Proof of Proposition 3. To show the result we make use of Lemmas A-7 and A-6 which we prove and state after the proof this result. We only provide a proof for i) the proof for ii) is analogous.

First we show that $\tau([-H, X_l(z|\,p,\tau)) \times [X_l(z|\,p,\tau), H]) = 0$. Suppose this is not true, then Lemma A-6 implies that there exists $(x, y) \in [-H, X_l(z|\,p,\tau)) \times [X_l(z|\,p,\tau), H]$ such that $y \in \mathcal{IR}(x)$ which, in turn, contradicts Lemma A-7 part *i)*. Observe that the same reasoning applies to the set $(\{z\} \cup (X_l(z|\,p,\tau), H]) \times ([-H, X_l(z|\,p,\tau)] \setminus \{z\})$. Thus in either case the result holds.

□

Lemma A-6. *Let $\mathcal{L}_1, \mathcal{L}_2 \subset \mathcal{C}$. If $\tau(\mathcal{L}_1 \times \mathcal{L}_2) > 0$ then there exists $(x, y) \in \mathcal{L}_1 \times \mathcal{L}_2$ such that $y \in \mathcal{IR}(x|p,\tau)$.*

Proof. Suppose $\tau(\mathcal{L}_1 \times \mathcal{L}_2) > 0$. This implies that there exists a pair $(x, y) \in \mathcal{L}_1 \times \mathcal{L}_2$ such that for all $\delta > 0$

$$\tau(B(x, \delta) \times B(y, \delta)) > 0. \tag{A-9}$$

If this is not true then for any $(x, y) \in \mathcal{L}_1 \times \mathcal{L}_2$ we can find $\delta_{x,y} > 0$ such that Eq. (A-9) does not hold when we replace δ with $\delta_{x,y}$. The collection $\mathcal{I}$ defined by

$$\mathcal{I} = \{B(x, \delta_{x,y}) \times B(y, \delta_{x,y})\}_{(x,y)\in\mathcal{L}_1\times\mathcal{L}_2}$$

is an open cover of $\mathcal{L}_1 \times \mathcal{L}_2$. Moreover the set $\mathcal{L}_1 \times \mathcal{L}_2$ is separable because $\mathcal{C} \times \mathcal{C}$ is separable. This implies that we can find a countable sub-cover of $\mathcal{L}_1 \times \mathcal{L}_2$ in $\mathcal{I}$, that is, there exists $\{B(x_n, \delta_{x_n,y_n}) \times B(y_n, \delta_{x_n,y_n})\}_{n\in\mathbb{N}}$ such that

$$\mathcal{L}_1 \times \mathcal{L}_2 \subset \bigcup_{n\in\mathbb{N}} B(x_n, \delta_{x_n,y_n}) \times B(y_n, \delta_{x_n,y_n}),$$

see, e.g., (Sierpiński & Krieger 1952, Theorem 69, p. 116). Using the subadditivity of the measure τ we have

$$\tau(\mathcal{L}_1 \times \mathcal{L}_2) \leq \tau\Big(\bigcup_{n\in\mathbb{N}} B(x_n, \delta_{x_n,y_n}) \times B(y_n, \delta_{x_n,y_n})\Big) \leq \sum_{n\in\mathbb{N}} \tau(B(x_n, \delta_{x_n,y_n}) \times B(y_n, \delta_{x_n,y_n})) = 0,$$

contradicting that $\tau(\mathcal{L}_1 \times \mathcal{L}_2) > 0$. This shows that for some $(x, y) \in \mathcal{L}_1 \times \mathcal{L}_2$, Eq. (A-9) holds for any $\delta > 0$.

We next show that $y \in \mathcal{IR}(x)$, that is,

$$\forall \epsilon > 0, \exists \delta_0 > 0 \;\text{ such that }\; \forall \delta < \delta_0 \quad \epsilon + V_{B(y,\delta)}(x) \geq V(x).$$

Let $\epsilon > 0$ and let $\delta_0 > 0$ small enough such that $B(x, \delta_0) \cap B(y, \delta_0) = \emptyset$, and $\delta_0 \leq \frac{\epsilon}{2}$. Consider $\delta \leq \delta_0$ then from Eq. (A-9) and the equilibrium definition we have

$$\begin{aligned}
0 &< \tau(B(x,\delta) \times B(y,\delta)) \\
&= \tau\Big(\Big\{(x', y') \in B(x,\delta) \times B(y,\delta) : \Pi(x', y') = V(x')\Big\}\Big) \\
&\leq \tau_2\Big(\underbrace{\Big\{y' \in B(y,\delta) : \exists x' \in B(x,\delta) \ \text{ such that } \ \Pi(x', y') = V(x')\Big\}}_{\triangleq R^{x,y,\delta}}\Big),
\end{aligned}$$

since $\tau_2 \ll \Gamma$ this implies that $\Gamma(R^{x,y,\delta}) > 0$. Now we argue that

$$R^{x,y,\delta} \subset \{y' \in B(y,\delta) : \epsilon + \Pi(x, y') \geq V(x)\}.$$

Indeed, let $y' \in R^{x,y,\delta}$ then there exists $x' \in B(x,\delta)$ for which

$$\begin{aligned}
U(y') &= V(x') + |y' - x'| \\
&\geq V(x) - \frac{\epsilon}{2} + |y' - x'| \\
&= V(x) - \frac{\epsilon}{2} + |y' - x'| - |y' - x| + |y' - x| \\
&\geq V(x) - \frac{\epsilon}{2} - \frac{\epsilon}{2} + |y' - x|,
\end{aligned}$$

where is the first inequality we used the Lipchitz property of V, and in the second we use that $(x', y') \in B(x,\delta) \times B(y,\delta)$ and that $\delta \leq \frac{\epsilon}{2}$. This yields $\epsilon + \Pi(x, y') \geq V(x)$, as desired. Therefore,

$$\Gamma(\{y' \in B(y,\delta) : \epsilon + \Pi(x, y') \geq V(x)\}) > 0,$$

which implies that $V_{B(y,\delta)}(x) \geq V(x) - \epsilon$. □

Lemma A-7. *Fix $p(\cdot)$ and let τ denote a corresponding equilibrium. Let z be a sink location then,*

i) For all $(x, y) \in [-H, X_l(z|\, p, \tau)) \times [X_l(z|\, p, \tau), H]$, $y \notin \mathcal{IR}(x|\, p, \tau)$.

ii) For all $(x, y) \in (\{z\} \cup (X_l(z|\, p, \tau), H]) \times ([-H, X_l(z|\, p, \tau)] \setminus \{z\})$, $y \notin \mathcal{IR}(x|\, p, \tau)$.

iii) For all $(x, y) \in (X_r(z|\, p, \tau), H] \times [-H, X_r(z|\, p, \tau)]$, $y \notin \mathcal{IR}(x|\, p, \tau)$.

iv) For all $(x, y) \in ([-H, X_r(z|\, p, \tau)) \cup \{z\}) \times ([X_r(z|\, p, \tau), H] \setminus \{z\})$, $y \notin \mathcal{IR}(x|\, p, \tau)$.

Proof. We provide a proof for i) and ii), the proofs for the other cases are analogous. We start with *i)*. We argue by contradiction. Suppose there exists $x \in [-H, X_l(z|\, p, \tau))$ and $y \in [X_l(z|\, p, \tau), H]$ such that $y \in \mathcal{IR}(x)$. We show that this would imply that $z \in \mathcal{IR}(x)$, which would contradict the minimality of $X_l(z|\,p,\tau)$. Let $\underline{w} = z \wedge y$ and $\bar{w} = z \vee y$. Let $\epsilon > 0$ and note that by Lemma A-3 we have that $\bar{w} \in \mathcal{IR}(\underline{w})$. Therefore,

$$\exists \delta_y > 0 \text{ such that } \forall \delta \leq \delta_y,\ \frac{\epsilon}{3} + V_{B(\bar{w},\delta)}(\underline{w}) \geq V(\underline{w}),$$

and since $y \in \mathcal{IR}(x)$

$$\exists \delta_x > 0 \text{ such that } \forall \delta \leq \delta_x,\ \frac{\epsilon}{3} + V_{B(y,\delta)}(x) \geq V(x).$$

Also, because $A(z) \neq \emptyset$ we have that $z \in \mathcal{IR}(z)$ which implies

$$\exists \delta_z > 0 \text{ such that } \forall \delta \leq \delta_z,\ \frac{\epsilon}{3} + V_{B(z,\delta)}(z) \geq V(z).$$

Furthermore, we have that (we provide a proof at the end)

$$V_{B(y,\delta)}(\underline{w}) \geq V_{B(y,\delta)}(x) + |\underline{w} - x| - \frac{\epsilon}{3}. \tag{A-10}$$

Putting these four inequalities together and considering $\delta \leq \min\{\delta_x, \delta_{\underline{w}}, \frac{\epsilon}{12}\}$ yields

$$\begin{aligned}
V(x) &\leq V_{B(y,\delta)}(x) + \frac{\epsilon}{3} \\
&\leq V_{B(y,\delta)}(\underline{w}) - |\underline{w} - x| + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&\leq V(\underline{w}) - |\underline{w} - x| + \frac{2}{3}\epsilon \\
&\leq V_{B(z,\delta)}(\underline{w}) + \frac{\epsilon}{3} - |\underline{w} - x| + \frac{2}{3}\epsilon \\
&\leq V_{B(z,\delta)}(x) + \epsilon
\end{aligned}$$

This implies that $\lim_{\delta \downarrow 0} V_{B(z,\delta)}(x) = V(x)$, that is, $z \in \mathcal{IR}(x)$.

Now we prove Eq.(A-10). First note that since $y' \in B(y,\delta)$ we have

$$\min_{y' \in B(y,\delta)} \{|y' - x| - |y' - \underline{w}|\} \geq -2\delta + |\underline{w} - x|,$$

which together with Lemma A-4, using $\frac{\epsilon}{6}$, delivers

$$V_{B(y,\delta)}(\underline{w}) \geq V_{B(y,\delta)}(x) - \frac{\epsilon}{6} - 2\delta + |\underline{w} - x|.$$

Finally, can use that $\delta \leq \frac{\epsilon}{12}$ in the inequality above to deduce Eq.(A-10).

Now we show *ii)*. We analyze two cases proceeding by contradiction. First, suppose that $X_l(z|p,\tau) < z$ and $y \in \mathcal{IR}(x|p,\tau)$ for some $y \in [-H, X_l(z|\,p,\tau)]$ and $x \in (X_l(z|\,p,\tau), H]$. There are two cases. Then by Lemma 3 $V(\cdot|p,\tau)$ is strictly increasing in $(X_l(z|p,\tau), z)$. However, because $y \in \mathcal{IR}(x|p,\tau)$

by Lemma 3 $V(\cdot|\, p, \tau)$ must be strictly decreasing in $(X_l(z|\, p, \tau), x]$. Since $V(\cdot|\, p, \tau)$ cannot be both strictly increasing and strictly decreasing in $(X_l(z|\, p, \tau), z \wedge x)$ a contradiction obtains.

Second, suppose that $X_l(z|\, p, \tau) = z$ and $y \in \mathcal{IR}(x|p, \tau)$ for some $y \in [-H, X_l(z|\, p, \tau))$ and $x \in [X_l(z|\, p, \tau), H]$. Then Lemma A-3 implies that $y \in \mathcal{IR}(z|\, p, \tau)$ or, equivalently $z \in A(y|\, p, \tau)$. However, z is a sink location and $y \neq z$, therefore, we cannot have $z \in A(y|\, p, \tau)$. Hence, we have a contradiction. □

Proof of Proposition 4. For ease of notation we use $X_l(z)$ and $X_r(z)$ to denote $X_l(z|\, p, \tau)$ and $X_r(z|\, p, \tau)$, respectively. We show that $\hat{\tau}$ belongs to $\mathcal{F}_{\mathcal{C}}(\mu)$ and that it is an equilibrium in $\mathcal{C}$. First we argue that $\hat{\tau} \in \mathcal{F}_{\mathcal{C}}(\mu)$. We clearly have that $\hat{\tau} \in \mathcal{M}(\mathcal{C} \times \mathcal{C})$. In order see why $\hat{\tau}_1$ coincides with μ, let B be a measurable subset of $\mathcal{C}$ then

$$\begin{aligned}
\hat{\tau}_1(B) &= \hat{\tau}(B \times \mathcal{C}) \\
&= \tau((B \cap \overline{\mathcal{A}^c}) \times \mathcal{A}^c) + \tilde{\tau}((B \cap \mathcal{A}) \times \mathcal{A}) \\
&\overset{(a)}{=} \tau((B \cap \overline{\mathcal{A}^c}) \times \mathcal{A}^c) + \tilde{\mu}(B \cap \mathcal{A}) \\
&= \tau((B \cap \overline{\mathcal{A}^c}) \times \mathcal{A}^c) + \tau((B \cap \mathcal{A}) \times \mathcal{A}) \\
&= \tau((B \cap \mathcal{A}^c) \times \mathcal{A}^c) + \tau((B \cap \partial\mathcal{A}) \times \mathcal{A}^c) + \tau((B \cap \mathcal{A}) \times \mathcal{A}) \\
&\overset{(b)}{=} \tau((B \cap \mathcal{A}^c) \times \mathcal{C}) + \tau((B \cap \mathcal{A}) \times \mathcal{C}) \\
&= \tau(B \times \mathcal{C}) \\
&= \mu(B),
\end{aligned}$$

where (a) comes from the fact that $\tilde{\tau}$ belongs to $\mathcal{F}_{\mathcal{A}}(\tilde{\mu})$, and (b) comes from Proposition 3 and the definition of $X_l(z)$ and $X_r(z)$. That is, $\hat{\tau}_1$ coincides with μ. Now, we show that $\hat{\tau}_2 \ll \mathit{\Gamma}$. Let B be as before and suppose $\mathit{\Gamma}(B)$ equals zero then

$$\begin{aligned}
\hat{\tau}_2(B) &= \hat{\tau}(\mathcal{C} \times B) \\
&= \tau(\overline{\mathcal{A}^c} \times (B \cap \mathcal{A}^c)) + \tilde{\tau}(\mathcal{A} \times (B \cap \mathcal{A})) \\
&\leq \tau_2(B \cap \mathcal{A}^c) + \tilde{\tau}_2(B \cap \mathcal{A}) \\
&= 0,
\end{aligned}$$

where the last line holds because $\tau_2 \ll \mathit{\Gamma}$ and $\tilde{\tau}_2 \ll \mathit{\Gamma}|_{\mathcal{A}}$. Now we show that $\hat{\tau}$ is an equilibrium. We need to verify that $\hat{\tau}(\hat{\mathcal{E}})$ equals $\mu(\mathcal{C})$, where

$$\hat{\mathcal{E}} \triangleq \Big\{(x, y) \in \mathcal{C} \times \mathcal{C} : \mathit{\Pi}(x, y, \hat{p}(y), s^{\hat{\tau}}(y)) = \operatorname*{ess\,sup}_{\mathcal{C}} \mathit{\Pi}\Big(x, \cdot, \hat{p}(\cdot), s^{\hat{\tau}}(\cdot)\Big)\Big\}.$$

First we show that $\Gamma-$a.e we have

$$s^{\hat{\tau}}(x) = \begin{cases} s^{\tau}(x) & \text{if } x \in \mathcal{A}^c \\ s^{\tilde{\tau}}(x) & \text{if } x \in \mathcal{A}. \end{cases}$$

Let B be a measurable subset of $\mathcal{A}^c$ then

$$\begin{aligned}
\hat{\tau}_2(B) &= \hat{\tau}(\mathcal{C} \times B) \\
&= \tau((\mathcal{C} \times B) \cap (\overline{\mathcal{A}^c} \times \mathcal{A}^c)) \\
&= \tau(\overline{\mathcal{A}^c} \times B) \\
&\overset{(a)}{=} \tau(\mathcal{C} \times B) \\
&= \tau_2(B),
\end{aligned}$$

where (a) comes from Proposition 3. Therefore, $s^{\hat{\tau}}(x)$ equals $s^{\tau}(x)$ $\;\Gamma - a.e.\; x\;$ in $\mathcal{A}^c$. Similarly, for B a measurable subset of $\mathcal{A}$ we have

$$\hat{\tau}_2(B) = \tilde{\tau}(\mathcal{A} \times B) = \tilde{\tau}_2(B),$$

where the second equality holds because $\tilde{\tau}$ is an equilibrium in $\mathcal{A}$.

Second, we show that $V(x|\,\hat{p},\hat{\tau})$ equals $V_{\mathcal{A}}(x|\,\tilde{p},\tilde{\tau})$ for all $x \in \mathcal{A}$. Let $x \in \mathcal{A}$, by definition

$$V(x|\,\hat{p},\hat{\tau}) \geq \Pi(x, y, \hat{p}(y), s^{\hat{\tau}}(y)), \quad \Gamma - a.e.\; y \;\text{ in }\; \mathcal{C}.$$

In particular, from our choice of $\hat{p}$ and $s^{\hat{\tau}}$ in $\mathcal{A}$ we have

$$V(x|\,\hat{p},\hat{\tau}) \geq \Pi(x, y, \tilde{p}(y), s^{\tilde{\tau}}(y)), \quad \Gamma - a.e.\; y \;\text{ in }\; \mathcal{A},$$

implying that $V(x|\,\hat{p},\hat{\tau}) \geq V_{\mathcal{A}}(x|\,\tilde{p},\tilde{\tau})$. Therefore, it remains to show $V(x|\,\hat{p},\hat{\tau}) \leq V_{\mathcal{A}}(x|\,\tilde{p},\tilde{\tau})$. To prove this, it is enough to argue that

$$V_{\mathcal{A}}(x|\,\tilde{p},\tilde{\tau}) \geq \Pi(x, y, p(y), s^{\tau}(y)), \quad \Gamma - a.e.\; y \;\text{ in }\; \mathcal{A}^c. \tag{A-11}$$

The Lipchitz property together with $V_{\mathcal{A}}(\cdot|\,\tilde{p},\tilde{\tau})$ being equal to $V(\cdot|\,p,\tau)$ in $\partial\mathcal{A}$ yield

$$\begin{aligned}
V_{\mathcal{A}}(x|\,\tilde{p},\tilde{\tau}) &\geq V_{\mathcal{A}}(X_l(z)|\,\tilde{p},\tilde{\tau}) - |x - X_l(z)| \\
&= V_{\mathcal{A}}(X_l|\,p,\tau) - (x - X_l(z)) \\
&\geq U(y, p(y), s^{\tau}(y)) - |y - X_l(z)| - (x - X_l(z)), \quad \Gamma - a.e.\; y \;\text{ in }\; [H, X_l(z)) \\
&= U(y, p(y), s^{\tau}(y)) - |x - y|, \quad \Gamma - a.e.\; y \;\text{ in }\; [H, X_l(z)).
\end{aligned}$$

Similarly, replacing $X_l(z)$ with $X_r(z)$ in the steps above we deduce

$$V_{\mathcal{A}}(x|\,\tilde{p},\tilde{\tau}) \geq U(y,p(y),s^{\tau}(y)) - |x-y|, \quad \Gamma - a.e.\ y\ \text{ in }\ (X_r(z),H].$$

Putting these two inequalities together delivers Eq. (A-11). In conclusion, $V(x|\,\hat{p},\hat{\tau})$ equals $V_{\mathcal{A}}(x|\,\tilde{p},\tilde{\tau})$ for all $x \in \mathcal{A}$. The same argument can be used to show that $V(x|\,\hat{p},\hat{\tau})$ equals $V(x|\,p,\tau)$ for all $x \in \mathcal{A}^c$. Indeed, let $x \in [H, X_l(z))$ then

$$\begin{aligned} V(x|\,\hat{p},\hat{\tau}) &\geq V(X_l(z)|\,\hat{p},\hat{\tau}) - |x - X_l(z)| \\ &= V(X_l(z)|\,\tilde{p},\tilde{\tau}) - (x - X_l(z)) \\ &= V(X_l(z)|\,p,\tau) - (x - X_l(z)) \\ &\geq U(y,p(y),s^{\tau}(y)) - |x-y|, \quad \Gamma - a.e.\ y\ \text{ in }\ \mathcal{A}, \end{aligned}$$

but we also know that

$$\begin{aligned} V(x|\,\hat{p},\hat{\tau}) &\geq U(y,\hat{p}(y),s^{\hat{\tau}}(y)) - |x-y|, \quad \Gamma - a.e.\ y\ \text{ in }\ \mathcal{A}^c \\ &= U(y,p(y),s^{\tau}(y)) - |x-y|, \quad \Gamma - a.e.\ y\ \text{ in }\ \mathcal{A}^c. \end{aligned}$$

Therefore, $V(x|\,\hat{p},\hat{\tau}) \geq V(x|\,p,\tau)$. Analogously,

$$\begin{aligned} V(x|\,p,\tau) &\geq V(X_l(z)|\,p,\tau) - |x - X_l(z)| \\ &= V(X_l(z)|\,\tilde{p},\tilde{\tau}) - (x - X_l(z)) \\ &\geq U(\tilde{p}(y),s^{\tilde{\tau}}(y),y) - |x-y|, \quad \Gamma - a.e.\ y\ \text{ in }\ \mathcal{A} \\ &= U(\hat{p}(y),s^{\hat{\tau}}(y),y) - |x-y|, \quad \Gamma - a.e.\ y\ \text{ in }\ \mathcal{A}, \end{aligned}$$

and

$$\begin{aligned} V(x|\,p,\tau) &\geq U(y,p(y),s^{\tau}(y)) - |x-y|, \quad \Gamma - a.e.\ y\ \text{ in }\ \mathcal{A}^c \\ &= U(y,\hat{p}(y),s^{\hat{\tau}}(y)) - |x-y|, \quad \Gamma - a.e.\ y\ \text{ in }\ \mathcal{A}^c. \end{aligned}$$

Thus $V(x|\hat{p},\hat{\tau}) \leq V(x|p,\tau)$. The same can be done for $x \in (X_r(z),H]$. Therefore, $V(x|\hat{p},\hat{\tau})$ equals $V(x|\,p,\tau)$ in $\mathcal{A}^c$.

Lastly, we verify that $\hat{\tau}(\hat{\mathcal{E}})$ equals $\mu(\mathcal{C})$. Define the sets

$$\begin{aligned} \mathcal{E}_1 &\triangleq \Big\{(x,y) \in \overline{\mathcal{A}^c} \times \mathcal{A}^c : \Pi(x,y,\hat{p}(y),s^{\hat{\tau}}(y)) = V(x|\,\hat{p},\hat{\tau})\Big\} \\ \mathcal{E}_2 &\triangleq \Big\{(x,y) \in \mathcal{A} \times \mathcal{A} : \Pi(x,y,\hat{p}(y),s^{\hat{\tau}}(y)) = V(x|\,\hat{p},\hat{\tau})\Big\} \end{aligned}$$

then $\hat{\tau}(\hat{\mathcal{E}}) = \tau(\mathcal{E}_1) + \tilde{\tau}(\mathcal{E}_2)$. Note that by our choice of prices and $s^{\hat{\tau}}$, and because $V(x|\,\hat{p},\hat{\tau})$ equals $V_{\mathcal{A}}(x|\,\tilde{p},\tilde{\tau})$ for all $x \in \mathcal{A}$, we have

$$\tau(\mathcal{E}_1) = \tau\Big(\Big\{(x,y) \in \overline{\mathcal{A}^c} \times \mathcal{A}^c : \Pi(x,y,p(y),s^{\tau}(y)) = V(x|\,p,\tau)\Big\}\Big),$$
$$\tilde{\tau}(\mathcal{E}_2) = \tilde{\tau}\Big(\Big\{(x,y) \in \mathcal{A} \times \mathcal{A} : \Pi(x,y,\tilde{p}(y),s^{\tilde{\tau}}(y)) = V(x|\,\tilde{p},\tilde{\tau})\Big\}\Big) = \tilde{\mu}(\mathcal{A}).$$

Then if we let $\mathcal{E}$ be defined analogously to $\hat{\mathcal{E}}$ but with $(\hat{p},\hat{\tau})$ replaced by (p,τ), this would yield that $\hat{\tau}(\hat{\mathcal{E}})$ equals $\mu(\mathcal{C})$. In fact,

$$\begin{aligned}
\hat{\tau}(\hat{\mathcal{E}}) &= \tau(\mathcal{E}_1) + \tilde{\mu}(\mathcal{A}) \\
&\overset{(a)}{=} \tau(\mathcal{E}_1) + \tau(\mathcal{A} \times \mathcal{A}) \\
&\overset{(b)}{=} \tau(\mathcal{E} \cap (\overline{\mathcal{A}^c} \times \mathcal{A}^c)) + \tau(\mathcal{E} \cap (\mathcal{A} \times \mathcal{A})) \\
&\overset{(c)}{=} \tau(\mathcal{E}) \\
&= \mu(\mathcal{C}),
\end{aligned}$$

where in (a) we use the definition of $\tilde{\mu}$. In (b) we use the fact that τ only puts mass in $\mathcal{E}$, and in (c) we use that

$$\tau(\mathcal{A}^{\circ} \times \mathcal{A}^c) = 0 \quad \text{and} \quad \tau(\mathcal{A}^c \times \mathcal{A}) = 0.$$

□

B Proofs for Section 5

Proof of Lemma 4. Suppose $\lambda(x) > 0$ and recall that the price achieving the maximum in the definition of $R_x^{loc}(s)$ is $\rho_x^{loc}(s) = \max\{\rho_x^{bal}(s), \rho_x^u\}$. Let s^u be equal to $\lambda(x) \cdot \overline{F}_x(\rho_x^u)$, that is, $\rho_x^{bal}(s^u) = \rho_x^u$. Then, since $\rho_x^{bal}(\cdot)$ is decreasing we have that $\rho_x^{loc}(s) = \rho_x^{bal}(s)$ for all $0 < s \le s^u$ and, therefore,

$$\frac{R_x^{loc}(s)}{s} = \rho_x^{bal}(s) = F^{-1}(1 - \frac{s}{\lambda(x)}), \quad \text{for all } 0 < s \le s^u. \tag{B-1}$$

Since F is strictly increasing, the quotient above is strictly decreasing for $s \in (0, s^u]$. Noting that $F^{-1}(1) = \overline{V}$, the point just made also includes $s = 0$. Now, for $s > s^u$ we have $\rho_x^{loc}(s) = \rho_x^u$, thus

$$\frac{R_x^{loc}(s)}{s} = \rho_x^u \cdot \frac{\lambda(x) \cdot \overline{F}_x(\rho_x^u)}{s}, \tag{B-2}$$

which is strictly decreasing. In any case, we conclude that $\psi(\cdot)$ is strictly decreasing over its domain. □

Proof of Proposition 5. Consider the set $R \triangleq \{x \in \mathcal{C} : \lambda(x) > 0\}$ and let $B \subset R$ be a set defined by

$$B \triangleq \{x \in R : V(x) > \psi_x(s^\tau(x))\}.$$

We want to show that $\mathit{\Gamma}(B) = 0$. First, note that for any $x \in B$ we have $U(x) \neq V(x)$. In fact, let $x \in B$ then

$$V(x) > \psi_x(s^\tau(x)) \geq U\Big(x, p(x), s^\tau(x)\Big),$$

that is, $V(x) > U(x)$. By Lemma A-2 we conclude that $\tau_2(B) = 0$. This yields,

$$0 = \tau_2(B) = \int_B s^\tau(x)\, d\mathit{\Gamma}(x). \tag{B-3}$$

If $\mathit{\Gamma}(B) = 0$ then we are done. Suppose $\mathit{\Gamma}(B) > 0$, from equation (B-3) we can conclude that

$$s^\tau(x) \;\; = \;\; 0 \qquad \mathit{\Gamma} - a.e. \;\, x \in B.$$

Since in B we have $\lambda(x) > 0$ this implies that $\mathit{\Gamma}$ almost everywhere in B, $\psi_x(s^\tau(x))$ equals $\alpha \cdot \overline{V}$. Because $\alpha \cdot \overline{V}$ is the maximum value that $V(\cdot)$ can attain, we conclude that

$$\alpha \cdot \overline{V} \geq V(x) > \psi_x(s^\tau(x)) = \alpha \cdot \overline{V} \qquad \mathit{\Gamma} - a.e. \;\, x \in B.$$

But since we are assuming that $\mathit{\Gamma}(B) > 0$, this yields a contradiction. $\square$

Before we move on to the proof of the main result of this section we provide a different version of the congestion bound stated in Lemma 5. This different version is an upper bound on the supply for all location ($\mathit{\Gamma} - a.e$) in an attraction region regardless of whether $\lambda(\cdot)$ is positive or not.

Lemma B-1. *Let (p, τ) be a price equilibrium pair and assume that $z \in \mathcal{C}$ is a sink location. Let*

$$X_l^{\text{supp}}(z|p,\tau) \triangleq \inf\{x \in A(z|p,\tau) \cap \text{supp}(\mathit{\Gamma})\} \;\; \textit{and} \;\; X_r^{\text{supp}}(z|p,\tau) \triangleq \sup\{x \in A(z|p,\tau) \cap \text{supp}(\mathit{\Gamma})\}.$$

Define the function

$$H_x(V) \triangleq \begin{cases} \psi_x^{-1}(V) & \textit{if } \lambda(x) > 0; \\ 0 & \textit{if } \lambda(x) = 0, \; x \in (X_l^{\text{supp}}(z|\,p,\tau), X_r^{\text{supp}}(z|\,p,\tau)); \\ \frac{d\mu}{d\mathit{\Gamma}}(x) & \textit{if } \lambda(x) = 0, \; x \in \{(X_l^{\text{supp}}(z|\,p,\tau), X_r^{\text{supp}}(z|\,p,\tau)\}; \\ 0 & \textit{if } x \in A(z|\,p,\tau) \setminus (X_l^{\text{supp}}(z|\,p,\tau), X_r^{\text{supp}}(z|\,p,\tau)), \end{cases}$$

then

$$s^\tau(x) \;\leq\; H_x(V(x|\,p,\tau)), \quad \mathit{\Gamma} - a.e. \;\, x \;\; in \;\; A(z|\,p,\tau).$$

Proof. From Lemma 5 we have that

$$s^{\tau}(x) \;\leq\; H_x(V(x|\,p,\tau)), \quad \Gamma - a.e.\; x \;\text{ in }\; A(z|\,p,\tau)\cap\{\lambda > 0\},$$

so we only need to show that the set

$$B \triangleq \{x \in A(z|\,p,\tau) : \lambda(x) = 0,\; s^{\tau}(x) > H_x(V(x|\,p,\tau))\},$$

satisfies $\Gamma(B) = 0$. From the definition of X_l^{supp} and X_r^{supp} we have

$$\Gamma(A(z|\,p,\tau) \setminus (X_l^{\text{supp}}(z|\,p,\tau), X_r^{\text{supp}}(z|\,p,\tau))) = 0,$$

therefore showing that $\Gamma(B)$ equals zero is equivalent to showing that $\Gamma(B_1 \cup B_2)$ equals zero, where

$$\begin{aligned}
B_1 &\triangleq \{x \in (X_l^{\text{supp}}, X_r^{\text{supp}}) : \lambda(x) = 0,\; s^{\tau}(x) > 0\},\\
B_2 &\triangleq \{x \in \{X_l^{\text{supp}}, X_r^{\text{supp}}\} : \lambda(x) = 0,\; s^{\tau}(x) > \frac{d\mu}{d\Gamma}(x)\}.
\end{aligned}$$

For the sake of contradiction assume that $\Gamma(B_1) > 0$ then

$$\tau_2(B_1) = \int_{B_1} s^{\tau}\, d\Gamma > 0.$$

This together with Lemma A-2 yields that $\tau_2(B_1 \cap \{x : U(x) = V(x)\}) > 0$, which in turn implies the existence of $x \in B_1 \cap \{x : U(x) = V(x)\}$. Such an x satisfies that $x \in (X_l^{\text{supp}}, X_r^{\text{supp}})$ and $V(x) = 0$ and, therefore, $V(x') < 0$ for some $x' \in \{X_l^{\text{supp}}, X_r^{\text{supp}}\}$. However, any x' in $\{X_l^{\text{supp}}, X_r^{\text{supp}}\}$ belongs to $\text{supp}(\Gamma)$ and, hence, Lemma A-1 guarantees that $V(x') \geq 0$, yielding a contradiction. Thus, $\Gamma(B_1) = 0$.

Also, if $\Gamma(B_2) > 0$ then WLOG we must have that $\Gamma(\{X_r^{\text{supp}}\}) > 0$ and $s^{\tau}(X_r^{\text{supp}}) > \frac{d\mu}{d\Gamma}(X_r^{\text{supp}})$. This implies that $\tau_2(\{X_r^{\text{supp}}\}) > \mu(\{X_r^{\text{supp}}\})$. However, by Lemma 3 part *ii)* we have that

$$\begin{aligned}
\tau_2(\{X_r^{\text{supp}}\}) &= \tau([X_r^{\text{supp}}, X_r] \times \{X_r^{\text{supp}}\})\\
&= \tau(\{X_r^{\text{supp}}\} \times \{X_r^{\text{supp}}\}) + \tau((X_r^{\text{supp}}, X_r] \times \{X_r^{\text{supp}}\}),
\end{aligned}$$

the second term in the last line is bounded above by $\mu((X_r^{\text{supp}}, X_r])$, because $\mu \ll \Gamma$ and $\Gamma((X_r^{\text{supp}}, X_r]) = 0$ we have that this second term equals zero. The first term is bounded above by $\mu(\{X_r^{\text{supp}}\})$ and, therefore, $\tau_2(\{X_r^{\text{supp}}\})$ must also be bounded above by $\mu(\{X_r^{\text{supp}}\})$, yielding a contradiction. In conclusion, $\Gamma(B) = 0$ and the result is proven.

□

Proof of Theorem 1. The proof of this theorem consists of several parts. First, we introduce an optimization problem which is a relaxation of platform's optimization problem restricted to the attraction region $A(z)$. Then we introduce some notation and pose technical properties. The proof of the properties is deferred to an additional lemma which proof is provided after the present theorem. Given this, we argue that the relaxation has a similar structure to a continuous bounded knapsack problem, and we show how to solve it. Next we construct a local price-equilibrium pair $(\tilde{p}, \tilde{\tau})$ in $A(z)$ that implements the relaxation's solution. We conclude by applying Proposition 4 to create the global price-equilibrium pair $(\hat{p}, \hat{\tau})$ in $\mathcal{C}$ as in the statement of the theorem.

Part 1: Relaxation. Consider the function $H_x(\cdot)$ defined in Lemma B-1 and the following problem relaxation in $[X_l, X_r]$

$$\max_{\tilde{s}(\cdot)} \quad \int_{[X_l,X_r]} V(x) \cdot \tilde{s}(x)\, d\Gamma(x) \qquad (\mathcal{P}_{KP}(z))$$

$$\text{s.t} \quad \tilde{s}(x) \leq H_x(V(x|\, p, \tau)), \quad \Gamma - a.e.\ x \ \text{ in } [X_l, X_r] \qquad \text{(CB)}$$

$$\int_{[X_l,X_r]} \tilde{s}(x)\, d\Gamma(x) = \tau([X_l, X_r] \times [X_l, X_r]) \qquad \text{(FC)}$$

$$\int_{(z,X_r]} \tilde{s}(x)\, d\Gamma(x) \leq \tau([X_l, X_r] \times (z, X_r]) \qquad \text{(FR)}$$

$$\int_{[X_l,z)} \tilde{s}(x)\, d\Gamma(x) \leq \tau([X_l, X_r] \times [X_l, z)) \qquad \text{(FL)}$$

Observe that s^τ (which defines τ_2) is a feasible solution for $(\mathcal{P}_{KP}(z))$. The supply density $s^{\hat{\tau}}$ will be shown to be an optimal solution for this relaxation.

Part 2: Notation. Next we define quantities that will simplify notation in the proof.

1. For any measurable set $B \subseteq [X_l, X_r]$ we define the measure

$$S^H(B) \triangleq \int_B H_x(V(x))\, d\Gamma(x),$$

   $S^H(\cdot)$ is the measure with density $H_x(V(x))$ with respect to the Γ measure.

2. Next we rename the quantities on the RHS of equations (FC), (FL) and (FR).

$$\begin{aligned}
\tau &= \tau([X_l, X_r] \times [X_l, X_r]),\\
\tau_l &= \tau([X_l, X_r] \times [X_l, z)),\\
\tau_r &= \tau([X_l, X_r] \times (z, X_r]),\\
\tau_c &= \tau([X_l, X_r] \times \{z\}).
\end{aligned}$$

   Note that τ equals $\tau_l + \tau_c + \tau_r$.

3. Before we define the thresholds z_l, z_r and quantities s_l, s_r as in the statement of the theorem, we need to identify the maximal intervals where the optimal solution is feasible. Let

$$\delta_l \triangleq \sup\left\{\delta \in [0, z - X_l] : S^H((z-\delta, z)) \leq \tau_l\right\},$$
$$\delta_r \triangleq \sup\left\{\delta \in [0, X_r - z] : S^H((z, z+\delta)) \leq \tau_r\right\},$$
$$\delta_c \triangleq \sup\left\{\delta \in [0, \delta_l \wedge \delta_r] : S^H((z-\delta, z+\delta)) \leq \tau\right\}.$$

Then $(z-\delta_l, z)$ is the maximum interval to the left of z in which our solution can put density $H_x(V(x))$ and satisfy equation (FL). A similar interpretation applies to $(z, z+\delta_r)$. The interval $(z-\delta, z+\delta)$ is the maximum symmetric interval in which we can put density $H_x(V(x))$ before we violate (FC), while satisfying both (FL) and (FR).

Without loss of generality let us assume that $\delta_r \leq \delta_l$, thus $\delta_c \in [0, \delta_r]$. We define

$$z_r \triangleq z + \delta_c.$$

4. We define s_r. Let $H_r \triangleq H_{z_r}(V(z_r))$ and $\Gamma_r \triangleq \Gamma(\{z_r\})$. Define

$$s_r \triangleq \min\left\{H_r, \frac{\tau_r - S^H((z, z+\delta_c))}{\Gamma_r}, \frac{\tau - S^H((z-\delta_c, z+\delta_c))}{\Gamma_r}\right\} \cdot \mathbf{1}_{\{\Gamma_r > 0\}}.$$

This is the largest amount of mass we can put on z_r and still be feasible.

5. Finally we define z_l (which will be lower or equal than $z - \delta_c$) and s_l. The quantity

$$S^H((z-\delta_c, z+\delta_c)) + s_r \cdot \Gamma_r,$$

corresponds to the total supply the optimal solution (as in the statement of the theorem) puts in the interval $(z-\delta_c, z+\delta_c]$. By the way we set s_r we cannot increase the interval to the right of z without violating one of the problem's constraints. However, we may still be able to increase this interval to the left of z. Let

$$\beta_l \triangleq \inf\{\beta \in [0, \delta_l - \delta_c] : S^H([z-\delta_c-\beta, z+\delta_c)) + s_r \cdot \Gamma_r \geq \tau\},$$

it corresponds to the tightest values to the left of $z - \delta_c$ at which we can put density $H_x(V(x))$ without violating the constraints (FC) and (FL). We define

$$z_l \triangleq z - \delta_c - \beta_l.$$

Let $\Gamma_l \triangleq \Gamma(\{z-\delta_c-\beta_l\})$. We define s_l by

$$s_l \triangleq \frac{\tau - S^H((z-\delta_c-\beta_l, z+\delta_c)) - s_r \cdot \Gamma_r}{\Gamma_l} \cdot \mathbf{1}_{\{\Gamma_l > 0\}}.$$

6. Recall that we will show that $s^{\hat{\tau}}$ is an optimal solution to $(\mathcal{P}_{KP}(z))$. It will be useful to denote by $S^{\hat{\tau}}(\cdot)$ to the measure associated with it $s^{\hat{\tau}}$. Note that $S^{\hat{\tau}}(\cdot)$ and $S^{H}(\cdot)$ coincide on $(z-\delta_c-\beta_l, z+\delta_c)$.

Part 3: Technical properties. In Lemma B-2 (stated and proved after the proof of the present theorem) we establish that the following properties hold:

1.

$$S^H((z,z+\delta_c)) \leq \tau_r, \tag{a}$$

$$S^H((z-\delta_c-\beta_l, z+\delta_c)) + s_r \cdot \mathit{\Gamma}_r \leq \tau, \tag{b}$$

$$S^H((z-\delta_c-\beta_l, z)) \leq \tau_l, \tag{c}$$

$$S^H(\{z\}) \geq \tau_c. \tag{d}$$

2. If $\delta_c < \delta_r$ then $\beta_l = 0$.

3. $\tau_c + \tau_r \leq S^H([z, z+\delta_c])$.

In what follows we will assume that these properties are satisfied.

Part 4: Knapsack. We show that $s^{\hat{\tau}}$ is an optimal solution to $(\mathcal{P}_{KP}(z))$. We divide the proof in two parts. First we prove that $s^{\hat{\tau}}$ is feasible, and then we show that any other s that is optimal equals $s^{\hat{\tau}}$, $\mathit{\Gamma}-$ a.e.

Feasibility: $s^{\hat{\tau}}$ is feasible.

1. First the congestion bound and non-negativity are clearly satisfied for all $x \in [X_l, X_r]$, except perhaps for $x = z_l$. In the latter case $s^{\hat{\tau}}(z_l)$ equals

$$s_l = \frac{\tau - S^H((z-\delta_c-\beta_l, z+\delta_c)) - s_r \cdot \mathit{\Gamma}_r}{\mathit{\Gamma}_l}.$$

Equation (b) in Lemma B-2 ensures that $s_l \geq 0$. Furthermore, the definition of β_l guarantees that

$$S^H([z-\delta_c-\beta_l, z+\delta_c)) + s_r \cdot \mathit{\Gamma}_r \geq \tau,$$

or equivalently

$$H_{z_l}(V(z_l)) \geq \frac{\tau - S^H((z-\delta_c-\beta_l, z+\delta_c)) - s_r \cdot \mathit{\Gamma}_r}{\mathit{\Gamma}_l}.$$

Thus, s_l satisfies the congestion bound.

2. We verify the global flow conservation constraint (FC). If $\Gamma_l = 0$ then from the definition of β_l and Eq. (b) the constraint is directly satisfied. If $\Gamma_l > 0$ then

$$\begin{aligned} S^{\hat{\tau}}([X_l, X_r]) &= s_l \cdot \Gamma_l + S^{\hat{\tau}}((z - \delta_c - \beta_l, z + \delta_c)) + s_r \cdot \Gamma_r \\ &= \tau - S^H((z - \delta_c - \beta_l, z + \delta_c)) - s_r \cdot \Gamma_r + S^{\hat{\tau}}((z - \delta_c - \beta_l, z + \delta_c)) + s_r \cdot \Gamma_r \\ &= \tau. \end{aligned}$$

3. Finally we verify both (FR) and (FL). Because of the definition of s_r, (FR) is immediately satisfied. In order to verify (FL) we need to show

$$s_l \cdot \Gamma_l + S^H((z - \delta_c - \beta_l, z)) \leq \tau_l.$$

If $\Gamma_l = 0$ then since $\beta_l + \delta_c \leq \delta_l$ the inequality is satisfied. If $\Gamma_l > 0$ to verify (FL) we need to show

$$\tau - S^H((z - \delta_c - \beta_l, z + \delta_c)) - s_r \cdot \Gamma_r + S^H((z - \delta_c - \beta_l, z)) \leq \tau_l,$$

or equivalently

$$S^H([z, z + \delta_c)) + s_r \cdot \Gamma_r \geq \tau_c + \tau_r.$$

The LHS in this equation is

$$S^H([z, z + \delta_c)) + s_r \cdot \Gamma_r = \min \Big\{ S^H([z, z + \delta_c)) + H_r \cdot \Gamma_r, \tau_r + S^H(\{z\}), \tau - S^H((z - \delta_c, z)) \Big\}.$$

Each term in the minimum above is larger or equal than $\tau_c + \tau_r$. The first term equals $S^H([z, z + \delta_c])$ which, from Lemma B-2, is larger than $\tau_c + \tau_r$. The second term is larger because of (d) and the third term because $\delta_c \leq \delta_l$.

Optimality: Next we show that $s^{\hat{\tau}}$ is indeed optimal. Consider another feasible solution s and let S be its associated measure. That is, for any measurable $B \subseteq A(z)$

$$S(B) \triangleq \int_B s(x) \, d\Gamma(x).$$

Suppose s is optimal. We show that s equals $s^{\hat{\tau}}$ $\Gamma-$ a.e in $[X_l, X_r]$. This would show that s and $s^{\hat{\tau}}$ yield the same objective and, therefore, $s^{\hat{\tau}}$ is optimal. We first show it for $(z - \beta_l - \delta_c, z + \delta_c)$. Define the set

$$R \triangleq \{x \in (z - \beta_l - \delta_c, z + \delta_c) : s(x) < s^{\hat{\tau}}(x)\}.$$

Using (CB) and the fact that $s^{\hat{\tau}}(x)$ equals $H_x(V(x))$ for $x \in (z - \beta_l - \delta_c, z + \delta_c)$, what we need to show is $\Gamma(R^c) = \Gamma((z - \beta_l - \delta_c, z + \delta_c))$. Assume otherwise, that is, $\Gamma(R) > 0$. Next we prove that there is a deviation from s yielding strictly larger objective.

Since s is feasible, from (FC) we must have

$$\int_{(z-\beta_l-\delta_c,z+\delta_c)} s(x)\, d\Gamma(x) + \int_{[X_l,X_r]\setminus(z-\beta_l-\delta_c,z+\delta_c)} s(x)\, d\Gamma(x) = \tau,$$

similarly for $s^{\hat{\tau}}$ we have

$$\int_{(z-\beta_l-\delta_c,z+\delta_c)} s^{\hat{\tau}}(x)\, d\Gamma(x) + \int_{[X_l,X_r]\setminus(z-\beta_l-\delta_c,z+\delta_c)} s^{\hat{\tau}}(x)\, d\Gamma(x) = \tau.$$

Putting these last two equations together yields

$$0 < \int_R (s^{\hat{\tau}}(x) - s(x))\, d\Gamma(x) = \int_{[X_l,X_r]\setminus(z-\beta_l-\delta_c,z+\delta_c)} (s(x) - s^{\hat{\tau}}(x)) d\Gamma(x), \tag{B-4}$$

implying that the set

$$Q \triangleq \{x \in [X_l, X_r] \setminus (z - \beta_l - \delta_c, z + \delta_c) : s(x) > s^{\hat{\tau}}(x)\},$$

has positive Γ measure, $\Gamma(Q) > 0$. Let $A_l = [X_l, z)$, $A_r = (z, X_r]$ and $A_c = \{z\}$ then one of the following must hold

$$\Gamma(R \cap A_l) > 0, \quad \Gamma(R \cap A_r) > 0 \quad \text{or} \quad \Gamma(R \cap A_c) > 0,$$

and the same holds if we replace R by Q. We use this two sets to create a feasible and profitable deviation.

Consider the case in which $\Gamma(R \cap (A_c \cup A_l)) = 0$ (the treatment of the case $\Gamma(R \cap (A_c \cup A_l)) > 0$ is analogous). In this case s equals $s^{\hat{\tau}}$ $\Gamma-$ a.e in $[z, z + \delta_c)$, and we must have $\Gamma(R \cap A_l) > 0$. Let us analyze two cases.

1. **Case 1**: $\Gamma(Q \cap A_l) > 0$. By the continuity of the measure we can always find $\epsilon > 0$ such that $\Gamma(R_\epsilon \cap A_l), \Gamma(Q_\epsilon \cap A_l) > 0$ where

$$\begin{aligned} R_\epsilon &\triangleq \{x \in (z - \beta_l - \delta_c, z + \delta_c) : s(x) + \epsilon \le s^{\hat{\tau}}(x)\}, \\ Q_\epsilon &\triangleq \{x \in [X_l, X_r] \setminus (z - \beta_l - \delta_c, z + \delta_c) : s(x) - \epsilon \ge s^{\hat{\tau}}(x)\}. \end{aligned} \tag{B-5}$$

   Let $\gamma = \Gamma(R_\epsilon \cap A_l)/\Gamma(Q_\epsilon \cap A_l)$. Consider modifying s as follows, in $R_\epsilon \cap A_l$ we increase it by $\epsilon_1 > 0$ and in $Q_\epsilon \cap A_l$ we decrease it by $\gamma \cdot \epsilon_1$, where $\epsilon_1 \le \epsilon$ is such that $\epsilon \ge \epsilon_1 \cdot \gamma$. Let us denote this modification $\tilde{s}$ which is given by

$$\tilde{s}(x) = \begin{cases} s(x) + \epsilon_1, & x \in R_\epsilon \cap A_l, \\ s(x) - \epsilon_1 \cdot \gamma, & x \in Q_\epsilon \cap A_l, \\ s(x), & \text{otherwise.} \end{cases}$$

Note that since $R_\epsilon \subseteq R_{\epsilon_1}$ and $Q_\epsilon \subseteq Q_{\epsilon_1 \cdot \gamma}$. We have that $0 \leq \tilde{s}$ and $\tilde{s} \leq H_x(V(x))$. So that $\tilde{s}$ satisfies the non-negativity constraint and (CB).

Next we verify that the constraints (FC),(FL) and (FR) are satisfied. We are modifying s only on A_l so (FR) is satisfied. For the other two constraints note that

$$\begin{aligned}
\int_{(R_\epsilon \cap A_l)\cup(Q_\epsilon \cap A_l)} \tilde{s}(x) d\mathit{\Gamma}(x) &= \int_{R_\epsilon \cap A_l} (s(x)+\epsilon_1) d\mathit{\Gamma}(x) + \int_{Q_\epsilon \cap A_l} (s(x) - \gamma \cdot \epsilon_1) d\mathit{\Gamma}(x), \\
&= \int_{(R_\epsilon \cap A_l)\cup(Q_\epsilon \cap A_l)} s(x)\, d\mathit{\Gamma}(x) + \epsilon_1 \cdot \Big(\mathit{\Gamma}(R_\epsilon \cap A_l) - \gamma \cdot \mathit{\Gamma}(Q_\epsilon \cap A_l)\Big) \\
&= \int_{(R_\epsilon \cap A_l)\cup(Q_\epsilon \cap A_l)} s(x)\, d\mathit{\Gamma}(x),
\end{aligned}$$

and, therefore, after modifying s both (FC) and (FL) are satisfied. Finally, we verify the objective improvement. The objective value only changes in $R_\epsilon \cap A_l$ and $Q_\epsilon \cap A_l$ then

$$\begin{aligned}
\int_{(R_\epsilon \cap A_l)\cup(Q_\epsilon \cap A_l)} V(x) \cdot \tilde{s}(x) d\mathit{\Gamma}(x) = \int_{(R_\epsilon \cap A_l)\cup(Q_\epsilon \cap A_l)} V(x) \cdot s(x) d\mathit{\Gamma}(x) + \epsilon_1 \int_{R_\epsilon \cap A_l} V(x) d\mathit{\Gamma}(x) \\
- \epsilon_1 \gamma \int_{Q_\epsilon \cap A_l} V(x) d\mathit{\Gamma}(x),
\end{aligned}$$

for the second term in on the RHS of the previos equality we have

$$\int_{R_\epsilon \cap A_l} V(x) d\mathit{\Gamma}(x) > V(z_l) \cdot \mathit{\Gamma}(R_\epsilon \cap A_l) = V(z_l) \cdot \gamma \cdot \mathit{\Gamma}(Q_\epsilon \cap A_l) \geq \gamma \int_{Q_\epsilon \cap A_l} V(x) d\mathit{\Gamma}(x).$$

Thus

$$\int_{(R_\epsilon \cap A_l)\cup(Q_\epsilon \cap A_l)} V(x) \cdot \tilde{s}(x) d\mathit{\Gamma}(x) > \int_{(R_\epsilon \cap A_l)\cup(Q_\epsilon \cap A_l)} V(x) \cdot s(x) d\mathit{\Gamma}(x).$$

This show that $\tilde{s}$ yields an strict objective improvement.

2. **Case 2**: $\mathit{\Gamma}(Q \cap A_l) = 0$. In this case $s \leq s^{\hat{\tau}}$ $\mathit{\Gamma}-$ a.e in $[X_l, z - \delta_c - \beta_l]$. Therefore, we must have that $\mathit{\Gamma}(Q \cap A_r) > 0$. Two more cases.

- $\mathit{\Gamma}(R \cap (z - \delta_c, z)) = 0$: First we show that

$$s_r = \frac{\tau_r - S^H((z, z+\delta_c))}{\mathit{\Gamma}_r}, \tag{B-6}$$

  and then, using this, we construct an objective improvement.

  Our current assumption implies $s = s^{\hat{\tau}}$ $\mathit{\Gamma}-$ a.e in $(z - \delta_c, z)$, and because $\mathit{\Gamma}(R \cap A_l) > 0$ we have $\mathit{\Gamma}(R \cap (z - \delta_c - \beta_l, z - \delta_c]) > 0$. Since this last set has positive $\mathit{\Gamma}$ measure it holds that $\beta_l > 0$. Lemma B-2 then implies that $\delta_c = \delta_r$. Definition of δ_r delivers

$$S^H((z, z+\delta_c)) \leq \tau_r \leq S^H((z, z+\delta_c]) = S^H((z, z+\delta_c)) + H_r \cdot \mathit{\Gamma}_r. \tag{B-7}$$

If $\Gamma_r = 0$ then $\tau_r = S^H((z, z+\delta_c))$ and also, from the feasibility of S,

$$\tau_r \geq S(z, X_r] = S(z, z+\delta_c) + S[z+\delta_c, X_r] = S^H((z, z+\delta_c)) + S[z+\delta_c, X_r] = \tau_r + S[z+\delta_c, X_r],$$

thus $S[z+\delta_c, X_r] = 0$ and, therefore, $\Lambda(Q \cap A_r) = 0$ which is not possible. So assume $\Gamma_r > 0$. Then from the definition of s_r and Eq. (B-7)

$$s_r = \min\Big\{\frac{\tau_r - S^H((z, z+\delta_c))}{\Gamma_r}, \frac{\tau - S^H((z-\delta_c, z+\delta_c))}{\Gamma_r}\Big\}.$$

So to verify Eq.(B-6) it is enough to show

$$\frac{\tau_r - S^H((z, z+\delta_c))}{\Gamma_r} \leq \frac{\tau - S^H((z-\delta_c, z+\delta_c))}{\Gamma_r}.$$

Suppose otherwise, then

$$S^H([z-\delta_c, z+\delta_c)) + s_r \cdot \Gamma_r \geq S^H((z-\delta_c, z+\delta_c)) + (\frac{\tau - S^H((z-\delta_c, z+\delta_c))}{\Gamma_r}) \cdot \Gamma_r = \tau,$$

and thus $\beta_l = 0$, which is not possible.

Next we construct the objective improvement. We have

$$S^H(z, z+\delta_r) + s_r \cdot \Gamma_r = \tau_r \geq S((z, z+\delta_c)) + S[z+\delta_c, X_r] = S^H((z, z+\delta_c)) + S[z+\delta_c, X_r],$$

hence

$$s_r \geq s(z+\delta_c) + \frac{S((z+\delta_c, X_r] \cap Q)}{\Gamma_r}.$$

This implies that $s_r > s(z+\delta_c)$ and $\Gamma(Q \cap (z+\delta_c, X_r]) > 0$. We can create an improvement by increasing $s(z+\delta_c)$ and reducing s in $Q \cap (z+\delta_c, X_r]$. There exists $\epsilon > 0$ such that $s_r \geq \epsilon + s(z+\delta_c)$ and $\Gamma(Q_\epsilon \cap (z+\delta_c, X_r]) > 0$, where Q_ϵ is as defined in Eq. (B-5). Let $\gamma = \Gamma_r / \Gamma(Q_\epsilon \cap (z+\delta_c, X_r])$ and consider $\epsilon_1 \in (0, \epsilon)$ such that $\epsilon \geq \epsilon_1 \cdot \gamma_1$. Define

$$\tilde{s}(x) = \begin{cases} s(x) + \epsilon_1, & x = z+\delta_c, \\ s(x) - \epsilon_1 \cdot \gamma, & x \in Q_\epsilon \cap (z+\delta_c, X_r], \\ s(x), & \text{otherwise.} \end{cases}$$

Following the same steps as in case 1 we can show that $\tilde{s}$ is a feasible strict improvement over s.

- $\Gamma(R\cap(z-\delta_c,z))>0$: Recall that in $R\cap(z-\delta_c,z)$ we have $r<s^{\hat{\tau}}$ and, thus,

$$\begin{aligned}
\tau_l &\geq S^{\hat{\tau}}((z,X_l]) \\
&= \int_{R\cap(z-\delta_c,z)} s^{\hat{\tau}}(x)\,d\Gamma(x) + \int_{(z,X_l]\setminus(R\cap(z-\delta_c,z))} s^{\hat{\tau}}(x)\,d\Gamma(x) \\
&> \int_{R\cap(z-\delta_c,z)} s(x)\,d\Gamma(x) + \int_{(z,X_l]\setminus(R\cap(z-\delta_c,z))} s^{\hat{\tau}}(x)\,d\Gamma(x) \\
&\geq \int_{R\cap(z-\delta_c,z)} s(x)\,d\Gamma(x) + \int_{(z,X_l]\setminus(R\cap(z-\delta_c,z))} s(x)\,d\Gamma(x) \\
&= S((z,X_l]).
\end{aligned}$$

  That is, $\tau_l>S((z,X_l])$. So we can increase s in $R\cap(z-\delta_c,z)$ and decrease s in $Q\cap A_r$ in a feasible manner and obtain an objective improvement. Choose $\epsilon>0$ such that $\Gamma(R_\epsilon\cap(z-\delta_c,z)),\Gamma(Q_\epsilon\cap A_r)>0$. Let $\gamma=\Gamma(R\cap(z-\delta_c,z))/\Gamma(Q\cap A_r)$ and

$$\tilde{s}(x)=\begin{cases} s(x)+\epsilon_1, & x\in R_\epsilon\cap(z-\delta_c,z),\\ s(x)-\epsilon_1\cdot\gamma, & x\in Q_\epsilon\cap A_r,\\ s(x), & \text{otherwise,}\end{cases}$$

  where $\epsilon_1\in[0,\epsilon]$ is such that $\epsilon\geq\epsilon_1\cdot\gamma$ and

$$\tau_l>S((z,X_l])=\int_{R_\epsilon\cap(z-\delta_c,z)}(s(x)+\epsilon_1)d\Gamma(x)+\int_{(z,X_l]\setminus(R_\epsilon\cap(z-\delta_c,z))} s(x)\,d\Gamma(x).$$

  That is, ϵ_1 is chosen so that $\tilde{s}$ satisfies (FL). Also note that (FR) is also satisfied because in A_r we have $\tilde{s}\leq r$. Moreover, the choice of γ ensures that $\tilde{s}$ verifies (FC). To see why this $\tilde{s}$ yields an objective improvement, it is enough to note that

$$\begin{aligned}
\int_{R_\epsilon\cap(z-\delta_c,z)} V(x)d\Gamma(x) &> V(z-\delta_c)\cdot\Gamma(R_\epsilon\cap(z-\delta_c,z)) \\
&= V(z-\delta_c)\cdot\gamma\cdot\Gamma(Q_\epsilon\cap A_r) \\
&\geq \gamma\cdot\int_{Q_\epsilon\cap A_r} V(x)d\Gamma(x),
\end{aligned}$$

  where the last inequality comes from $V(z-\delta_c)$ being larger than $V(x)$ for any $x\in Q_\epsilon\cap A_r$. This shows that $\tilde{s}$ yields an strict objective improvement.

We proved in $(z-\beta_l-\delta_c,z+\delta_c)$ we must have that any optimal solution s equals $s^{\hat{\tau}}$ $\Gamma-$ a.e; otherwise, we can find an strict objective improvement. Next we argue that the same result holds in $[X_l,z-\beta_l-\delta_c)\cap(z+\delta_c,X_r]$. To see this consider Eq. (B-4). if s does not equal $s^{\hat{\tau}}$ in $[X_l,z-\beta_l-\delta_c)\cap(z+\delta_c,X_r]$ $\Gamma-$ a.e, then Eq. (B-4) ensures that s does not equal $s^{\hat{\tau}}$ $\Gamma-$ a.e

$(z-\beta_l-\delta_c, z+\delta_c)$. Thus, we can use the previous analysis to find an strict objective improvement. In conclusion, s equals $s^{\hat{\tau}}$ $\Gamma-$ a.e in $[X_l, X_r] \setminus \{z-\beta_l-\delta_c, z+\delta_c\}$.

Finally, we show that the previous conclusion also holds in $\{z-\beta_l-\delta_c, z+\delta_c\}$. Let s be any optimal solution, since both s and $s^{\hat{\tau}}$ verify (FC) we must have

$$s_l\Gamma_l + s_r\Gamma_r = s(z-\beta_l-\delta_c)\Gamma_l + s(z+\delta_c)\Gamma_r.$$

Furthermore, from the other constraints we can conclude that $s(z+\delta_c) \leq s_r$. Hence, if we denote by $L(r, z-\beta_l-\delta_c, z+\delta_c)$ the objective function under s in the set $\{z-\beta_l-\delta_c, z+\delta_c\}$ we have that

$$\begin{aligned}
L(r, z-\beta_l-\delta_c, z+\delta_c) &= V(z-\beta_l-\delta_c)\cdot s(z-\beta_l-\delta_c)\Gamma_l + V(z+\delta_c)\cdot s(z+\delta_c)\Gamma_r \\
&= V(z-\beta_l-\delta_c)\cdot \Big((s_l\Gamma_l + s_r\Gamma_r) - s(z+\delta_c)\Gamma_r\Big) + V(z+\delta_c)\cdot s(z+\delta_c)\Gamma_r \\
&= V(z-\beta_l-\delta_c)\cdot \Big(s_l\Gamma_l + s_r\Gamma_r\Big) + \Big(V(z+\delta_c) - V(z-\beta_l-\delta_c)\Big)\cdot s(z+\delta_c)\Gamma_r \\
&\leq V(z-\beta_l-\delta_c)\cdot \Big(s_l\Gamma_l + s_r\Gamma_r\Big) + \Big(V(z+\delta_c) - V(z-\beta_l-\delta_c)\Big)\cdot s_r\Gamma_r \\
&= V(z-\beta_l-\delta_c)\cdot s_l\Gamma_l + V(z+\delta_c)\cdot s_r\Gamma_r \\
&= L(s^{\hat{\tau}}, z-\beta_l-\delta_c, z+\delta_c).
\end{aligned}$$

This means that the best possible way to pick $s(z-\beta_l-\delta_c)$ and $s(z+\delta_c)$ is to set them equal to s_l and s_r respectively and, therefore, at any optimal solution s we must have r equals $s^{\hat{\tau}}$ $\Gamma-$ a.e in $\{z-\beta_l-\delta_c, z+\delta_c\}$. This concludes part 3 of the proof.

Part 5: Implementation. We construct a price-equilibrium pair $(\tilde{p}, \tilde{\tau})$ in $A(z)$ with $\tilde{\tau} \in \mathcal{F}_{A(z)}(\tilde{\mu})$ and

$$\tilde{\mu}(\mathcal{B}) \triangleq \tau((\mathcal{B} \cap A(z)) \times A(z)), \quad \mathcal{B} \subseteq \mathcal{C} \text{ measurable.}$$

Define $\tilde{p} : A(z) \to [0, \overline{V}]$ by

$$\tilde{p}(x) = \begin{cases} \rho_x^{loc}(s^{\hat{\tau}}(x)) & \text{if } x \in (z_l, z_r); \\ p_i & \text{if } x = z_i,\ i \in \{l, r\}; \\ \overline{V} & \text{otherwise,} \end{cases}$$

where p_i is such that $U(z_i, p_i, s_i) = V(z_i|p,\tau)\cdot\mathbf{1}_{\{\lambda(z_i)>0\}}$ for $i \in \{l, r\}$. By the way s_i was constructed, the value p_i is always well defined.

Next, we define the flow $\tilde{\tau}$. We do this in three steps. Firstly, we define a measure that transports flow from $A(z)$ to z. Secondly, we construct a measure that sends flow from $[X_l, z)$ to $[X_l, z)$. This last steps is analogous for $(z, X_r]$. Finally, we put together the measures constructed in the first two steps to create a measure in $A(z)$.

- **Step 1:** For any measurable set $\mathcal{B} \subseteq A(z)$ define the measures

$$\mu_l(\mathcal{B}) \triangleq \tau(\mathcal{B} \times [X_l, z)) \quad \text{and} \quad \mu_r(\mathcal{B}) \triangleq \tau(\mathcal{B} \times (z, X_r]),$$

  and the quantities

$$\varDelta_l \triangleq \mu_l([X_l, z)) - S^{\hat{\tau}}([X_l, z)) \quad \text{and} \quad \varDelta_r \triangleq \mu_r((z, X_r]) - S^{\hat{\tau}}((z, X_r]),$$

  note that because of (FL) and (FR), $\varDelta_l, \varDelta_r \geq 0$. Further define

$$q_l \triangleq z - \inf\{\delta \geq 0 : \mu_l([z-\delta, z)) \geq \varDelta_l\} \quad \text{and} \quad q_r \triangleq z + \inf\{\delta \geq 0 : \mu_r((z, z+\delta]) \geq \varDelta_r\}.$$

  For any set $\mathcal{B} \subseteq A(z)$ we define the left and right mass going to the center by the measures

$$\mu_l^c(\mathcal{B}) \triangleq \mu_l(\mathcal{B} \cap (q_l, z)) + \mathbf{1}_{\{q_l \in \mathcal{B}\}} \cdot (\varDelta_l - \mu_l(q_l, z)),$$
$$\mu_r^c(\mathcal{B}) \triangleq \mu_r(\mathcal{B} \cap (z, q_r)) + \mathbf{1}_{\{q_r \in \mathcal{B}\}} \cdot (\varDelta_r - \mu_r(z, q_r)),$$

  observe that by the definition of q_l and q_r, the atoms above have non-negative mass. Let $H_z \triangleq A(z) \times \{z\}$. For any measurable set $\mathcal{L} \subseteq A(z) \times A(z)$, the measure that sends flow to the origin is defined by

$$\tau^c(\mathcal{L}) \triangleq \mu_l^c(\pi_1(\mathcal{L} \cap H_z)) + \tau(\mathcal{L} \cap H_z) + \mu_r^c(\pi_1(\mathcal{L} \cap H_z)).$$

  From Lemma B-3 (which we state and prove after the present proof) we verify that τ^c is indeed a non-negative measure.

- **Step 2:** As mentioned earlier we show how to create the flow on the left side of z. We create a measure, τ^l in $[X_l, z) \times [X_l, z)$, that transports the mass in $[X_l, z)$ that we are not sending to the origin to the mass created by our knapsack solution in $[X_l, z)$. The construction of an analogous measure τ^r in $(z, X_r]$ is identical to the one of τ^l and, thus, omitted.

  For any set $\mathcal{B} \subseteq [X_l, z)$, define the measure

$$\mu_l^l(\mathcal{B}) \triangleq \mu_l(\mathcal{B} \cap [X_l, q_l)) + \mathbf{1}_{\{q_l \in \mathcal{B}\}} \cdot (\mu_l([q_l, z)) - \varDelta_l)$$

  We denote by S^l de measure $S^{\hat{\tau}}|_{[X_l, z)}$. Both measures μ_l^l and S^l satisfy the following property

$$\mu_l^l([a, z)) \leq S^l([b, z)), \quad \forall a, b \in [X_l, z),\ b \leq a. \tag{B-8}$$

To see why this is true note that

$$
\begin{aligned}
\mu_l^l([a,z)) &= \mu_l([a,z)\cap[X_l,q_l)) + \mathbf{1}_{\{q_l\in[a,z)\}}\cdot(\mu_l([q_l,z))-\Delta_l)\\
&= \mathbf{1}_{\{q_l\in[a,z)\}}\cdot(\mu_l([a,q_l))+\mu_l([q_l,z))-\Delta_l)\\
&= \mathbf{1}_{\{q_l\in[a,z)\}}\cdot(\mu_l([a,z))-\Delta_l)\\
&\le \mathbf{1}_{\{q_l\in[a,z)\}}\cdot(\mu_l([a,z))+\tau([X_l,a)\times[b,z))-\Delta_l)\\
&= \mathbf{1}_{\{q_l\in[a,z)\}}\cdot(\tau([a,z)\times[X_l,z))+\tau([X_l,a)\times[b,z))-\Delta_l)\\
&\overset{(a)}{=} \mathbf{1}_{\{q_l\in[a,z)\}}\cdot(\tau([a,z)\times[b,z))+\tau([X_l,a)\times[b,z))-\Delta_l)\\
&= \mathbf{1}_{\{q_l\in[a,z)\}}\cdot(\tau_2([b,z))-\Delta_l),
\end{aligned}
$$

where (a) follows from the fact that $b\le a$ and, therefore, $\tau([a,z)\times[X_l,b))$ equals zero. Next, if $b\in(z_l,z)$ then from the congestion bound we must have that $\tau_2([b,z))\le S^{\hat\tau}([b,z))$ which, together with the fact that $\Delta_l\ge 0$, yields Eq. (B-8). If $b\in[X_l,z_l]$ then

$$\tau_2([b,z))-\Delta_l = S^{\hat\tau}([b,z))+S^{\hat\tau}([X_l,b))-\tau_2([X_l,b)) = S^{\hat\tau}([b,z))-\tau_2([X_l,b))\le S^{\hat\tau}([b,z)),$$

in either case we conclude that Eq. (B-8) holds.

Let $\hat t\triangleq S^l([X_l,z)$, note that $\mu_l^l([X_l,z))$ also equals $\hat t$. For any measure ν defined in $[X_l,z)$ we define its cumulative function and pseudo-inverse by

$$F_\nu(y)\triangleq\nu([y,z)),\quad \forall\, y\le z \quad\text{and}\quad F_\nu^{[-1]}(t)\triangleq\sup\{y\le z: F_\nu(y)\ge t\},\quad \forall t\in[0,\hat t].$$

It is important to note that the knowledge of F_μ is enough to characterize the measure ν, see e.g, Santambrogio (2015). Similarly, the knowledge of the value a measure in the product space assigns to the sets of the type $[y_1,z)\times[y_2,z)$ is enough to characterize that measure.

We want to transport μ_l^l in to S^l. Before we provide the formal definition we introduce the push-forward notation (very typical in the literature of optimal transport). For a map $T:[X_l,z)\to[X_l,z)$ and a measure μ in $[X_l,z)$, we define the push-forward measure $T_\#\mu$, by

$$T_\#\mu(E)\triangleq\mu(T^{-1}(E)),\quad \text{for all } E\subseteq[X_l,z).$$

Let m be the Lebesgue measure in $[0,\hat t]$, then we define the transport $\tau^l(\mathcal{L})$ by

$$\tau^l(\mathcal{L})\triangleq(F_{\mu_l^l}^{[-1]},F_{S^l}^{[-1]})_\# m(\mathcal{L}),\quad \text{for all } \mathcal{L}\subseteq[X_l,z)\times[X_l,z).$$

Next we argue that $\tau_1^l=\mu_l^l$ and $\tau_2^l=S^l$. Recall that is enough to show this for the cumulative distribution function. Consider any $y \in [X_l, z]$ and the set $[y, z)$ then

$$\begin{aligned}\tau_1^l([y,z)) &= \tau^l([y,z) \times [X_l, z)) \\ &= m\Big(t \in [0,\hat{t}] : F^{[-1]}_{\mu_l^l}(t) \in [y,z)\Big) \\ &= m\Big(t \in [0,\hat{t}] : F_{\mu_l^l}(z) < t \leq F_{\mu_l^l}(y)\Big) \\ &= F_{\mu_l^l}(y),\end{aligned}$$

and the same argument holds for τ_2^l and S^l. Furthermore, it's not difficult to see that for $y_1, y_2 \in [X_l, z]$ we have

$$\tau^l([y_1,z)\times[y_2,z)) = m\Big(t \in [0,\hat{t}] : F_{\mu_l^l}(z) < t \leq F_{\mu_l^l}(y_1), F_{S^l}(z) < t \leq F_{S^l}(y_2)\Big) = F_{\mu_l^l}(y_1) \wedge F_{S^l}(y_2). \tag{B-9}$$

- **Step 3:** Now we are ready to define the measure $\tilde{\tau}$ in $A(z)$. For any measurable set $\mathcal{L} \subseteq A(z) \times A(z)$, $\tilde{\tau}$ is defined by

$$\tilde{\tau}(\mathcal{L}) \triangleq \tau^l(\mathcal{L} \cap [X_l, z) \times [X_l, z)) + \tau^c(\mathcal{L}) + \tau^r(\mathcal{L} \cap (z, X_r] \times (z, X_r]).$$

Next, we show that $\tilde{\tau} \in \mathcal{F}_{A(z)}(\tilde{\mu})$ is an equilibrium in $A(z)$ for the prices $\tilde{p}$ defined above. In order to do so we first show that $\tilde{\tau} \in \mathcal{F}_{A(z)}(\tilde{\mu})$. Second, we compute the supply density of $\tilde{\tau}$ and corroborate they coincide with $s^{\hat{\tau}}$. Third, we compute $V_{A(z)}(\cdot|\, \tilde{p}, \tilde{\tau})$ and verify is coincides with $V(\cdot|\, p, \tau)$ in $A(z)$. Finally, we check the equilibrium condition.

- $\tilde{\tau} \in \mathcal{F}_{A(z)}(\tilde{\mu})$: Clearly $\tilde{\tau}$ is a non-negative measure in $A(z) \times A(z)$ because is the sum of non-negative measures. Now we check that $\tilde{\tau}_1 = \tilde{\mu}$. Consider a measurable set $\mathcal{B} \subseteq A(z)$ then

$$\begin{aligned}\tilde{\tau}_1(\mathcal{B}) &= \tilde{\tau}(\mathcal{B} \times A(z)) \\ &= \tau^l((\mathcal{B} \cap [X_l, z)) \times [X_l, z)) + \tau^c(\mathcal{B} \times A(z)) + \tau^r((\mathcal{B} \cap (z, X_r]) \times (z, X_r]) \\ &= \mu_l^l(\mathcal{B} \cap [X_l, z)) + \tau^c(\mathcal{B} \times A(z)) + \mu_r^s(\mathcal{B} \cap (z, X_r]) \\ &= \mu_l^l(\mathcal{B} \cap [X_l, z)) + \mu_l^c(\mathcal{B}) + \tau(\mathcal{B} \times \{z\}) + \mu_r^c(\mathcal{B}) + \mu_r^s(\mathcal{B} \cap (z, X_r]) \\ &= \mu_l(\mathcal{B} \cap [X_l, q_l)) + \mathbf{1}_{\{q_l \in \mathcal{B} \cap [X_l, z)\}} \cdot (\mu_l([q_l, z)) - \Delta_l) + \mu_l^c(\mathcal{B}) + \tau(\mathcal{B} \times \{z\}) + \mu_r^c(\mathcal{B}) \\ &+ \mu_r(\mathcal{B} \cap (q_r, X_r]) + \mathbf{1}_{\{q_r \in \mathcal{B} \cap (z, X_r]\}} \cdot (\mu_r((z, q_r]) - \Delta_r) \\ &= \mu_l(\mathcal{B} \cap [X_l, z)) - \mu_l(\mathcal{B} \cap \{q_l\}) + \mathbf{1}_{\{q_l \in \mathcal{B}\}} \cdot \mu_l(\{q_l\}) \\ &+ \tau(\mathcal{B} \times \{z\}) + \\ &+ \mu_r(\mathcal{B} \cap (z, X_r]) - \mu_r(\mathcal{B} \cap \{q_r\}) + \mathbf{1}_{\{q_r \in \mathcal{B}\}} \cdot \mu_r(\{q_r\}) \\ &= \tau(\mathcal{B} \times A(z)) \\ &= \tilde{\mu}(\mathcal{B}),\end{aligned}$$

and from the definition of $\tilde{\mu}$ is clear that $\tilde{\tau}_1 \ll \Gamma$. For the second marginal of $\tilde{\tau}$ we have

$$
\begin{aligned}
\tilde{\tau}_2(\mathcal{B}) &= \tilde{\tau}(A(z) \times \mathcal{B}) \\
&= S^l(\mathcal{B}) + \tau^c(A(z) \times \mathcal{B}) + S^r(\mathcal{B}) \\
&= S^l(\mathcal{B}) + \mathbf{1}_{\{z \in \mathcal{B}\}} \cdot \Big(\mu_l^c(A(z)) + \tau(A(z) \times \{z\}) + \mu_r^c(A(z))\Big) + S^r(\mathcal{B}) \\
&= S^l(\mathcal{B}) + \mathbf{1}_{\{z \in \mathcal{B}\}} \cdot \Big(\Delta_l + \tau(A(z) \times \{z\}) + \Delta_r\Big) + S^r(\mathcal{B}) \\
&= S^l(\mathcal{B}) + S^r(\mathcal{B}) \\
&+ \mathbf{1}_{\{z \in \mathcal{B}\}} \cdot \Big(\mu_l([X_l, z)) - S^{\hat{\tau}}([X_l, z)) + \tau(A(z) \times \{z\}) + \mu_r((z, X_r]) - S^{\hat{\tau}}((z, X_r])\Big) \\
&= S^{\hat{\tau}}(\mathcal{B}) - S^{\hat{\tau}}(\mathcal{B} \cap \{z\}) \\
&+ \mathbf{1}_{\{z \in \mathcal{B}\}} \cdot \Big(\mu_l([X_l, z)) - S^{\hat{\tau}}([X_l, z)) + \tau(A(z) \times \{z\}) + \mu_r((z, X_r]) - S^{\hat{\tau}}((z, X_r])\Big) \\
&= S^{\hat{\tau}}(\mathcal{B}) + \mathbf{1}_{\{z \in \mathcal{B}\}} \cdot \Big(\mu_l([X_l, z)) + \tau(A(z) \times \{z\}) + \mu_r((z, X_r]) - S^{\hat{\tau}}([X_l, X_r])\Big) \\
&= S^{\hat{\tau}}(\mathcal{B})
\end{aligned}
$$

Since $S^{\hat{\tau}}$ is such that $S^{\hat{\tau}} \ll \Gamma$, we conclude that $\tilde{\tau} \in \mathcal{F}_{A(z)}(\tilde{\mu})$.

- **Supply density:** We just proved that for any measurable set $\mathcal{B} \subseteq A(z)$ $\tilde{\tau}_2(\mathcal{B}) = S^{\hat{\tau}}(\mathcal{B})$. This in turn, implies that
$$\frac{d\tilde{\tau}_2}{d\Gamma}(x) = s^{\hat{\tau}}(x), \quad \Gamma - a.e.\ x \text{ in } A(z).$$

- **Equilibrium utilities:** We show that $V_{A(z)}(x|\tilde{p}, \tilde{\tau})$ equals $V(z|p,\tau) - |z-x|$ for all $x \in [X_l, X_r]$. First, observe that $\Gamma - a.e.\ y$ in $A(z)$
$$U(y, \tilde{p}(y), s^{\tilde{\tau}}(y)) = \begin{cases} (V(z|\, p, \tau) - |z-y|) \cdot \mathbf{1}_{\{\lambda(y) > 0\}} & \text{if } y \in [z_l, z_r], \\ 0 & \text{if } y \in A(z) \setminus [z_l, z_r]. \end{cases}$$
Second, for any $x \in [X_l, X_r]$ we argue that $V(z|\, p, \tau) - |z - x| \geq V_{A(z)}(x|\, \tilde{p}, \tilde{\tau})$. Suppose
$$\Gamma(y \in A(z) : U(y, \tilde{p}(y), s^{\tilde{\tau}}(y)) - |y - x| > V(x|\, p, \tau)) > 0,$$
then, since by Lemma A-1 $V(y|p,\tau)$ is non-negative $\Gamma - a.e$ and $V(y|p,\tau)$ equals $V(z|p,\tau) - |z-y|$ for any $y \in A(z)$, it must be true that $V(y|\, p, \tau)$ is larger or equal than $U(y, \tilde{p}(y), s^{\tilde{\tau}}(y))$ $\Gamma - a.e.$ Thus
$$\Gamma(y \in A(z) : V(y|\, p, \tau) - |y - x| > V(x|\, p, \tau)) > 0,$$
but this contradicts the Lipschitz property of $V(\cdot|\, p, \tau)$.

Third, we show that the upper bound we just proved is tight, that is, for all $\epsilon > 0$
$$\Gamma(y \in A(z) : U(y, \tilde{p}(y), s^{\tilde{\tau}}(y)) - |y - x| > V(x|\, p, \tau) - \epsilon) > 0. \tag{B-10}$$

Fix $\epsilon > 0$, there are two cases. Suppose WLOG that $z \neq z_r$ then since $\Gamma([z, z+\delta) \cap \{\lambda > 0\}) > 0$ for any $\delta > 0$, we can take $\delta \in (0, \min\{\epsilon/2, z_r - z\})$ which yields

$$\begin{aligned}
0 &< \Gamma(y \in [z, z+\delta) : \lambda(y) > 0) \\
&= \Gamma(y \in [z, z+\delta) : \lambda(y) > 0,\ 2\delta + |z-x| \geq |y-x| + |y-z|) \\
&\leq \Gamma(y \in [z, z+\delta) : \lambda(y) > 0,\ \epsilon + |z-x| > |y-x| + |y-z|) \\
&\leq \Gamma(y \in [z_l, z_r] : \lambda(y) > 0,\ \epsilon + |z-x| > |y-x| + |y-z|) \\
&= \Gamma(y \in [z_l, z_r] : \lambda(y) > 0,\ U(y, \tilde{p}(y), s^{\tilde{\tau}}(y)) - |y-x| > V(x) - \epsilon) \\
&\leq \Gamma(y \in A(z) : U(y, \tilde{p}(y), s^{\tilde{\tau}}(y)) - |y-x| > V(x) - \epsilon),
\end{aligned}$$

this shows that Eq. (B-10) holds. For the other case suppose that both z_l and z_r equal z. Then, we must have

$$0 < \tau([X_l, X_r] \times [X_l, X_r]) = \int_{[z_l, z_r]} s^{\hat{\tau}}(x)\, d\Gamma(x) = s^{\hat{\tau}}(z) \cdot \Gamma(\{z\}).$$

Thus both $s^{\hat{\tau}}(z)$ and $\Gamma(\{z\})$ are strictly positive. If $\lambda(z) > 0$ then the same series of inequalities that we used for the previous case applies to this case, and so the desired Eq. (B-10) holds. If $\lambda(z) = 0$ then since by feasibility we have

$$0 < s^{\hat{\tau}}(z) \leq H_z(V(z|\, p, \tau)),$$

it must be the case that z belongs to $\{X_l^{\text{supp}}, X_r^{\text{supp}}\}$. WLOG suppose that $z = X_r^{\text{supp}}$ then by the previous inequality we have that $0 < s^{\hat{\tau}}(X_r^{\text{supp}}) \leq \frac{d\mu}{d\Gamma}(X_r^{\text{supp}})$. In turn, this implies that $\mu(\{X_r^{\text{supp}}\}) > 0$. This means that z has and initial mass of supply. Since z is a sink location, it does not belong to the indifference region of any other location and, therefore, by Lemma A-6 it does not send flow to any other location. Hence, $\tau_2(\{z\}) > 0$ and by Lemma A-2 we conclude that $U(y, p(y), s^{\tau}(y)) = V(z|\, p, \tau)$. Since, $\lambda(z) = 0$ this implies that $V(z|\, p, \tau) = 0$. To conclude, note that

$$\begin{aligned}
\Gamma(y \in A(z) : U(y, \tilde{p}(y), s^{\tilde{\tau}}(y)) - |y-x| > V(x|\, p, \tau) - \epsilon) &\geq \Gamma(y \in \{z\} : -|y-x| > -|z-x| - \epsilon) \\
&= \Gamma(\{z\}) \\
&> 0,
\end{aligned}$$

hence Eq. (B-10) holds.

- **Equilibrium condition:** Consider the equilibrium set

$$\tilde{\mathcal{E}} \triangleq \Big\{(x, y) \in A(z) \times A(z) : U(y, \tilde{p}(y), s^{\tilde{\tau}}(y)) - |y-x| = V_{A(z)}(x|\, \tilde{p}, \tilde{\tau})\Big\},$$

we need to verify that $\tilde{\tau}(\tilde{\mathcal{E}})$ equals $\tilde{\mu}(A(z))$. First, for $\tilde{\tau}(\tilde{\mathcal{E}})$ we have

$$\begin{aligned}\tilde{\tau}(\tilde{\mathcal{E}}) &= \tilde{\tau}\Big(\Big\{(x,y) \in A(z) \times [z_l, z_r] : \lambda(y) > 0,\ |z-y| + |y-x| = |z-x|\Big\}\Big)\\ &\quad + \tilde{\tau}\Big(\Big\{(x,y) \in A(z) \times \{z_l, z_r\} : \lambda(y) = 0,\ U(y, \tilde{p}(y), s^{\tilde{\tau}}(y)) - |y-x| = V_{A(z)}(x|\,\tilde{p}, \tilde{\tau})\Big\}\Big)\\ &= \tilde{\tau}\Big(\Big\{(x,y) \in A(z) \times [z_l, z_r] : \lambda(y) > 0,\ |z-y| + |y-x| = |z-x|\Big\}\Big)\\ &\quad + \sum_{i \in \{l,r\}} \tilde{\tau}\Big(\Big\{(x,y) \in A(z) \times \{z_i\} : -|z_i - x| = V_{A(z)}(x|\,\tilde{p}, \tilde{\tau})\Big\}\Big) \cdot \mathbf{1}_{\{\lambda(z_i)=0\}},\end{aligned}$$

denote by Z_i the i term in the summation above. Then, for $i = l$ since $\tilde{\tau}((z_l, X_r] \times \{z_l\}) = 0$ and the property we showed for the equilibrium utilities we have

$$\begin{aligned}Z_l &= \tilde{\tau}\Big(\Big\{(x,y) \in [X_l, z_l] \times \{z_l\} : -|z_l - x| = V_{A(z)}(x|\,\tilde{p}, \tilde{\tau})\Big\}\Big) \cdot \mathbf{1}_{\{\lambda(z_l)=0\}}\\ &= \tilde{\tau}\Big(\Big\{(x,y) \in [X_l, z_l] \times \{z_l\} : -|z_l - x| = V(z_l) - |z_l - x|\Big\}\Big) \cdot \mathbf{1}_{\{\lambda(z_l)=0\}}\\ &= \tilde{\tau}\Big(\Big\{(x,y) \in [X_l, z_l] \times \{z_l\} : 0 = V(z_l)\Big\}\Big) \cdot \mathbf{1}_{\{\lambda(z_l)=0\}}\\ &= s_l \cdot \mathit{\Gamma}_l \cdot \mathbf{1}_{\{V(z_l)=0, \lambda(z_l)=0\}}.\end{aligned}$$

Similarly, we can show that Z_r equals $s_r \cdot \mathit{\Gamma}_r \cdot \mathbf{1}_{\{V(z_r)=0, \lambda(z_r)=0\}}$. Consider the sets

$$\begin{aligned}\tilde{\mathcal{E}}_c &\triangleq A(z) \times \{y \in \{z\} : \lambda(y) > 0\},\\ \tilde{\mathcal{E}}_l &\triangleq \Big\{(x,y) \in [X_l, z) \times [z_l, z) : \lambda(y) > 0,\ x \le y\Big\},\\ \tilde{\mathcal{E}}_r &\triangleq \Big\{(x,y) \in (z, X_r] \times (z, z_r] : \lambda(y) > 0,\ y \le x\Big\}.\end{aligned}$$

Then,

$$\tilde{\tau}(\tilde{\mathcal{E}}) = \tilde{\tau}(\tilde{\mathcal{E}}_c) + \sum_{i \in \{l,r\}} \tilde{\tau}(\tilde{\mathcal{E}}_i) + s_i \cdot \mathit{\Gamma}_i \cdot \mathbf{1}_{\{V(z_i)=0, \lambda(z_i)=0\}}.$$

For the first term we have

$$\tilde{\tau}(\tilde{\mathcal{E}}_c) = \tilde{\tau}_2(\{y \in \{z\} : \lambda(y) > 0\}) = \tilde{\tau}_2(\{z\}) \cdot \mathbf{1}_{\{\lambda(z)>0\}}$$

Next we show that $\tilde{\tau}(\tilde{\mathcal{E}}_l)$ equals $\tilde{\tau}_2([z_l, z) \cap \{y : \lambda(y) > 0\})$, the same argument applies to $\tilde{\mathcal{E}}_r$. Indeed, observe that

$$[X_l, z) \times ([z_l, z) \cap \{y : \lambda(y) > 0\}) = \tilde{\mathcal{E}}_l \cup \Big\{(x,y) \in [z_l, z) \times [z_l, z) : \lambda(y) > 0,\ x > y\Big\},$$

and that $\tilde{\tau}([X_l, z) \times ([z_l, z) \cap \{y : \lambda(y) > 0\}))$ equals $\tilde{\tau}_2([z_l, z) \cap \{y : \lambda(y) > 0\})$. We show

$$\tilde{\tau}\Big(\underbrace{\Big\{(x,y) \in [z_l, z) \times [z_l, z) : \lambda(y) > 0, x > y\Big\}}_{\triangleq \mathcal{O}}\Big) = 0.$$

Note that $\tilde{\tau}(\mathcal{O}) = \tau^l(\mathcal{O})$ and

$$\mathcal{O} \subseteq \bigcup_{a \in [z_l, z) \cap \mathbb{Q}} [a, z) \times [z_l, z),$$

therefore,

$$\begin{aligned}
\tau^l(\mathcal{O}) &\leq \sum_{a \in (z_l, z) \cap \mathbb{Q}} \tau^l([a, z) \times [z_l, a)) \\
&= \sum_{a \in (z_l, z) \cap \mathbb{Q}} \tau^l([a, z) \times [z_l, z)) - \tau^l([a, z) \times [a, z)) \\
&\overset{(a)}{=} \sum_{a \in (z_l, z) \cap \mathbb{Q}} F_{\mu_l^l}(a) \wedge F_{S^l}(z_l) - F_{\mu_l^l}(a) \wedge F_{S^l}(a) \\
&\overset{(b)}{=} \sum_{a \in (z_l, z) \cap \mathbb{Q}} F_{\mu_l^l}(a) - F_{\mu_l^l}(a) \\
&= 0,
\end{aligned}$$

where (a) comes from Eq. (C-28) and (b) from Eq. (B-8). Therefore,

$$\begin{aligned}
\tilde{\tau}(\tilde{\mathcal{E}}) &= \tilde{\tau}_2(\{z\}) \cdot \mathbf{1}_{\{\lambda(z) > 0\}} + \tilde{\tau}_2(\{y \in [z_l, z) : \lambda(y) > 0\}) + \tilde{\tau}_2(\{y \in (z, z_r] : \lambda(y) > 0\}) \\
&\quad + \sum_{i \in \{l, r\}} \tilde{\tau}(\{z_i\} \times \{z_i\}) \cdot \mathit{\Gamma}_i \cdot \mathbf{1}_{\{V(z_i) = 0, \lambda(z_i) = 0\}} \\
&\quad = \tilde{\tau}_2(\{y \in [z_l, z_r] : \lambda(y) > 0\}) + \sum_{i \in \{l, r\}} s_i \cdot \mathit{\Gamma}_i \cdot \mathbf{1}_{\{V(z_i) = 0, \lambda(z_i) = 0\}}.
\end{aligned}$$

Now, recall that

$$\tilde{\mu}(A(z)) = \tau(A(z) \times A(z)) = \int_{[z_l, z_r]} s^{\hat{\tau}} \, d\mathit{\Gamma},$$

and for the integral above we have

$$\begin{aligned}
\int_{[z_l, z_r]} s^{\hat{\tau}} \, d\mathit{\Gamma} &= \int_{[z_l, z_r] \cap \{y : \lambda(y) > 0\}} s^{\hat{\tau}} \, d\mathit{\Gamma} + \int_{\{z_l, z_r\} \cap \{y : \lambda(y) = 0\}} s^{\hat{\tau}} \, d\mathit{\Gamma} \\
&\overset{(a)}{=} \tilde{\tau}_2(\{y \in [z_l, z_r] : \lambda(y) > 0\}) + \int_{\{z_l, z_r\} \cap \{y : \lambda(y) = 0\}} s^{\hat{\tau}} \, d\mathit{\Gamma} \\
&\overset{(b)}{=} \tilde{\tau}_2(\{y \in [z_l, z_r] : \lambda(y) > 0\}) + \sum_{i \in \{l, r\}} s_i \cdot \mathit{\Gamma}_i \cdot \mathbf{1}_{\{\lambda(z_i) = 0\}}.
\end{aligned}$$

Hence, if we show that

$$\sum_{i \in \{l, r\}} s_i \cdot \mathit{\Gamma}_i \cdot \mathbf{1}_{\{V(z_i) = 0, \lambda(z_i) = 0\}} = \sum_{i \in \{l, r\}} s_i \cdot \mathit{\Gamma}_i \cdot \mathbf{1}_{\{\lambda(z_i) = 0\}},$$

the proof will be complete. It is enough to show that if $\lambda(z_i) = 0$, $\mathit{\Gamma}_i > 0$ and $V(z_i) > 0$ then $s_i \cdot \mathbf{1}_{\{V(z_i) = 0\}}$ equals s_i. If $V(z_i) > 0$ then if $s_i = 0$ then we are done; however, if $s_i > 0$ then since $\lambda(z_i) = 0$ from the congestion bound we deduce that $z_i \in \{X_l^{\text{supp}}, X_r^{\text{supp}}\}$. WLOG suppose z_i equals X_l^{supp} then

$$\begin{aligned}
\tau_2((X_l^{\text{supp}}, z)) &\leq S^H((X_l^{\text{supp}}, z)) \\
&< S^H((X_l^{\text{supp}}, z)) + s_i \cdot \mathit{\Gamma}_i \\
&= \tilde{\tau}_2([X_l^{\text{supp}}, z)) \\
&\leq \tau_l \\
&= \tau_2([X_l, z)) \\
&= \tau_2([X_l^{\text{supp}}, z))
\end{aligned}$$

where the first inequality comes from the congestion bound of Lemma B-1, the second from $s_i, \mathit{\Gamma}_i > 0$ and the last from the feasibility of $s^{\hat{\tau}}$. Therefore $\tau_2(\{z_i\}) > 0$ and, therefore, Lemma A-2 implies that $U(z_i, p(z_i), s^\tau(z_i)) = V(z_i)$. Since, $\lambda(z_i) = 0$ we conclude that in this case we cannot have $s_i > 0$. This completes the proof.

Part 6: Conclusion. We conclude by applying Proposition 4. The price-equilibrium pair $(\tilde{p}, \tilde{\tau})$ satisfies the hypothesis in Proposition 4, so we can create a global price-equilibrium pair $(\hat{p}, \hat{\tau})$ in $\mathcal{C}$. This new solution has the same objective that (p, τ) in $A(z)^c$, but it dominates the platform revenue in $A(z)$. Therefore, $(\hat{p}, \hat{\tau})$ revenue dominates (p, τ). □

Lemma B-2. *The following properties hold.*

1.

$$S^H((z, z + \delta_c)) \leq \tau_r, \tag{a}$$

$$S^H((z - \delta_c - \beta_l, z + \delta_c)) + s_r \cdot \mathit{\Gamma}_r \leq \tau, \tag{b}$$

$$S^H((z - \delta_c - \beta_l, z)) \leq \tau_l, \tag{c}$$

$$S^H(\{z\}) \geq \tau_c. \tag{d}$$

2. *If* $\delta_c < \delta_r$ *then* $\beta_l = 0$.

3. $\tau_c + \tau_r \leq S^H([z, z + \delta_c])$.

Proof of Lemma B-2. We provide a proof for each statement separately.

1. Inequality (a) comes from the definition of δ_r and that $\delta_c \leq \delta_r$. To prove (b), first note when $\beta_l = 0$ our choice of s_r make the inequality true. Suppose $\beta_l > 0$, by the definition of β_l we must have
$$S^H([z - \delta_c - \beta_l - \frac{1}{n}, z + \delta_c)) + s_r \cdot \mathit{\Gamma}_r < \tau, \quad \forall n \in \mathbb{N}.$$

Taking $\lim_{n\to\infty}$ yields the the desired result. Inequality (c) comes from the definition of δ_l and the fact that $\beta_l + \delta_c \leq \delta_l$.

Next we prove (d). Note that from the feasibility of s^τ we have $\tau_2([X_l, X_r]) = \tau$, in turn

$$\tau_2(\{z\}) = \tau_c + (\tau_r - \tau_2((z, X_r]) + (\tau_l - \tau_2([X_l, z))).$$

Again by the feasibility of s^τ the last two terms in this equation are non-negative. Hence, $\tau_2(\{z\}) \geq \tau_c$ and since $\tau_2(\{z\}) \leq S^H(\{z\})$ we conclude the result.

2. We have $\delta_c < \delta_r \leq \delta_l$ and, therefore,

$$S^H([z-\delta_c, z+\delta_c]) \geq \tau \quad \text{and} \quad S^H((z, z+\delta_c]) \leq \tau_r.$$

The first inequality comes from $\delta_c < \delta_r$ and the definition of δ_c. The second comes from $\delta_c < \delta_r$ and the definition of δ_r.

If $\varGamma_r = 0$, because $S^H([z-\delta_c, z+\delta_c]) \geq \tau$, we can conclude that $\beta_l = 0$. If $\varGamma_r > 0$ then $S^H((z, z+\delta_c]) \leq \tau_r$ implies

$$H_r \leq \frac{\tau_r - S^H((z, z+\delta_c))}{\varGamma_r},$$

and, therefore,

$$s_r = \min\Big\{H_r, \frac{\tau - S^H((z-\delta_c, z+\delta_c))}{\varGamma_r}\Big\}.$$

Hence,

$$S^H((z-\delta_c, z+\delta_c)) + s_r \cdot \varGamma_r = \min\{S^H((z-\delta_c, z+\delta_c]), \tau\},$$

if the minimum equals τ then, since

$$S^H([z-\delta_c, z+\delta_c)) + s_r \cdot \varGamma_r \geq S^H((z-\delta_c, z+\delta_c)) + s_r \cdot \varGamma_r = \tau,$$

we would have $\beta_l = 0$. If the minimum equals $S^H((z-\delta_c, z+\delta_c])$, then s_r equals H_r which together with $S^H([z-\delta_c, z+\delta_c]) \geq \tau$ imply that $\beta_l = 0$. In any case β_l equals zero.

3. Suppose otherwise, thar is, $\tau_c + \tau_r > S^H([z, z+\delta_c])$. This together with (d) imply that $\tau_r > S^H((z, z+\delta_c])$. In turn, this yields $\delta_c < \delta_r \leq \delta_l$ and, therefore, $\beta_l = 0$. Since $\delta_c < \delta_l$ we also have that $S^H([z-\delta_c, z)) \leq \tau_l$. Hence, $\tau > S^H([z-\delta_c, z+\delta_c])$ which contradicts the definition of δ_c when β_l equals zero.

$\square$

Lemma B-3. *Let ν be a non-negative measure in $\mathcal{C}$. Consider any measurable subset K of $\mathcal{C}$ and some $z \in \mathcal{C}$ then the mappings $\nu(\pi_1(\,\cdot\, \cap \mathcal{D}) \cap K)$ and $\nu(\pi_1(\,\cdot\, \cap (K \times \{z\})))$, defined on the Borel sets of $\mathcal{C} \times \mathcal{C}$, belong to $\mathcal{M}(\mathcal{C} \times \mathcal{C})$.*

Proof. For any Borel set $\mathcal{L} \subset \mathcal{C} \times \mathcal{C}$ define

$$\tau_a(\mathcal{L}) \triangleq \nu(\pi_1(\mathcal{L} \cap \mathcal{D}) \cap K) \text{ and } \tau_b(\mathcal{L}) \triangleq \nu(\pi_1(\mathcal{L} \cap (K \times \{z\}))).$$

We show that $\tau_a, \tau_b \in \mathcal{M}(\mathcal{C} \times \mathcal{C})$. Note that because $\nu \in \mathcal{M}(\mathcal{C})$ for $i \in \{a, b\}$ we have that $\tau_i(\emptyset) = 0$, and for any Borel set $\mathcal{L} \subseteq \mathcal{C} \times \mathcal{C}$ that $\tau_i(\mathcal{L}) \in [0, \infty)$. To verify $\sigma-$ additivity consider a countable partition $\{\mathcal{L}_n\}_{n \in \mathbb{N}} \subseteq \mathcal{C} \times \mathcal{C}$, we need to show that

$$\tau_i\Big(\bigcup_{n \in \mathbb{N}} \mathcal{L}_n\Big) = \sum_{n \in \mathbb{N}} \tau_i(\mathcal{L}_n).$$

Note that from the definition of $\mathcal{D}$ and the fact the set $K \times \{z\}$ has second component equal to 0, both collections $\{\pi_1(\mathcal{L}_n \cap \mathcal{D})\}_{n \in \mathbb{N}}$ and $\{\pi_1(\mathcal{L}_n \cap (K \times \{z\}))\}_{n \in \mathbb{N}}$ form a partition. Given this we can verify $\sigma-$ additivity, we do it for both τ_a and τ_b at the same time

$$\begin{aligned}
\tau_a\Big(\bigcup_{n \in \mathbb{N}} \mathcal{L}_n\Big) + \tau_b\Big(\bigcup_{n \in \mathbb{N}} \mathcal{L}_n\Big) &= \nu(\pi_1(\bigcup_{n \in \mathbb{N}} \mathcal{L}_n \cap \mathcal{D}) \cap K) + \nu(\pi_1(\bigcup_{n \in \mathbb{N}} \mathcal{L}_n \cap K \times \{z\})) \\
&= \nu(\bigcup_{n \in \mathbb{N}} \pi_1(\mathcal{L}_n \cap \mathcal{D}) \cap K) + \nu(\bigcup_{n \in \mathbb{N}} \pi_1(\mathcal{L}_n \cap K \times \{z\})) \\
&= \sum_{n \in \mathbb{N}} \nu(\pi_1(\mathcal{L}_n \cap \mathcal{D}) \cap K) + \sum_{n \in \mathbb{N}} \nu(\pi_1(\mathcal{L}_n \cap K \times \{z\})) \\
&= \sum_{n \in \mathbb{N}} \tau_a(\mathcal{L}_n) + \sum_{n \in \mathbb{N}} \tau_b(\mathcal{L}_n),
\end{aligned}$$

where the third line comes from the $\sigma-$additivity of the ν measure. This shows that $\tau \in \mathcal{M}(\mathcal{C} \times \mathcal{C})$. □

C Proofs for Section 6

To simplify the exposition of our result in this section we define the quantities

$$\rho_1 \triangleq \rho_x^{loc}(\mu_1) \quad \text{and} \quad \psi_1 \triangleq \psi_1(\mu_1),$$

where the function $\psi_1(\cdot)$ correspond to $\psi_x(\cdot)$ for some $x \neq 0$ (at locations other that the origin, $\psi_x(\cdot)$ is the same function.) We use X_l, X_r and $V(0)$ to denote $X_l(0|p,\tau), X_r(0|p,\tau)$ and $V(0|p,\tau)$, respectively. Also, $m \in \mathcal{M}(\mathcal{C})$ denotes the Lebesgue measure in $\mathcal{C}$. We use $\mathcal{D}$ to denote the subset of $\mathcal{C} \times \mathcal{C}$ with equal first and second components, that is, $\mathcal{D} = \{(x, y) \in \mathcal{C} \times \mathcal{C} : x = y\}$. For any measurable set $\mathcal{B} \subseteq \mathcal{C}$ and a price-equilibrium pair (p, τ) we denote the platform's revenue in $\mathcal{B}$ under (p, τ) by $\mathbf{Rev}_{\mathcal{B}}(p, \tau)$. In case that $\mathcal{B}$ is $\mathcal{C}$ we simply use $\mathbf{Rev}(p, \tau)$.

Furthermore, recall that Lemma 4 shows that for fix $x \in \mathcal{C}$ whenever the density $\Gamma(x)$ is positive, the congestion function $\psi_x(\cdot)$ is strictly decreasing. In the current setting we assume that both λ_0 and λ_1 are strictly positive and, therefore, the monotonicity property of the congestion function applies to every location in the city.

Lemma C-1. *Let p be any price mapping and τ a corresponding equilibrium flow. Then for any measurable set $\mathcal{B} \subseteq \mathcal{C}_\lambda$ such that $0 \notin \mathcal{B}$ and $\tau(\mathcal{B} \times \mathcal{B}^c) = 0$ we have*

$$V(x|p,\tau) \;\leq\; \psi_1, \quad \Gamma - a.e.\ x \;\; in\ \mathcal{B}.$$

Furthermore, in the pre-shock environment we can replace $\mathcal{B}$ with $\mathcal{C}_\lambda$ in the inequality above.

Proof. Define the set

$$\mathcal{L} \triangleq \{x \in \mathcal{B} : V(x|p,\tau) \;\leq\; \psi_1\}.$$

We would like to show that $\Gamma(\mathcal{L}^c) = 0$ where the complement is taken with respect to $\mathcal{B}$. Suppose this is not the case, and note that

$$\mu_1 \cdot m(\mathcal{L}^c) = \mu(\mathcal{L}^c) = \tau(\mathcal{L}^c \times \mathcal{C}) = \tau(\mathcal{L}^c \times \mathcal{B}) + \tau(\mathcal{L}^c \times \mathcal{B}^c),$$

since $\mathcal{L}^c \subseteq \mathcal{B}$ and $\tau(\mathcal{B} \times \mathcal{B}^c) = 0$, the second term in the expression above is zero. This yields,

$$\begin{aligned}
\mu_1 \cdot m(\mathcal{L}^c) &= \tau(\mathcal{L}^c \times \mathcal{B}) \\
&= \tau(\mathcal{L}^c \times \mathcal{B} \cap \mathcal{L}^c) + \tau(\mathcal{L}^c \times \mathcal{B} \cap \mathcal{L}) \\
&= \tau(\mathcal{L}^c \times \mathcal{L}^c) + \tau(\mathcal{L}^c \times \mathcal{L})
\end{aligned}$$

There are two cases. First, if $\tau(\mathcal{L}^c \times \mathcal{L}) > 0$ then by Lemma A-6 there exists a pair $(x,y) \in \mathcal{L}^c \times \mathcal{L}$ such that $y \in \mathcal{IR}(x|p,\tau)$. Therefore, by Lemma 3 we have

$$V(y|p,\tau) = V(x|p,\tau) + |x-y|.$$

However, since $(x,y) \in \mathcal{L}^c \times \mathcal{L}$

$$V(y|p,\tau) \leq \psi_1 \text{ and } V(x|p,\tau) > \psi_1.$$

Using the previous equation we can deduce that $\psi_1 > \psi_1$, which is not possible. The second case is $\tau(\mathcal{L}^c \times \mathcal{L}) = 0$. Note that

$$\tau_2(\mathcal{L}^c) = \tau(\mathcal{C} \times \mathcal{L}^c) \geq \tau(\mathcal{L}^c \times \mathcal{L}^c) = \mu_1 \cdot m(\mathcal{L}^c).$$

We also have that

$$\tau_2(\mathcal{L}^c) = \int_{\mathcal{L}^c} s^\tau(x) d\Gamma(x) \leq \int_{\mathcal{L}^c} \psi_x^{-1}(V(x|\,p,\tau)) d\Gamma(x) < \mu_1 \cdot \Gamma(\mathcal{L}^c),$$

where the first inequality comes from Proposition 5, and the second from the fact that $\psi_x(\cdot)$ is a strictly decreasing function, the definition of $\mathcal{L}^c$ and $\mathit{\Gamma}(\mathcal{L}^c) > 0$. Note that this inequality holds in both of the cases in the statement of the lemma. In both cases we have $0 \notin \mathcal{B}$ so $\mathit{\Gamma}(\mathcal{L}^c)$ equals $m(\mathcal{L}^c)$, yielding

$$\mu_1 \cdot m(\mathcal{L}^c) \leq \tau_2(\mathcal{L}^c) < \mu_1 \cdot \mathit{\Gamma}(\mathcal{L}^c) = \mu_1 \cdot m(\mathcal{L}^c),$$

a contradiction. □

C.1 Proofs for Section 6.1

Proof of Proposition 6. Let (p, τ) be any feasible price-equilibrium pair by Lemma C-1 we have $V(x|\, p, \tau) \leq \psi_1$, $\mathit{\Gamma}$ almost everywhere in $\mathcal{C}_\lambda = \mathcal{C} \setminus \{0\}$. This yields the following upper bound for the platform's objective

$$\int_{\mathcal{C}_\lambda} V(x|\, p, \tau) \cdot s^\tau(x)\, dx \leq \psi_1 \cdot \int_{\mathcal{C}_\lambda} s^\tau(x)\, dx \leq \psi_1 \cdot \mu_1 \cdot m(\mathcal{C}).$$

The maximum revenue the platform can achieve in this case is bounded above by $\gamma \cdot \psi_1 \cdot \mu_1 \cdot m(\mathcal{C})$. Next, we show that the solution given in the statement of the lemma is feasible and achieves the upper bound.

Flow feasibility. We show that $\tau \in \mathcal{F}(\mu)$. A complete definition of the measure τ is $\tau(\mathcal{L}) = \mu(\pi_1(\mathcal{L} \cap \mathcal{D})$. From the definition of τ it is clear that $\tau \in \mathcal{M}(\mathcal{C})$. Furthermore, τ_1 coincides with μ and so does τ_2. Since μ is the Lebesgue measure times a constant and $\mathit{\Gamma}$ is the Lebesgue measure plus an atom, we have $\tau_1, \tau_2 \ll \mathit{\Gamma}$. From this we can deduce that $m-$ a.e in $\mathcal{C}_\lambda$ we have $s^\tau(x)$ equals μ_1.

Equilibrium utilities. We show that $V(x|\, p, \tau)$ equals ψ_1. Note that

$$U(y, p(y), s^\tau(y)) = \psi_1, \quad \mathit{\Gamma} - a.e.\ y \text{ in } \mathcal{C}_\lambda.$$

Fix $x \in \mathcal{C}$, we have that

$$\mathit{\Gamma}(\{y \in \mathcal{C} : U(y, p(y), s^\tau(y)) - |y - x| > \psi_1\}) = \mathbf{1}_{\{0 - |0 - x| > \psi_1\}} + \mathit{\Gamma}(\{y \in \mathcal{C} \setminus \{0\} : -|y - x| > 0\}) = 0.$$

Moreover, for any $\epsilon > 0$

$$\mathit{\Gamma}(\{y \in \mathcal{C} : U(y, p(y), s^\tau(y)) - |y - x| > \psi_1 - \epsilon\}) \geq \mathit{\Gamma}(\{y \in \mathcal{C}_\lambda : -|y - x| > \epsilon\}) > 0,$$

where the last inequality comes from the fact that $\mathit{\Gamma}$ corresponds to the Lebesgue measure (plus an atom). That is, $V(x|\, p, \tau)$ equals ψ_1.

Equilibrium condition. Consider the equilibrium set

$$\mathcal{E} \triangleq \Big\{(x,y) \in \mathcal{C} \times \mathcal{C} : U(y, p(y), s^{\tau}(y)) - |y - x| = V(x|\, p, \tau)\Big\}.$$

Then,

$$\tau(\mathcal{E}) = \tau\Big(\Big\{(x,y) \in \mathcal{C} \times \{0\} : -|y-x| = \psi_1\Big\}\Big) + \tau\Big(\Big\{(x,y) \in \mathcal{C} \times \mathcal{C}_{\lambda} : -|y-x| = 0\Big\}\Big) = \mu(\mathcal{C}).$$

We have proven that the solution is the statement is feasible, and because of the values of $V(\cdot|\, p, \tau)$ and $s^{\tau}(\cdot)$ we conclude that this solution achieves the upper bound. □

C.2 Proofs for Section 6.2

Proof of Proposition 7. The proof of this proposition consists of several steps. In the first step we establish that the origin is an attraction region, characterize some properties of it and compute the value of the equilibrium utility function outside the attraction region. After this step, the drivers utility function will be pinned down in the entire city as a function of its value in the origin, $V(0|\, p, \tau)$. The second step supplies us with a full characterization, up to $V(0|\, p, \tau)$, of the post-relocation supply τ_2 in the entire city. Finally, in step three we show how to solve for the optimal value of $V(0|\, p, \tau)$ and, therefore, we pin down both $V(\cdot|\, p, \tau)$ and τ_2. We further show how to find the optimal $p(0)$ and the corresponding optimal flow τ.

Step 1: We show that we can restrict attention to solutions (p, τ) such that $X_l < 0 < X_r$, $X_r = V(0) - \psi_1$ and $X_l = -X_r$. Furthermore, such solutions have $V(x|\, p, \tau) = \psi_1$ for all $x \in \mathcal{C} \setminus [X_l, X_r]$.

Proof of Step 1: Let (p, τ) be a feasible solution. First, we show that at any optimal solution we must have $X_l < 0 < X_r$. By Lemma C-2 (which we state and prove after the proof of the present proposition) we have that if either of the sets $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \tau)\}$ or $\{x \in [-H, 0) : 0 \in \mathcal{IR}(x|p, \tau)\}$ is empty then the revenue the platform makes satisfies

$$\frac{1}{\gamma} \cdot \mathbf{Rev}(p, \tau) \leq \psi_1 \cdot \mu_1 \cdot 2 \cdot H.$$

Now we construct a new feasible solution $(\tilde{p}, \tilde{\tau})$ for which both sets are non-empty and such that

$$\frac{1}{\gamma} \cdot \mathbf{Rev}(\tilde{p}, \tilde{\tau}) > \psi_1 \cdot \mu_1 \cdot 2 \cdot H, \tag{C-1}$$

where $\tilde{p}$ equals ρ_1 in $\mathcal{C} \setminus \{0\}$ and $p(0)$ is appropriately chosen. This will imply that any optimal solution must satisfy $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \tau)\} \neq \emptyset$ and $\{x \in [-H, 0) : 0 \in \mathcal{IR}(x|p, \tau)\} \neq \emptyset$ and, therefore, $X_l < 0 < X_r$. This also implies that the optimal revenue in this case is strictly larger than the one in the pre-shock environment.

Our solution will send flow in $[-h, h]$ to the origin, where $h > 0$ is to be determined. Inside this interval, all the flow in the subinterval $[-\bar{h}(h), \bar{h}(h)]$ goes to the origin where $0 \le \bar{h}(h) \le h$. The rest of the flow in $[-h, h]$ partially stays at its original position and partially goes to the origin. We now show how to determine $\bar{h}(h)$ and h. For any given $h > 0$ we define

$$\bar{h}(h) \triangleq (\psi_1 + h - \alpha \cdot \rho_1)^+,$$

note that when ψ_1 equals $\alpha \cdot \rho_1$ we have that $\bar{h}(h)$ equals h, and we will send all the flow in $[-h, h]$ to the origin. However, when $\psi_1 < \alpha \cdot \rho_1$ not all the flow will be sent to the origin. Define

$$\mu_1(x) \triangleq \alpha \cdot \rho_1 \cdot \frac{\lambda_1 \overline{F}(\rho_1)}{\psi_1 + h - |x|},$$

then

$$\frac{\lambda_1 \overline{F}(\rho_1)}{\mu_1(x)} \le 1, \quad x \in [-h, h] \setminus [-\bar{h}(h), \bar{h}(h)].$$

The idea is that for every location $x \in K(h) \triangleq [-h, h] \setminus [-\bar{h}(h), \bar{h}(h)]$ we will leave a density $\mu_1(x)$ of flow there and send $\mu_1 - \mu_1(x)$ (note that this difference is non-negative) to the origin. In order to make this possible, we need to chose h appropriately. Observe that the total supply we will send to the origin is

$$S_T(h) = 2\bar{h}(h)\mu_1 + 2\int_{\bar{h}(h)}^{h} (\mu_1 - \mu_1(x))\, dm(x),$$

where $\lim_{h \to 0} S_T(h) = 0$. Hence, since $\psi_1 < \overline{\alpha} \cdot \overline{V}$, we can always find $h > 0$ such that

$$\alpha \cdot \overline{V} - h \ge \alpha \cdot F^{-1}\Big(1 - \frac{S_T(h)}{\lambda_0}\Big) - h \ge \psi_1. \tag{C-2}$$

This yields

$$\overline{F}\Big(\frac{\psi_1 + h}{\alpha}\Big) \ge \frac{S_T(h)}{\lambda_0}.$$

Now we construct the solution $(\tilde{p}, \tilde{\tau})$. Fix any h satisfying Eq. (C-2) and consider prices defined by

$$\tilde{p}(x) = \begin{cases} \frac{\psi_1 + h}{\alpha} & \text{if } x = 0 \\ \rho_1 & \text{if } x \in \mathcal{C} \setminus \{0\}, \end{cases}$$

and flows for any measurable set $\mathcal{L} \subseteq \mathcal{C} \times \mathcal{C}$ defined by

$$\begin{aligned} \tilde{\tau}(\mathcal{L}) &= \mu(\pi_1(\mathcal{L} \cap \mathcal{D}) \cap [-h, h]^c) + \mu(\pi_1(\mathcal{L} \cap [-\bar{h}(h), \bar{h}(h)] \times \{0\})) \\ &\quad + G_0(\pi_1(\mathcal{L} \cap K(h) \times \{0\})) + G_1(\pi_1(\mathcal{L} \cap \mathcal{D}) \cap K(h)), \end{aligned}$$

where G_0, G_1 are measures defined for any measurable set $\mathcal{B} \subseteq K(h)$ by

$$G_0(\mathcal{B}) \triangleq \int_{\mathcal{B}} (\mu_1 - \mu_1(x))\, dm(x), \quad G_1(\mathcal{B}) \triangleq \int_{\mathcal{B}} \mu_1(x)\, dm(x).$$

We argue that $(\tilde{p}, \tilde{\tau})$ is a feasible solution that complies with Eq. (C-1). From Lemma B-3 we have that $\tilde{\tau} \in \mathcal{M}(\mathcal{C} \times \mathcal{C})$, also note that for any measurable set $\mathcal{B} \subseteq \mathcal{C}$ the first marginal of $\tilde{\tau}$ satisfies

$$\tilde{\tau}_1(\mathcal{B}) = \mu(\mathcal{B} \cap [-h, h]^c) + \mu(\mathcal{B} \cap [-\bar{h}(h), \bar{h}(h)]) + G_0(\mathcal{B} \cap K(h)) + G_1(\mathcal{B} \cap K(h)) = \mu(\mathcal{B}).$$

The post-relocation supply measure is

$$\tilde{\tau}_2(\mathcal{B}) = \mu(\mathcal{B} \cap [-h, h]^c) + S_T(h) \cdot \mathbf{1}_{\{0 \in \mathcal{B}\}} + G_1(\mathcal{B} \cap K(h)),$$

clearly $\tilde{\tau}_2 \ll \mathit{\Gamma}$. Therefore, $\tilde{\tau} \in \mathcal{F}(\mu)$. Next, we need to show that $\tilde{\tau}$ is a supply equilibrium. The Radon-Nikodym derivative of $\tilde{\tau}_2$ with respect the city measure is ($\mathit{\Gamma}$-a.e)

$$s(x) = \begin{cases} S_T(h) & \text{if } x = 0 \\ 0 & \text{if } x \in [-\bar{h}(h), \bar{h}(h)] \setminus \{0\} \\ \mu_1(x) & \text{if } x \in K(h) \\ \mu_1 & \text{if } x \in [-h, h]^c. \end{cases}$$

Indeed,

$$\int_{\mathcal{L}} s(x)\, d\mathit{\Gamma}(x) = S_T(h) \mathbf{1}_{\{0 \in \mathcal{L}\}} + \int_{\mathcal{L} \cap [-h,h]^c} \mu_1 \, dm(x) + \int_{\mathcal{L} \cap K(h)} \mu_1(x)\, dm(x) = \tilde{\tau}_2(\mathcal{L}),$$

that is, $\frac{d\tilde{\tau}_2}{d\mathit{\Gamma}}(\cdot)$ equals $s(\cdot)$ $\mathit{\Gamma}$-a.e. From this we can compute $V(\cdot|\tilde{p}, \tilde{\tau})$. Note that ($\mathit{\Gamma}$-a.e)

$$\tilde{U}(y) = U\Big(y, \tilde{p}(y), \frac{d\tilde{\tau}_2}{d\mathit{\Gamma}}(y)\Big) = \begin{cases} \psi_1 + h & \text{if } y = 0; \\ \alpha \cdot \rho_1 & \text{if } y \in [-\bar{h}(h), \bar{h}(h)] \setminus \{0\}; \\ \alpha \cdot \rho_1 \cdot \frac{\lambda_1 \overline{F}(\rho_1)}{\mu_1(x)} & \text{if } y \in K(h); \\ \psi_1 & \text{if } y \in [-h, h]^c. \end{cases}$$

Let $a(x)$ be defined by

$$a(x) \triangleq \begin{cases} \psi_1 + h - |x| & \text{if } x \in [-h, h], \\ \psi_1 & \text{if } x \in [-h, h]^c. \end{cases}$$

We argue that $V(\cdot|\tilde{p}, \tilde{\tau}) \equiv a(\cdot)$. Fix $x \in \mathcal{C}$, it is not hard to verify that

$$\mathit{\Gamma}(y \in \mathcal{C} : \tilde{U}(y) - |y - x| > a(x)) = 0,$$

and, thus, $a(x) \geq V(x|\, \tilde{p}, \tilde{\tau})$. Suppose that $x \in [-h, h]$ and $a(x) > V(x|\, \tilde{p}, \tilde{\tau})$ then, because $\mathit{\Gamma}(\{0\}) > 0$, we have that

$$\psi_1 + h - |x| = a(x) > V(x|\, \tilde{p}, \tilde{\tau}) \geq \mathit{\Pi}(x, 0) = \psi_1 + h - |0 - x|,$$

a contradiction. Thus, for $x \in [-h, h]$ we have $a(x) = V(x|\tilde{p}, \tilde{\tau})$. For any other x we can use a similar argument to conclude that $a(x) = V(x|\tilde{p}, \tilde{\tau})$.

Now we are ready to verify the equilibrium condition. Observe that

$$\mathcal{E} = \Big\{(x, y) \in \mathcal{C}\times\mathcal{C} : \varPi(x, y) = V(x|\tilde{p}, \tilde{\tau})\Big)\Big\} = ([-h, h]\times\{0\})\cup([-h, h]^c\times[-h, h]^c\cap\mathcal{D})\cup(K(h)\times K(h)\cap\mathcal{D}),$$

then

$$\begin{aligned}\tilde{\tau}(\mathcal{E}) &= \mu(\pi_1(\mathcal{E}\cap\mathcal{D})\cap[-h, h]^c) + \mu(\pi_1(\mathcal{E}\cap[-\bar{h}(h), \bar{h}(h)]\times\{0\}))\\ &\quad + G_1(\pi_1(\mathcal{E}\cap\mathcal{D})\cap K(h)) + G_0(\pi_1(\mathcal{E}\cap K(h)\times\{0\}))\\ &= \mu([-h, h]^c) + \mu([-\bar{h}(h), \bar{h}(h)]) + G_1(K(h)) + G_0(K(h))\\ &= \mu(\mathcal{C}).\end{aligned}$$

This proves that $\tilde{\tau}$ is an equilibrium. Next we need to show $(\tilde{p}, \tilde{\tau})$ satisfies Eq. (C-1). From Proposition 1 we have

$$\begin{aligned}\gamma\mathbf{Rev}(\tilde{p}, \tilde{\tau}) &= \int_{\mathcal{C}} V(x)\cdot\frac{d\tilde{\tau}_2}{d\varGamma}(x)d\varGamma(x)\\ &= (\psi_1 + h)\cdot S_T(h) + 2\int_{\bar{h}(h)}^{h}(\psi_1 + h - x)\mu_1(x)\,dm(x) + \psi_1\cdot\mu_1\cdot 2(H - h)\\ &\geq h\cdot S_T(h) + \psi_1\Big(S_T(h) + 2\int_{\bar{h}(h)}^{h}\mu_1(x)dx\Big) + \psi_1\cdot\mu_1\cdot 2(H - h)\\ &= h\cdot S_T(h) + \psi_1\Big(2\bar{h}(h)\mu_1 + 2\int_{\bar{h}(h)}^{h}(\mu_1 - \mu_1(x))dx + 2\int_{\bar{h}(h)}^{h}\mu_1(x)dx\Big) + \psi_1\cdot\mu_1\cdot 2(H - h)\\ &= h\cdot S_T(h) + \psi_1\cdot\mu_1\cdot 2\cdot H.\end{aligned}$$

Since $h\cdot S_T(h) > 0$, Eq. (C-1) obtains. This proves that $X_l < 0 < X_r$ in any optimal solution.

The next step of the proof of Step 1 consists on arguing that given $V(0)$, $X_r = V(0) - \psi_1$ and $X_l = -(V(0) - \psi_1)$. Consider a feasible solution (p, τ) where $p(\cdot)$ equals ρ_1 everywhere but at the origin, and $X_l < 0 < X_r$. From Proposition 3 and the fact that $\mu(\{X_r\}) = 0$ we have that

$$\tau([X_r, H]\times[X_r, H]^c) \leq \mu(\{X_r\}) + \tau((X_r, H]\times[X_r, H]^c) = 0.$$

Then by Lemma C-1 we have that $V(x) \leq \psi_1$, $\varGamma - a.e.\ x$ in $[X_r, H]$. This, together with the continuity of $V(\cdot)$ imply that $V(x) \leq \psi_1$ for all $x \in [X_r, H]$.

Suppose first that $X_r < V(0) - \psi_1$ then

$$V(X_r|\, p, \tau) = V(0) - X_r > \psi_1,$$

but this violates the continuity of V to the right of X_r. Thus $X_r \geq V(0) - \psi_1$. On the other hand, suppose $X_r > V(0) - \psi_1$ then we must have that $\psi_1 > V(x|\, p, \tau) = V(0) - x$ for all $x \in (V(0) - \psi_1, X_r]$. Observe that

$$\mu([V(0) - \psi_1, X_r]) \geq \tau_2([V(0) - \psi_1, X_r]) = \int_{[V(0)-\psi_1, X_r]} s^\tau(x)\, d\Gamma(x). \tag{C-3}$$

Define the set

$$K \triangleq \{y \in [V(0) - \psi_1, X_r] : s^\tau(y) \leq \mu_1\},$$

it must be that $\Gamma(K) = 0$; otherwise, from the definition of $V(X_r|\, p, \tau)$ we have

$$\begin{aligned} V(0) - X_r = V(X_r) &\geq U(y, \rho_1, s^\tau(y)) - |y - X_r|, \quad \Gamma - a.e.\ y \text{ in } K \\ &\geq U(y, \rho_1, \mu_1) - |y - X_r|, \quad \Gamma - a.e.\ y \text{ in } K \\ &= \psi_1 - (X_r - y), \quad \Gamma - a.e.\ y \text{ in } K, \end{aligned}$$

and $\Gamma(K) > 0$ implies that $V(0) - y \geq \psi_1$ for some $y \in (V(0) - \psi_1, X_r]$. However, we know that $\psi_1 > V(0) - y$ for $y \in (V(0) - \psi_1, X_r]$ and, therefore, we must have $\Gamma(K) = 0$. Using this in Eq. (C-3) yields

$$\mu([V(0) - \psi_1, X_r]) > \mu_1 \cdot \Gamma([V(0) - \psi_1, X_r]) = \mu([V(0) - \psi_1, X_r]),$$

which is not possible. Hence, $X_r = V(0) - \psi_1$ and the same arguments applies to X_l, yielding $X_l = -(V(0) - \psi_1)$.

In order to conclude the proof for Step 1 we show that we can restrict attention to solutions (p, τ) such that $V(x|\, p, \tau)$ equals ψ_1 for all $x \in [X_l, X_r]^c$. In turn, this will show that $s^\tau(x)$ equals μ_1, $\Gamma - a.e.\ x$ in $[X_l, X_r]^c$. We base the proof of the latter statements in Lemma C-3 (which we state and prove after the proof of the present result), this lemma enables us to separate the city into two regions $[X_l, X_r]$ and $[X_l, X_r]^c$. For each region we can modify the prices and equilibria, and then paste them together to obtain a new solution that is an equilibrium for the entire city.

Consider a feasible solution (p, τ) such that $X_l < 0 < X_r$, $X_r = V(0) - \psi_1$ and $X_l = -X_r$. Since $\tau([X_l, X_r] \times [X_l, X_r]^c) = 0$ and $0 \notin [X_l, X_r]^c$, Lemma C-1 delivers

$$\frac{1}{\gamma} \cdot \mathbf{Rev}(p, \tau) \leq \frac{1}{\gamma} \cdot \mathbf{Rev}_{[X_l, X_r]}(p, \tau) + 2 \cdot \mu_1 \cdot \psi_1 \cdot (H - X_r). \tag{C-4}$$

We show that we can always modify (p, τ) so that the previous upper bound is achieve. Let $\mathcal{B} = [X_l, X_r]$, since $\tau(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\tau(\mathcal{B}^c \times \mathcal{B}) = 0$, Lemma C-3 ensures that $(p, \tau)|_{\mathcal{B}}$ is a price equilibrium pair in $\mathcal{B}$. Such equilibrium satisfies $V_{\mathcal{B}}(x) = \psi_1$ for $x \in \partial\mathcal{B}$.

Now, we choose prices $p^{\mathcal{B}^c}(x)$ equal to ρ_1 for all $x \in \mathcal{B}^c$ and a flow $\tau^{\mathcal{B}^c}$ defines by for any measurable set $\mathcal{L}_1 \times \mathcal{L}_2 \subseteq \mathcal{B}^c \times \mathcal{B}^c$

$$\tau^{\mathcal{B}^c}(\mathcal{L}_1 \times \mathcal{L}_2) = \mu(\mathcal{L}_1 \cap \mathcal{L}_2).$$

Then, it is easy to verify (as we did in the pre-shock environment, see Proposition 6) that $(p^{\mathcal{B}^c}, \tau^{\mathcal{B}^c})$ forms a price-equilibrium pair in $\mathcal{B}^c$. This solution satisfy that $V_{\mathcal{B}^c}(x) = \psi_1$ for $x \in \mathcal{B}^c$, and that $s^{\tau^{\mathcal{B}^c}}(x)$ equals μ_1, $\Gamma - a.e.\ x$ in $\mathcal{B}^c$.

Lemma C-3 enables us to paste the solutions $(p,\tau)|_{\mathcal{B}}$ and $(p^{\mathcal{B}^c}, \tau^{\mathcal{B}^c})$, and generate a new solution in the entire city. Such solution preserve the prices and flows in both $\mathcal{B}$ and $\mathcal{B}^c$ and, therefore, the upper bound in Eq. (C-4) is achieved. In conclusion, we can restrict attention to solutions (p,τ) such that $V(x|\,p,\tau)$ equals ψ_1 for all $x \in [X_l, X_r]^c$, and that $s^\tau(x)$ equals μ_1, $\Gamma - a.e.\ x$ in $[X_l, X_r]^c$.

Step 2: We characterize $s^\tau(\cdot)$ (this completely characterizes τ_2). Let

$$X_r^0 = (V(0) - \alpha \cdot \rho_1))^+ \quad \text{and} \quad X_l^0 = -X_r^0,$$

and

$$\mu_1(y) \triangleq \alpha \cdot \rho_1 \cdot \frac{\lambda_1 \cdot \overline{F}(\rho_1)}{V(0) - |y|}, \quad S_T = 2 \cdot \mu_1 \cdot X_r^0 + 2\int_{X_r^0}^{X_r} (\mu_1 - \mu_1(x))dx.$$

In this step we show that $(\Gamma - a.e)$

$$s^\tau(y) = \begin{cases} S_T & \text{if } y = 0 \\ 0 & \text{if } y \in [X_l^0, X_r^0] \setminus \{0\} \\ \mu_1(y) & \text{if } y \in [X_l, X_r] \setminus [X_l^0, X_r^0] \\ \mu_1 & \text{if } y \in [X_l, X_r]^c. \end{cases}$$

Proof of Step 2: Note that at the end of the previous step we showed the result for $y \in [X_l, X_r]^c$. So first we show

$$s^\tau(y) = 0, \quad \Gamma - a.e.\ x \text{ in } [X_l^0, X_r^0] \setminus \{0\}.$$

Define the set $K_1 \triangleq \{y \in [X_l^0, X_r^0] \setminus \{0\} : s^\tau(y) > 0\}$. We argue that $\Gamma(K_1) = 0$. If this is not the case then $\Gamma(K_1) > 0$ and, therefore,

$$\tau_2(K_1) = \int_{K_1} s^\tau(x)\, d\Gamma(x) > 0.$$

Then Lemma A-2 ensures that

$$U\Big(x, \rho_1, s^\tau(x)\Big) \;=\; V(x|\,p,\tau) \qquad \tau_2 - a.e.\ x \in K_1, \tag{C-5}$$

but for $x \in K_1 \subseteq [X_l^0, X_r^0] \setminus \{0\}$ we have $V(x|\, p, \tau) = V(0) - |x|$ and $V(0) - |x| \geq \alpha \cdot \rho_1$. Then Eq. (C-5) implies the existence of $x \in (X_l^0, X_r^0) \setminus \{0\}$ such that

$$\alpha \cdot \rho_1 < U\Big(x, \rho_1, s^\tau(x)\Big) \leq \alpha \cdot \rho_1,$$

yielding a contradiction. Next we show that

$$s^\tau(y) = \mu_1(y), \quad \Gamma - a.e.\ y \ \text{ in } [X_l, X_r] \setminus [X_l^0, X_r^0].$$

By Lemma A-2 we have that

$$U\Big(x, \rho_1, s^\tau(x)\Big) = V(x) = V(0) - |x|, \quad \Gamma - a.e.\ x \ \text{ in } [X_l, X_r] \setminus [X_l^0, X_r^0], \tag{C-6}$$

but for any $x \in [X_l, X_r] \setminus [X_l^0, X_r^0]$ the definition of X_l^0 and X_r^0 imply that $V(0) - |x| < \alpha \cdot \rho_1$. Thus Eq. (C-6) and the definition of $U\Big(x, \rho_1, s^\tau(x)\Big)$ deliver

$$\lambda_1 \cdot \overline{F}(\rho_1)/s^\tau(x) < 1, \quad \Gamma - a.e.\ x \ \text{ in } [X_l, X_r] \setminus [X_l^0, X_r^0].$$

Using the again Eq. (C-6) and the definition of $U\Big(x, \rho_1, s^\tau(x)\Big)$ we conclude that

$$s^\tau(x) = \alpha \cdot \rho_1 \cdot \frac{\overline{F}(\rho_1)}{V(0) - |x|}, \quad \Gamma - a.e.\ x \ \text{ in } [X_l, X_r] \setminus [X_l^0, X_r^0],$$

as needed. Next we compute $s^\tau(0)$,

$$\begin{aligned}
s^\tau(0) \cdot \Gamma(\{0\}) &= \int_{\{0\}} s^\tau(x)\, d\Gamma \\
&= \tau_2(\{0\}) \\
&= \tau(\mathcal{C} \times \{0\}) \\
&= \tau([X_l, X_r] \times \{0\}) \\
&= \underbrace{\tau([X_l^0, X_r^0] \times \{0\})}_{(1)} + \underbrace{\tau([X_l, X_r] \setminus [X_l^0, X_r^0] \times \{0\})}_{(2)},
\end{aligned}$$

for (1) we have

$$\begin{aligned}
\tau([X_l^0, X_r^0] \times \{0\}) &= \mu([X_l^0, X_r^0]) - \tau([X_l^0, X_r^0] \times \mathcal{C} \setminus \{0\}) \\
&= 2\mu_1 \cdot X_r^0 - \tau([X_l^0, X_r^0] \times [X_l^0, X_r^0] \setminus \{0\}) \\
&\overset{(a)}{=} 2\mu_1 \cdot X_r^0,
\end{aligned}$$

in (a) we use $s^\tau(x) = 0,\ \Gamma - a.e.\ x$ in $[X_l^0, X_r^0] \setminus \{0\}$. For (2) we have

$$\begin{aligned}
\tau([X_l, X_r] \setminus [X_l^0, X_r^0] \times \{0\}) &= \mu([X_l, X_r] \setminus [X_l^0, X_r^0]) - \tau([X_l, X_r] \setminus [X_l^0, X_r^0] \times [X_l, X_r] \setminus \{0\}) \\
&= 2\mu_1 \cdot (X_r - X_r^0) - \tau([X_l, X_r] \setminus [X_l^0, X_r^0] \times [X_l^0, X_r^0] \setminus \{0\}) \\
&- \tau([X_l, X_r] \setminus [X_l^0, X_r^0] \times [X_l, X_r] \setminus [X_l^0, X_r^0]) \\
&= 2\mu_1 \cdot (X_r - X_r^0) - 0 - \tau_2([X_l, X_r] \setminus [X_l^0, X_r^0]) \\
&= 2\mu_1 \cdot (X_r - X_r^0) - \int_{[X_l, X_r] \setminus [X_l^0, X_r^0]} \mu_1(x)\, d\Gamma,
\end{aligned}$$

from this we conclude that

$$s^\tau(0) = 2 \cdot \mu_1 \cdot X_r^0 + 2 \int_{X_r^0}^{X_r} (\mu_1 - \mu_1(x))\, dx.$$

Step 3: Now we can provide a full solution for the optimization problem. Recall that we are only optimizing over $p(0)$ or, equivalently, over $V(0)$. By our congestion bound (see Proposition 5), any solution has to satisfy $V(0|\, p, \tau) \leq \psi_0(s^\tau(0))$. Moreover, Step 2 characterizes the supply-demand ratio at every location as a function of $V(0)$. Thus, the following formulation is a natural relaxation for the platform's problem

$$\begin{aligned}
\max_{V(0)} \quad & V(0) \cdot S_T + 2 \cdot \psi_1 \cdot \mu_1 \cdot (H - X_r^0) && (\mathcal{P}_{loc-reac}) \\
\text{s.t} \quad & X_r^0 = (V(0) - \alpha \cdot \rho_1))^+, \quad X_r = V(0) - \psi_1 \\
& S_T = 2X_r^0 \mu_1 + 2 \int_{X_r^0}^{X_r} (\mu_1 - \mu_1(x)) dx \\
& \psi_1 < V(0) \leq \psi_0(S_T)
\end{aligned}$$

We show that the optimal $V^\star(0)$ in $(\mathcal{P}_{loc-reac})$ is the unique solution to

$$V^\star(0) = \psi_0(S_T(V^\star(0))).$$

The optimal solution to the platform's problem set price at the origin $p^\star(0) = \rho_0^{loc}(S_T(V^\star(0)))$ such that $p^\star(0) \geq \rho_1$, and flows for any measurable set $\mathcal{B} \subset \mathcal{C} \times \mathcal{C}$ given by

$$\begin{aligned}
\tau(\mathcal{B}) &= \mu(\pi_1(\mathcal{B} \cap \mathcal{D}) \cap [X_l, X_r]^c) + \mu(\pi_1(\mathcal{B} \cap [X_l^0, X_r^0] \times \{0\})) \\
&+ G_1(\pi_1(\mathcal{B} \cap \mathcal{D}) \cap [X_l, X_r] \setminus [X_l^0, X_r^0]) + G_0(\pi_1(\mathcal{B} \cap [X_l, X_r] \setminus [X_l^0, X_r^0] \times \{0\})),
\end{aligned}$$

where G_0, G_1 are measures defined for any measurable set $\mathcal{L} \subset [X_l, X_r] \setminus [X_l^0, X_r^0]$ by

$$G_0(\mathcal{L}) \triangleq \int_{\mathcal{L}} (\mu_1 - \mu_1(x))\, dm(x), \quad G_1(\mathcal{L}) \triangleq \int_{\mathcal{L}} \mu_1(x)\, dm(x).$$

Proof of Step 3: The proof consists of two parts. First, we show that $V^\star(0)$ as stated above is an optimal solution for $(\mathcal{P}_{loc-reac})$. To do this we prove that $S_T(V(0))$ is increasing for $V(0) > \psi_1$, with $S_T(\psi_1) = 0$. This implies that $\psi_0(S_T(V(0)))$ is decreasing and, therefore, it crosses with $V(0)$ at only one point. Then, we show the objective function increases with $V(0)$. These two facts imply the optimality of $V^\star(0)$. Second, we show that (p, τ) with $p(0) = p^\star(0)$ (and equal to ρ_1 for $x \neq 0$) and and τ as stated above, are a feasible price-equilibrium pair that achieve the same revenue than the optimal solution of $(\mathcal{P}_{loc-reac})$. Since this problem is a relaxation to our original optimization problem we have optimality.

We begin with the first part. Note that

$$S_T(V(0)) = 2\mu_1 \cdot (V(0) - \psi_1) + 2\psi_1 \cdot \mu_1 \cdot \log\Big(\frac{\psi_1}{V(0) - (V(0) - \alpha\rho_1)^+}\Big).$$

From this it follows that $S_T(\psi_1) = 0$. If $V(0) \geq \alpha\rho_1$ then $S_T(V(0))$ is clearly increasing. If $V(0) \in (\psi_1, \alpha\rho_1)$ then the derivative of $S_T(V(0))$ with respect to $V(0)$ equals

$$2\mu_1 - 2\psi_1 \cdot \mu_1 \cdot \frac{V(0)}{\psi_1} \cdot \frac{\psi_1}{V(0)^2} = 2\mu_1 - 2\psi_1 \cdot \mu_1 \cdot \frac{1}{V(0)},$$

which is nonnegative if and only if $V(0) \geq \psi_1$. Since this is in our domain, we conclude that $S_T(\cdot)$ is increasing in $(\psi_1, \alpha\rho_1)$ and, therefore, is increasing for all $V(0) > \psi_1$.

Next, we show the objective is increasing in $V(0)$, the objective function is

$$V(0) \cdot S_T(V(0)) + 2 \cdot \psi_1 \cdot \mu_1 \cdot (H - (V(0) - \alpha\rho_1)^+),$$

when $V(0) \geq \alpha \cdot \rho_1$, the objective becomes

$$2\mu_1 \cdot V(0) \cdot (V(0) - \psi_1) + 2\psi_1 \cdot \mu_1 \cdot V(0) \cdot \log\Big(\frac{\psi_1}{\alpha\rho_1}\Big) + 2 \cdot \psi_1 \cdot \mu_1 \cdot (H - V(0) + \alpha\rho_1).$$

Its derivative is non-negative if and only if

$$2\frac{V(0)}{\psi_1} \geq 2 + \log\Big(\frac{\alpha\rho_1}{\psi_1}\Big),$$

but from the fact that $V(0) \geq \alpha\cdot\rho_1$ and that the logarithm is a concave function the latter inequality is always true. Similarly, for $V(0) \in (\psi_1, \alpha \cdot \rho_1)$ the objective's derivative is non-negative if and only if

$$2\frac{V(0)}{\psi_1} \geq 2 + \log\Big(\frac{V(0)}{\psi_1}\Big),$$

which, since $V(0) > \psi_1$, is always true. Observe that in both cases the inequalities for the sign of the objective's derivative is strict except when $V(0) = \psi_1$. Thus, the objective is strictly increasing in the domain.

For the second part we need to show that (p, τ) with $p(0) = p^\star(0)$ (and equal to ρ_1 for $x \neq 0$) and and τ, implement the solution of $(\mathcal{P}_{loc-reac})$. To do this we first need to argue that this solution is feasible. It can be easily seen that this flow yields the exact same flows as in Step 2, only this time we replace $V^\star(0)$ in all the quantities that depend on $V(0)$. Given the value of s^τ and the fact that under $p^\star(0)$ we have $U(0, p(0), s^\tau(0)) = V(0|\, p, \tau) = V^\star(0)$, we can do the same as we did in Step 1(to show that $\tilde{\tau}$ is an equilibrium) and show that τ is an equilibrium. Since we have pinned the value of $V(0|\, p, \tau)$ (and thus the value of $V(|\, p, \tau)$ in the entire city) and the value of $s^\tau(\cdot)$, it is easy to see (using Proposition 1) that $\frac{1}{\gamma} \cdot \mathbf{Rev}(p, \tau)$ coincides with the optimal value of $(\mathcal{P}_{loc-reac})$. Therefore, (p, τ) is the optimal solution.

To conclude we argue that $p^\star(0) \geq \rho_1$. There are two cases. If $\mu_1 \leq \lambda_1 \cdot \overline{F}(\rho_1)$ then ψ_1 equals $\alpha \cdot \rho_1$. Since $V^\star(0) > \psi$ and $V^\star(0) = \psi_0(S_T(V^\star(0)))$ we have have that

$$\alpha \cdot \rho_1 = \psi_1 < V^\star(0) = \psi_0(S_T(V^\star(0))) \leq \alpha \cdot \rho_0^{loc}(S_T(V^\star(0))) = \alpha \cdot p^\star(0),$$

that is, $\rho_1 < p^\star(0)$. The second case is $\mu_1 > \lambda_1 \cdot \overline{F}(\rho_1)$. Here ρ_1 equals ρ^u and, since $\rho_0^{loc}(S_T(V^\star(0)))$ equals $\max\{\rho_0^{bal}, \rho^u\}$, we have that $\rho_1 \leq p^\star(0)$. □

Lemma C-2. *Let (p, τ) be a feasible price-equilibrium pair for either the local price response environment (Section 6.2) or the global price response environment (Section 6.3). If either $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \tau)\} = \emptyset$ or $\{x \in [-H, 0) : 0 \in \mathcal{IR}(x|p, \tau)\} = \emptyset$, then the platform's objective satisfies*

$$\gamma \cdot \mathbf{Rev}(p, \tau) \leq \psi_1 \cdot \mu_1 \cdot 2 \cdot H.$$

Proof. WLOG let us just assume that $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \tau)\} = \emptyset$. That is, for all $x \in (0, H]$ we have $0 \notin \mathcal{IR}(x|p, \tau)$. In turn, this implies that $\tau((0, H] \times [-H, 0]) = 0$ and, therefore, by Lemma C-1 we conclude that

$$V(x|p, \tau) \leq \psi_1 \qquad \Gamma - a.e. \text{ in } \quad (0, H],$$

which, from the continuity of $V(\cdot|p, \tau)$, implies that $V(x|p, \tau) \leq \psi_1$ for all $x \in [0, H]$. Now, we show that the same bound holds for $x \in [-H, 0)$. If $\tau([-H, 0) \times \mathcal{B}) = 0$ for any $\mathcal{B} \subset [0, H]$, we can use Lemma C-1 to obtain the upper bound. On the other hand, if there exists $\mathcal{B} \subset [0, H]$ such that $\tau([-H, 0) \times \mathcal{B}) > 0$ then by Lemma A-6 we know there exists a pair $(x, y) \in [-H, 0) \times \mathcal{B}$ for which $y \in \mathcal{IR}(x|\, p, \tau)$. Thus, we can define

$$\underline{x} = \inf\{z \in [-H, 0) : y \in \mathcal{IR}(z|\, p, \tau)\},$$

and by Lemma A-7, $y \in \mathcal{IR}(\underline{x}|\, p, \tau)$. Also, from Lemma A-5 we have

$$V(z|p, \tau) = V(\underline{x}|p, \tau) + z - \underline{x}, \quad \forall\, z \in [\underline{x}, y].$$

This implies $V(z|p,\tau) \leq V(y|p,\tau)$ for all $z \in [\underline{x}, y]$, and because $y \in \mathcal{B} \subset [0,H]$ we have $V(y|p,\tau) \leq \psi_1$ yielding

$$V(z|p,\tau) \leq \psi_1 \qquad \forall\, z \in [\underline{x}, y].$$

Furthermore, from Lemma A-6 and the definition of $\underline{x}$ we can conclude that $\tau([-H,\underline{x}] \times (\underline{x}, H]) = 0$ which together with Lemma C-1 and the continuity of V imply that $V(x|p,\tau) \leq \psi_1$ for all $x \in [-H, \underline{x}]$. This completes the argument for the upper bound.

In order to bound the revenue, simply note that

$$\frac{1}{\gamma} \cdot \mathbf{Rev}(p,\tau) = \int_{\mathcal{C}} V(x) s^{\tau}(x)\, d\mathit{\Gamma}(x) \leq \psi_1 \cdot \int_{\mathcal{C}} s^{\tau}(x)\, d\mathit{\Gamma}(x) = \psi_1 \cdot \mu_1 \cdot 2 \cdot H.$$

□

Lemma C-3. *(Equilibria Separation and Pasting) Consider a set $\mathcal{B} \subset \mathcal{C}$ such that both $\mathcal{B}$ and $\mathcal{B}^c$ are intervals or union of intervals with $\mathit{\Gamma}(\partial\mathcal{B}) = 0$.*

1. *(Separation) Let (p,τ) be a price-equilibrium in $\mathcal{C}$, if $\tau(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\tau(\mathcal{B}^c \times \mathcal{B}) = 0$ then $(p|_{\mathcal{B}}, \tau|_{\mathcal{B}\times\mathcal{B}})$ and $(p|_{\mathcal{B}^c}, \tau|_{\mathcal{B}^c\times\mathcal{B}^c})$ are price-equilibrium pairs in $\mathcal{B}$ and $\mathcal{B}^c$, respectively. Moreover, $V(\cdot|\, p|_{\mathcal{B}}, \tau|_{\mathcal{B}\times\mathcal{B}})$ equals $V(\cdot|\, p|_{\mathcal{B}^c}, \tau|_{\mathcal{B}^c\times\mathcal{B}^c})$ in $\partial\mathcal{B}$, $V(\cdot|\, p|_{\mathcal{B}}, \tau|_{\mathcal{B}\times\mathcal{B}})$ coincides with $V(\cdot|\, p,\tau)|_{\mathcal{B}}$ and the same holds for $\mathcal{B}^c$.*

2. *(Pasting) Suppose we have two price-equilibrium pairs $(p^{\mathcal{B}}, \tau^{\mathcal{B}})$ and $(p^{\mathcal{B}^c}, \tau^{\mathcal{B}^c})$ in $\mathcal{B}$ and $\mathcal{B}^c$ such that $\tau^{\mathcal{B}} \in \mathcal{F}_{\mathcal{B}}(\mu|_{\mathcal{B}})$ and $\tau^{\mathcal{B}^c} \in \mathcal{F}_{\mathcal{B}^c}(\mu|_{\mathcal{B}^c})$, respectively. If $V(\cdot|\, p^{\mathcal{B}}, \tau^{\mathcal{B}})$ equals $V(\cdot|\, p^{\mathcal{B}^c}, \tau^{\mathcal{B}^c})$ in $\partial\mathcal{B}$ then the flow τ defined by for any measurable set $\mathcal{L} \subseteq \mathcal{C} \times \mathcal{C}$*

$$\tau(\mathcal{L}) = \tau^{\mathcal{B}}(\mathcal{L} \cap \mathcal{B} \times \mathcal{B}) + \tau^{\mathcal{B}^c}(\mathcal{L} \cap \mathcal{B}^c \times \mathcal{B}^c),$$

belongs to $\mathcal{F}(\mu)$ and is an equilibrium in $\mathcal{C}$ for a price p equal to $p^{\mathcal{B}}$ in $\mathcal{B}$ and equal to $p^{\mathcal{B}^c}$ in $\mathcal{B}^c$. Moreover, $V(x|p,\tau) = V(x|\, p^{\mathcal{B}}, \tau^{\mathcal{B}})$ in $\mathcal{B}$ and $V(x|p,\tau) = V(x|\, p^{\mathcal{B}^c}, \tau^{\mathcal{B}^c})$ in $\mathcal{B}^c$.

Proof. **Separation.** Suppose that $\tau(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\tau(\mathcal{B}^c \times \mathcal{B}) = 0$. Let $\tau^{\mathcal{B}} = \tau|_{\mathcal{B}\times\mathcal{B}}$ and $p^{\mathcal{B}} = p|_{\mathcal{B}}$, we show that $(p^{\mathcal{B}}, \tau^{\mathcal{B}})$ is a price-equilibrium pair. The proof for $(p|_{\mathcal{B}^c}, \tau|_{\mathcal{B}^c\times\mathcal{B}^c})$ is analogous and, thus, omitted. We need to prove that $\tau^{\mathcal{B}} \in \mathcal{F}_{\mathcal{B}}(\mu^{\mathcal{B}})$, where $\mu^{\mathcal{B}}$ coincides with $\mu|_{\mathcal{B}}$, and that the set

$$\mathcal{E}|_{\mathcal{B}} \triangleq \left\{ (x,y) \in \mathcal{B} \times \mathcal{B} : \mathit{\Pi}(x,y,p^{\mathcal{B}}(y), \frac{d\tau_2^{\mathcal{B}}}{d\mathit{\Gamma}|_{\mathcal{B}}}(y)) = \operatorname*{ess\,sup}^{\mathcal{B}} \mathit{\Pi}(x,\cdot,p^{\mathcal{B}}(\cdot), \frac{d\tau_2^{\mathcal{B}}}{d\mathit{\Gamma}|_{\mathcal{B}}}(\cdot)\Big) \right\},$$

satisfies $\tau_{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}) = \mu|_{\mathcal{B}}(\mathcal{B})$.

First we verify that $\tau^{\mathcal{B}} \in \mathcal{F}_{\mathcal{B}}(\mu^{\mathcal{B}})$. Since $\tau^{\mathcal{B}}$ is the restriction of τ to $\mathcal{B} \times \mathcal{B}$ it clearly belongs to $\mathcal{M}(\mathcal{B} \times \mathcal{B})$. Also, for any $\mathcal{L}_1$ measurable subset of $\mathcal{B}$ we have that

$$\begin{aligned}\tau_1^{\mathcal{B}}(\mathcal{L}_1) &= \tau^{\mathcal{B}}(\mathcal{L}_1 \times \mathcal{B}) \\ &= \tau((\mathcal{L}_1 \times \mathcal{B}) \cap (\mathcal{B} \times \mathcal{B})) \\ &= \tau(\mathcal{L}_1 \times \mathcal{B}) \\ &= \tau(\mathcal{L}_1 \times \mathcal{C}) \\ &= \tau_1(\mathcal{L}_1) \\ &= \mu(\mathcal{L}_1).\end{aligned}$$

Thus, $\tau_1^{\mathcal{B}} = \mu|_{\mathcal{B}}$. Now we need to prove that $\tau_2^{\mathcal{B}} \ll \Gamma|_{\mathcal{B}}$. Observe that for any $\mathcal{L}_2$ measurable subset of $\mathcal{B}$ we have that

$$\begin{aligned}\tau_2^{\mathcal{B}}(\mathcal{L}_2) &= \tau_{\mathcal{B}}(\mathcal{B} \times \mathcal{L}_2) \\ &= \tau((\mathcal{B} \times \mathcal{L}_2) \cap (\mathcal{B} \times \mathcal{B})) \\ &= \tau(\mathcal{B} \times \mathcal{L}_2) \\ &= \tau(\mathcal{C} \times \mathcal{L}_2) \\ &= \tau_2(\mathcal{L}_2),\end{aligned}$$

that is, $\tau_2^{\mathcal{B}} = \tau_2|_{\mathcal{B}}$. Therefore, since $\tau_2 \ll \Gamma$, we have that $\tau_2^{\mathcal{B}} \ll \Gamma|_{\mathcal{B}}$. In turn, $\tau^{\mathcal{B}} \in \mathcal{F}_{\mathcal{B}}$.

Now we show $\tau^{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}) = \mu|_{\mathcal{B}}(\mathcal{B})$. It suffices to prove that $\tau^{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}^c) = 0$ where the complement is taken with respect to $\mathcal{B} \times \mathcal{B}$, we do this by contradiction. Assume that $\tau^{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}^c) > 0$, this implies that

$$0 < \tau^{\mathcal{B}}(\mathcal{E}|_{\mathcal{B}}^c) = \tau(\mathcal{E}|_{\mathcal{B}}^c),$$

and we must have that $\tau_2(\mathcal{B}) > 0$, indeed

$$0 < \tau(\mathcal{E}|_{\mathcal{B}}^c) \leq \tau(\mathcal{C} \times \mathcal{B}) = \tau_2(\mathcal{B}).$$

Next, observe that for any $\mathcal{L}_2$ measurable subset of $\mathcal{B}$

$$\tau_2^{\mathcal{B}}(\mathcal{L}_2) = \tau_2(\mathcal{L}_2) = \int_{\mathcal{L}_2} s^{\tau}(x)\, d\Gamma(x) = \int_{\mathcal{L}_2} s^{\tau}(x)\, d\Gamma|_{\mathcal{B}}(x),$$

therefore,

$$\frac{d\tau_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(x) \;=\; s^{\tau}(x), \qquad \Gamma - a.e.\ x \text{ in } \mathcal{B}. \tag{C-7}$$

This implies that

$$V(x|p^{\mathcal{B}}, \tau^{\mathcal{B}}) = \operatorname*{ess\,sup}_{\mathcal{B}} \Pi\Big(x, \cdot, p^{\mathcal{B}}(\cdot), \frac{d\tau_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(\cdot)\Big) = \operatorname*{ess\,sup}_{\mathcal{B}} \Pi\Big(x, \cdot, p(\cdot), \frac{d\tau_2}{d\Gamma}(\cdot)\Big) = V_{\mathcal{B}}(x|p, \tau). \tag{C-8}$$

Consider the set $\mathcal{G} \triangleq \{y \in \mathcal{B} : \frac{d\tau_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(y) = s^{\tau}(y)\}$. Then, by Eq. (C-7) we have

$$\tau(\mathcal{E}|_{\mathcal{B}}^c \cap (\mathcal{B} \times \mathcal{G}^c)) \leq \tau(\mathcal{C} \times \mathcal{G}^c) = \tau_2(\mathcal{G}^c) = 0,$$

where the complement is take with respect to $\mathcal{B}$. Therefore, $0 < \tau(\mathcal{E}|_{\mathcal{B}}^c) = \tau(\mathcal{E}|_{\mathcal{B}}^c \cap (\mathcal{B} \times \mathcal{G}))$ and we can conclude that

$$\tau\Big(\underbrace{\{(x,y) \in \mathcal{B} \times \mathcal{B} : \Pi(x, y, p(y), \frac{d\tau_2}{d\Gamma}(y)) \neq V_{\mathcal{B}}(x|p,\tau)\}}_{\triangleq R}\Big) > 0.$$

Define the sets R^- and R^+ by

$$R^- = \{(x,y) \in \mathcal{B} \times \mathcal{B} : \Pi(x, y, p(y), \frac{d\tau_2}{d\Gamma}(y)) > V_{\mathcal{B}}(x|p,\tau)\}$$
$$R^+ = \{(x,y) \in \mathcal{B} \times \mathcal{B} : \Pi(x, y, p(y), \frac{d\tau_2}{d\Gamma}(y)) < V_{\mathcal{B}}(x|p,\tau)\},$$

and note that $R = R^- \cup R^+$. To obtain a contradiction we argue that $\tau(R^- \cup R^+) = 0$. Consider first the set R^+, and note that $\tau(R^+) = \tau(R^+ \cap \mathcal{E})$. However, any $(x,y) \in R^+ \cap \mathcal{E}$ satisfies

$$\Pi(x, y, (p(y), \frac{d\tau_2}{d\Gamma}(y)) < V_{\mathcal{B}}(x|p,\tau) \text{ and } \Pi(x, y, (p(y), \frac{d\tau_2}{d\Gamma}(y)) = V(x|p,\tau),$$

but $V(x) \geq V_{\mathcal{B}}(x)$ implies that $R^+ \cap \mathcal{E} = \emptyset$ and, therefore, $\tau(R^+) = 0$.

Consider R^- and define

$$A \triangleq \{y \in \mathcal{B} : U(y) = V_{\mathcal{B}}(y|p,\tau)\},$$

then by Lemma A-2 we have $\tau(R^-) = \tau(R^- \cap (\mathcal{B} \times A))$. Take any $(x,y) \in R^- \cap (\mathcal{B} \times A)$ then

$$V_{\mathcal{B}}(y|p,\tau) - |y - x| > V_{\mathcal{B}}(x|p,\tau),$$

which, because of the Lipchitz property (see Lemma 1), is not possible. Thus, $R^- \cap (\mathcal{B} \times A)) = \emptyset$ and we have that $\tau(R^-) = 0$. This proves that $\tau^{\mathcal{B}}$ is an equilibrium in $\mathcal{B}$.

Now we show that $V(x|p^{\mathcal{B}}, \tau^{\mathcal{B}})$ equals $V(x|p^{\mathcal{B}^c}, \tau^{\mathcal{B}^c})$ for all $x \in \partial\mathcal{B}$. Recall that from equation (C-8) we have

$$V(x|p^{\mathcal{B}}, \tau^{\mathcal{B}}) = V_{\mathcal{B}}(x|p,\tau) \quad \text{and} \quad V(x|p^{\mathcal{B}^c}, \tau^{\mathcal{B}^c}) = V_{\mathcal{B}^c}(x|p,\tau),$$

so we just need to show $V_{\mathcal{B}}(x|p,\tau)$ equals $V_{\mathcal{B}^c}(x|p,\tau)$ for all $x \in \partial B$. We first show that $V_{\mathcal{B}}(x|p,\tau) = V(x|p,\tau)$ for all $x \in \mathcal{B}$. Let $x \in \mathcal{B}$, since $\mathcal{B}$ is an interval or a union of intervals we must have $\mu(B(x, \frac{1}{n}) \cap \mathcal{B}) > 0$ for all $n \in \mathbb{N}$. In turn, this implies

$$\begin{aligned}
0 &< \tau(B(x, \frac{1}{n}) \cap \mathcal{B} \times \mathcal{B}) \\
&= \tau(B(x, \frac{1}{n}) \cap \mathcal{B} \times \mathcal{B}^{\circ}) + \tau(B(x, \frac{1}{n}) \cap \mathcal{B} \times \partial\mathcal{B}) \\
&= \tau(B(x, \frac{1}{n}) \cap \mathcal{B} \times \mathcal{B}^{\circ}),
\end{aligned}$$

where the third line comes from $\tau_2 \ll \mathit{\Gamma}$ and $\mathit{\Gamma}(\partial\mathcal{B}) = 0$. Thus, from Lemma A-6 there exists $(z_n, y_n) \in B(x, \frac{1}{n}) \cap \mathcal{B} \times \mathcal{B}^{\circ}$ such that $y_n \in \mathcal{IR}(z_n|\, p, \tau)$. Then,

$$\forall n \in \mathbb{N},\ \exists\, \delta(n) > 0 \ \text{ such that } \forall \delta \leq \delta(n) \quad \frac{1}{n} + V_{B(y_n,\delta)}(z_n) \geq V(z_n). \tag{C-9}$$

Note that since $y_n \in \mathcal{B}^{\circ}$ we can always find δ_0 such that $B(y_n, \delta) \subseteq \mathcal{B}$ for all $\delta \leq \delta_0$. So we can consider $\delta \leq \delta_0 \vee \delta(n)$ in Eq. (C-9). Using that $z_n \in B(x, \frac{1}{n})$ and the Lipschitz property (see Lemma 1) we have

$$V_{B(y_n,\delta)}(z_n) - V_{B(y_n,\delta)}(x) \leq \frac{1}{n} \quad \text{and} \quad V(z_n) - V(x) \geq -\frac{1}{n},$$

plugging this into Eq. (C-9) yields

$$\forall n \in \mathbb{N},\ \exists\, \delta(n) > 0 \ \text{ such that } \forall \delta \leq \delta_0 \vee \delta(n) \quad \frac{3}{n} + V_{B(y_n,\delta)}(x) \geq V(x).$$

Since $B(y_n, \delta) \subseteq \mathcal{B}$ we have $V_{\mathcal{B}}(x) \geq V_{B(y_n,\delta)}(x)$ thus the former expression implies that $V_{\mathcal{B}}(x) \geq V(x)$. But we always have that $V(x) \geq V_{\mathcal{B}}(x)$ and, therefore, $V(x) = V_{\mathcal{B}}(x)$. The same argument shows that $V(x) = V_{\mathcal{B}^c}(x)$ for all $x \in \mathcal{B}^c$.

To conclude we need to prove that $V_{\mathcal{B}}(x|p, \tau)$ equals $V_{\mathcal{B}^c}(x|p, \tau)$ for all $x \in \partial B$. Consider $x \in \partial\mathcal{B}$. Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{B}$ be a sequence converging to x. Then the continuity of $V_{\mathcal{B}}$ implies $V_{\mathcal{B}}(x_n) \to V_{\mathcal{B}}(x)$. At the same time, since $x_n \in \mathcal{B}$ we have $V_{\mathcal{B}}(x_n) = V(x_n)$ and by continuity $V(x_n) \to V(x)$. Then $V_{\mathcal{B}}(x) = V(x)$ and the same is true for $\mathcal{B}^c$, which implies $V_{\mathcal{B}}(x|p, \tau) = V_{\mathcal{B}^c}(x|p, \tau)$ for all $x \in \partial B$.

Pasting. First we check that $\tau \in \mathcal{F}(\mu)$. Let $\mathcal{L}_1$ be any measurable subset of $\mathcal{C}$ we have that

$$\begin{aligned}
\tau_1(\mathcal{L}_1) &= \tau(\mathcal{L}_1 \times \mathcal{C}) \\
&= \tau^{\mathcal{B}}((\mathcal{L}_1 \times \mathcal{C}) \cap (\mathcal{B} \times \mathcal{B})) + \tau^{\mathcal{B}^c}((\mathcal{L}_1 \times \mathcal{C}) \cap (\mathcal{B}^c \times \mathcal{B}^c)) \\
&= \tau^{\mathcal{B}}((\mathcal{L}_1 \cap \mathcal{B}) \times \mathcal{B}) + \tau^{\mathcal{B}^c}((\mathcal{L}_1 \cap \mathcal{B}^c) \times \mathcal{B}^c) \\
&= \mu|_{\mathcal{B}}(\mathcal{L}_1 \cap \mathcal{B}) + \mu|_{\mathcal{B}^c}(\mathcal{L}_1 \cap \mathcal{B}^c) \\
&= \mu(\mathcal{L}_1).
\end{aligned}$$

Also, if $\mathit{\Gamma}(\mathcal{L}_1) = 0$ then $\mathit{\Gamma}|_{\mathcal{B}}(\mathcal{L}_1) = \mathit{\Gamma}|_{\mathcal{B}^c}(\mathcal{L}_1) = 0$. Therefore, $\tau_2^{\mathcal{B}}(\mathcal{L}_1) = \tau_2^{\mathcal{B}^c}(\mathcal{L}_1) = 0$, which in turn implies $\tau_2 \ll \mathit{\Gamma}$. Hence $\tau \in \mathcal{F}(\mu)$.

Now we show the set

$$\mathcal{E} \triangleq \Big\{(x, y) \in \mathcal{C} \times \mathcal{C} : \mathit{\Pi}(x, y, p(y), s^{\tau}(y)) = \operatorname*{ess\,sup}_{\mathcal{C}} \mathit{\Pi}(x, \cdot, p(\cdot), s^{\tau}(\cdot)\Big)\Big\},$$

satisfies $\tau(\mathcal{E}) = \mu(\mathcal{C})$. Note that

$$\mathcal{E} \cap \mathcal{B} \times \mathcal{B} = \Big\{(x, y) \in \mathcal{B} \times \mathcal{B} : \mathit{\Pi}(x, y, p(y), s^{\tau}(y)) = V(x|p, \tau)\Big\}.$$

It is enough to prove that $\tau^{\mathcal{B}}(\mathcal{E} \cap \mathcal{B} \times \mathcal{B}) = \mu(\mathcal{B})$. As we did in the first part of the proof (see Eq. (C-7)) we can show that

$$\frac{d\tau_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(x) \;=\; s^{\tau}(x), \qquad \Gamma - a.e.\ x \text{ in } \mathcal{B},$$

so if we prove that $V(\cdot|p,\tau)|_{\mathcal{B}} \equiv V(\cdot|p^{\mathcal{B}},\tau^{\mathcal{B}})$ we will be done (the proof for $\mathcal{B}^c$ is analogous). Fix $x \in \mathcal{B}$, as in Eq. (C-8) we have

$$V(x|p^{\mathcal{B}},\tau^{\mathcal{B}}) = \operatorname*{ess\,sup}_{\mathcal{B}} \Pi\Big(x,\cdot,p^{\mathcal{B}}(\cdot),\frac{d\tau_2^{\mathcal{B}}}{d\Gamma|_{\mathcal{B}}}(\cdot)\Big) = \operatorname*{ess\,sup}_{\mathcal{B}} \Pi\Big(x,\cdot,p(\cdot),\frac{d\tau_2}{d\Gamma}(\cdot)\Big) = V_{\mathcal{B}}(x|p,\tau).$$

So we just need to verify that $V(x|p,\tau) = V_{\mathcal{B}}(x|p,\tau)$. We show that $V(x|p,\tau) \le V_{\mathcal{B}}(x|p,\tau)$, the other inequality always holds. Let $I(x)$ be the interval in $\mathcal{B}$ to which x belongs to. Let $y_L = \inf I(x)$ and $y_U = \sup I(x)$, note that y_L and y_U do not necessarily belong to $\mathcal{B}$ but they do belong to $\partial\mathcal{B}$. Then by assumption we have $V(y|\,p^{\mathcal{B}},\tau_{\mathcal{B}})$ equals $V(y|\,p^{\mathcal{B}^c},\tau^{\mathcal{B}^c})$ for $y \in \{y_L,y_U\}$, in turn this implies that $V_{\mathcal{B}}(y|p,\tau)$ equals $V_{\mathcal{B}^c}(y|p,\tau)$ for $y \in \{y_L,y_U\}$. Now, consider the sets $\mathcal{B}^c_L = [H,y_L] \cap \mathcal{B}^c$ and $\mathcal{B}^c_U = [y_U,H] \cap \mathcal{B}^c$ then

$$\begin{aligned}
V_{\mathcal{B}}(x|p,\tau) &\overset{(a)}{\ge} V_{\mathcal{B}}(y_U|p,\tau) - |x - y_U| \\
&= V_{\mathcal{B}}(y_U|p,\tau) - (y_U - x) \\
&\overset{(b)}{\ge} U(w,s^{\tau}(w)) - |y_U - w| - (y_U - x), \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}^c_U \\
&\overset{(c)}{\ge} U(w,s^{\tau}(w)) - (w - y_U) - (y_U - x), \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}^c_U \\
&\overset{(d)}{\ge} U(w,s^{\tau}(w)) - |w - x|, \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}^c_U,
\end{aligned}$$

where (a) comes from the Lipschitz property (see Lemma 1), (b) comes from the definition of $V_{\mathcal{B}}(y_U|p,\tau)$ together with $\Gamma(\mathcal{B}^c_U) > 0$, and $(c),(d)$ hold because for $w \in \mathcal{B}^c_U$ we have $x \le y_U \le w$. Similarly,

$$\begin{aligned}
V_{\mathcal{B}}(x|p,\tau) &\ge V_{\mathcal{B}}(y_L|p,\tau) - |x - y_L| \\
&= V_{\mathcal{B}}(y_L|p,\tau) - (x - y_L) \\
&\ge U(w,s^{\tau}(w)) - |y_L - w| - (x - y_L), \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}^c_L \\
&= U(w,s^{\tau}(w)) - (y_L - w) - (x - y_L), \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}^c_L \\
&= U(w,s^{\tau}(w)) - |w - x|, \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}^c_L.
\end{aligned}$$

Since $\mathcal{B}^c_L \cup \mathcal{B}^c_U = \mathcal{B}^c$ this implies that $V_{\mathcal{B}}(x|p,\tau) \ge V(x|p,\tau)$. This concludes the proof. $\square$

C.3 Proofs for Section 6.3

Proof of Lemma 5. Let (p, τ) be a feasible solution. We show that at any optimal solution we must have $X_l < 0 < X_r$, in turn this implies that 0 is a sink location. By Lemma C-2 we have that if either of the sets $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \tau)\}$ or $\{x \in [-H, 0) : 0 \in \mathcal{IR}(x|p, \tau)\}$ is empty then the revenue the platform makes satisfies

$$\frac{1}{\gamma} \cdot \mathbf{Rev}(p, \tau) \leq \psi_1 \cdot \mu_1 \cdot 2 \cdot H.$$

However, the solution (p, τ) given in Proposition 7 has both sets non-empty because $0 \in \mathcal{IR}(X_r|p, \tau)$ and $0 \in \mathcal{IR}(-X_r|\, p, \tau)$ with $X_r > 0$. Furthermore, $\mathbf{Rev}(p, \tau)$ is strictly large than the revenue of the pre-demand shock environment or, equivalently, strictly larger than $\psi_1 \cdot \mu_1 \cdot 2 \cdot H$. This implies that any optimal solution must satisfy $\{x \in (0, H] : 0 \in \mathcal{IR}(x|p, \tau)\} \neq \emptyset$ and $\{x \in [-H, 0) : 0 \in \mathcal{IR}(x|p, \tau)\} \neq \emptyset$ and, therefore, $X_l < 0 < X_r$.

□

Proof of Lemma 6. If $X_r = H$ there is nothing to prove, so let's assume $X_r < H$. Fix $x \in [X_r, H]$. From the Lipschitz property (see Lemma 1) we have that $V(x|p, \tau) \leq V(X_r|p, \tau) + (x - X_r)$. Moreover, Proposition 3 ensures that $\tau([X_r, H] \times [X_r, H]^c) = 0$ and, hence, because $0 \notin [X_r, H]$ we can apply Lemma C-1 to deduce that

$$V(x|p, \tau) \; \leq \; \psi_1, \quad \mathit{\Gamma} - a.e.\ x \text{ in } [X_r, H]. \tag{C-10}$$

To show that the previous inequality holds everywhere, notice that if $V(x|\, p, \tau) > \psi_1$ the from the Lipschitz continuity property of $V(\cdot|\, p, \tau)$ we could find a subset of of $[X_r, H]$ with positive $\mathit{\Gamma}$ measure (in this set $\mathit{\Gamma}$ coincides with the Lebesgue measure) in which $V(\cdot|\, p, \tau)$ is strictly larger than ψ_1. This is not possible because it would contradict Eq. (C-10). Putting together both upper bounds yields the desire result.

□

Proof of Proposition 8. Let (p, τ) be optimal for problem $(\mathcal{P}_2)$ as in Lemma 5 so we have $0 < X_r$. Note that is $X_r = H$ then the result trivially holds, so let's assume $X_r < H$. Before we begin note that for any $x \geq X_r$, by Lemma 6 and the Lipschitz continuity property of $V(\cdot|\, p, \tau)$ (see Lemma 1), we must have $V(x) \leq \psi_1$.

We first prove the second statement of the proposition. Suppose $V(X_r) = \psi_1$ and define the set

$$R \triangleq \{x \in [X_r, H] : V(x) = \psi_1\}.$$

We show by contradiction that we cannot have $\tau_2(R^c) > 0$ (the complement is taken with respect to $[X_r, H]$). If $\tau_2(R^c) > 0$ then because ψ_1 is an upper bound from Proposition 1 we have the following

$$
\begin{aligned}
\frac{1}{\gamma} \cdot \mathbf{Rev}_{[X_r,H]}(p,\tau) &= \int_{[X_r,H]} V(x)\, d\tau_2(x) \\
&= \int_R V(x)\, d\tau_2(x) + \int_{R^c} V(x)\, d\tau_2(x) \\
&< \int_R V(x)\, d\tau_2(x) + \int_{R^c} \psi_1\, d\tau_2(x) \\
&\leq \psi_1 \cdot \tau_2([X_r, H]) \\
&= \psi_1 \cdot \mu_1 \cdot (H - X_r),
\end{aligned}
$$

where the last line comes Proposition 3. Thus, the quantity

$$
\mathbf{Rev}_{[-H,X_r]}(p,\tau) + \gamma \cdot \psi_1 \cdot \mu_1 \cdot (H - X_r),
$$

strictly upper bounds the platform's objective. So if we are able to construct a solution such that attains the upper bound, we will contradict the optimality of (p,τ). Observe that Lemma C-3 enables us to separate the solution (p,τ) in $[-H, X_r]$ and $(X_r, H]$. The separated solution $(p^{[-H,X_r]}, \tau^{[-H,X_r]})$ (see Lemma C-3 for notation) in $[-H, X_r]$ has revenue equal to $\mathbf{Rev}_{[-H,X_r]}(p,\tau)$, and $V(X_r|\, p^{[-H,X_r]}, \tau^{[-H,X_r]})$ coincides with $V(X_r|\, p,\tau)$ which equals ψ_1. For $(X_r, H]$ consider prices $\tilde{p}(x) = \rho_1$ for all $x \in (X_r, H]^c$, and flows $\tilde{\tau}(\mathcal{L}) = \mu(\pi_1(\mathcal{L} \cap \mathcal{D}))$ for any measurable set $\mathcal{L} \subset (X_r, H] \times (X_r, H]$. The pair $(\tilde{p}, \tilde{\tau})$ is the same solution as in Proposition 6 with the sole difference that we have changed the city to be $(X_r, H]$ instead of $\mathcal{C}$. Therefore, $(\tilde{p}, \tilde{\tau})$ is a feasible price-equilibrium in $(X_r, H]$ with revenue equal to $\gamma \cdot \psi_1 \cdot \mu_1 \cdot (H - X_r)$, and such that $V(x|\, \tilde{p}, \tilde{\tau})$ equal to ψ_1 for all $x \in (X_r, H]$. Thus we can use Lemma C-3 to paste both solution and obtain an equilibrium in the entire city. This new equilibrium achieves the upper bound.

Suppose that $\tau_2(R^c) = 0$ and define the sets

$$
L_+ \triangleq \{x : \mu_1 > s^\tau(x)\}, \quad L_0 \triangleq \{x : \mu_1 = s^\tau(x)\}, \quad L_- \triangleq \{x : \mu_1 < s^\tau(x)\}.
$$

Then by Lemma 5 it holds that $\Gamma(R \cap L_-) = 0$. Moreover, if $\Gamma(R \cap L_+) > 0$ we have

$$
\mu([X_r, H]) = \tau_2([X_r, H]) \overset{(a)}{=} \tau_2(R) = \int_{R \cap L_+} s^\tau(x) d\Gamma(x) + \int_{R \cap L_0} s^\tau(x) d\Gamma(x) < \mu_1 \Gamma(R) \leq \mu([X_r, H]),
$$

not possible, where (a) comes from Proposition 3. Thus $\Gamma(R \cap L_+) = 0$. This implies that $\Gamma(R \cap L_0) = \Gamma(R)$ and

$$
\mu_1 \Gamma([X_r, H]) = \mu([X_r, H]) = \int_{R \cap L_0} s^\tau(x)\, d\Gamma(x) = \mu_1 \Gamma(R),
$$

that is, $\mathit{\Gamma}(R) = \mathit{\Gamma}([X_r, H])$ or $\mathit{\Gamma}(R^c) = 0$. In turn, $\mathit{\Gamma} - a.e.$ $x \in [X_r, H]$ we have that $V(x)$ equals ψ_1. Since, $V(\cdot)$ is continuous and $\mathit{\Gamma}|_{[X_r,H]}$ has full support in $[X_r, H]$ which has non-empty interior we conclude that $V(x) = \psi_1$ for all $x \in [X_r, H]$.

For the reminder of the proof we assume $V(X_r) < \psi_1$. We show that if $V(\cdot)$ is not non-decreasing in $[X_r, H]$ then there is an strict objective improvement. In the proof we define several critical points in the interval $[X_r, H]$ which will help us to create a flow separated region (no flow leaves this region). Then we show the objective strict improvement in this region. In Figure 11 we provide a graphical representation of the points just mentioned.

So assume that $V(x)$ is not non-decreasing in $[X_r, H]$, then there exists $\hat{x} > \hat{y} \geq X_r$ such that $V(\hat{x}) < V(\hat{y})$. Let,

$$\bar{y} \triangleq \sup\{z \in [\hat{y}, \hat{x}] : V(z) = V(\hat{y})\},$$

note that since for $z = \hat{y}$, $V(z) = V(\hat{y})$ thus the set over which we take the supremum above is both bounded and non-empty. Hence, $\bar{y}$ is well defined and it corresponds to the last point z in $[\hat{y}, \hat{x}]$ such that $V(z)$ equals $V(\hat{y})$. Moreover, because $V(\cdot)$ is continuous $\bar{y} < \hat{x}$, and for all $z \in (\bar{y}, \hat{x}]$ we have $V(z) < V(\hat{y}) = V(\bar{y})$. Let

$$y_0 \triangleq \inf\{z \in [X_r, \bar{y}] : \exists x \in (\bar{y}, H] \quad \text{such that} \quad z \in \mathcal{IR}(x)\},$$

if for all $z \in [X_r, \bar{y}]$ and for all $x \in (\bar{y}, H]$ we have $z \notin \mathcal{IR}(x)$, we let $y_0 = \bar{y}$. That is, y_0 is the smallest z in $[X_r, \bar{y}]$ to which some location in $(\bar{y}, H]$ is indifferent to travel to. Note that for all $z \in (y_0, \hat{x}]$ we have $V(z) < V(y_0)$. Also, the definition of y_0 and Lemma A-6 imply that $\tau([-H, y_0] \times (y_0, H]) = 0$ and $\tau((y_0, H] \times [-H, y_0]) = 0$. Let

$$y_1 \triangleq \inf\{z \in [\hat{x}, H] : V(z) > V(y_0)\},$$

that is, y_1 is the first value after $\hat{x}$ for which $V(\cdot)$ hits $V(y_0)$. Note that when well defined y_1 satisfies that $\tau([y_1, H] \times [-H, y_1]) = 0$. If this is not the case then since atoms do not have measure we would have $\tau((y_1, H] \times [-H, y_1)) > 0$ and, therefore, by Lemma A-6 we can find $(x, y) \in (y_1, H] \times [-H, y_1)$ such that $y \in \mathcal{IR}(x)$. Then Lemma A-5 would contradict the minimality of y_1.

There are two cases:

1. y_1 is not well defined: In this case we have that for all $z \in [\hat{x}, H]$, $V(z) \leq V(y_0)$. Recall that from our previous discussion we have that $V(z) < V(y_0)$ for all $z \in (y_0, \hat{x}]$. Also, Property 1 (which we prove at the end of the present proof) establishes that $\tau_2((y_0, \hat{x}]) > 0$. Using this observations we create a new solution $(\tilde{p}, \tilde{\tau})$ with revenue strictly larger than that of (p, τ).

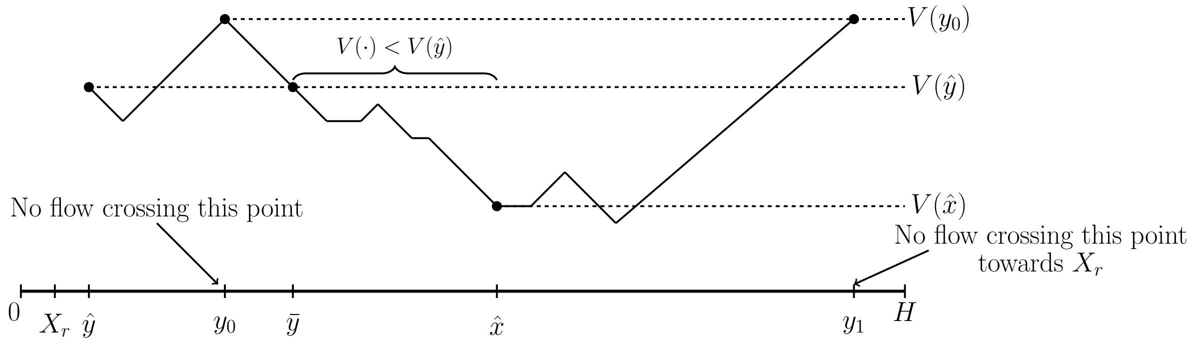

Figure 11: Graphical representation of $\hat{y}, \hat{x}, \bar{y}, y_0$ and y_1.

Let $\mathcal{B} = [-H, y_0]$ and note that we have both $\tau(\mathcal{B} \times \mathcal{B}^c) = 0$ and $\tau(\mathcal{B}^c \times \mathcal{B}) = 0$, so we can use the separation result in Lemma C-3. Hence $(p^{\mathcal{B}}, \tau^{\mathcal{B}})$ (see Lemma C-3 for notation) is a price-equilibrium pair in $\mathcal{B}$. Its revenue equals the revenue of (p, τ) in $\mathcal{B}$, and $V(y_0 |\, p^{\mathcal{B}}, \tau^{\mathcal{B}}) = V(y_0)$.

For $\mathcal{B}^c$ we choose flows $\tau^{\mathcal{B}^c}(\mathcal{L}) = \mu(\pi_1(\mathcal{L} \cap \mathcal{D}))$ for all $\mathcal{L} \subset \mathcal{B}^c \times \mathcal{B}^c$. That is all drivers stay at their initial location. It is not hard to see that $s^{\tau^{\mathcal{B}^c}}(x)$ equals μ_1, $\Gamma - a.e.\ x$ in $\mathcal{B}^c$. We choose prices $p^{\mathcal{B}^c}(x) = p_0$ for all $x \in \mathcal{B}^c$, where p_0 is such that

$$\alpha \cdot p_0 \cdot \min\{1, \frac{\lambda_1 \cdot \overline{F}(p_0)}{\mu_1}\} = V(y_0), \tag{C-11}$$

note that since $V(y_0) \leq \psi_1$, p_0 is well defined. That is, the solution $(p^{\mathcal{B}^c}, \tau^{\mathcal{B}^c})$ is the same solution as in pre-demand shock environment but in smaller city, $\mathcal{B}^c$ and with a larger price across all locations. Using Proposition 1 it is not hard to see that the revenue associated with this solution is $\gamma \cdot V(y_0) \cdot \mu_1 \cdot (H - y_0)$.

By Lemma C-3, we can paste the two previous solutions to create a new solution $(\tilde{p}, \tilde{\tau})$ in entire city. This new solution yields a strict objective improvement. Indeed,

$$\begin{aligned}
\mathbf{Rev}_{[y_0,H]}(p,\tau) &= \int_{[y_0,H]} V(x)\,d\tau_2(x) \\
&= \int_{(y_0,\hat{x}]} V(x)\,d\tau_2(x) + \int_{(\hat{x},H]} V(x)\,d\tau_2(x) \\
&\overset{(a)}{<} V(y_0)\cdot\tau_2((y_0,\hat{x}]) + \int_{(\hat{x},H]} V(x)\,d\tau_2(x) \\
&\le V(y_0)\cdot\tau_2((y_0,\hat{x}]) + V(y_0)\cdot\tau_2((\hat{x},H]) \\
&= V(y_0)\cdot\tau_2([y_0,H]) \\
&\overset{(b)}{=} V(y_0)\cdot\mu([y_0,H]) \\
&= V(y_0)\cdot\mu_1\cdot(H-y_0) \\
&= \mathbf{Rev}_{[y_0,H]}(\tilde{p},\tilde{\tau}),
\end{aligned}$$

where (a) comes from $\tau_2((y_0,\hat{x}]) > 0$, (b) comes from the fact that under τ no flow leaves or enters $[y_0,H]$, and the last two lines from the definition of $(\tilde{p},\tilde{\tau})$ restricted to $[y_0,H]$.

2. y_1 is well defined: In this case there exists $z \in [\hat{x},H]$ such that $V(z) > V(y_0)$. Also, we must have $y_1 > \hat{x}$, and we already argued that $\tau([y_1,H]\times[-H,y_1]) = 0$. There are two more cases.

   a) $\forall y \in (y_0,y_1]$, $\forall x > y_1$, $x \notin \mathcal{IR}(y)$: This together with Lemma A-6 imply that $\tau([y_0,y_1]\times([-H,y_0]\cup[y_1,H])) = 0$, and we also have $\tau(([-H,y_0]\cup[y_1,H])\times[y_0,y_1]) = 0$. From this we can construct a new feasible solution $(\tilde{p},\tilde{\tau})$ with revenue strictly larger than that of (p,τ).

   Let $\mathcal{B} = [-H,y_0)\cup(y_1,H]$ and note that we have both $\tau(\mathcal{B}\times\mathcal{B}^c) = 0$ and $\tau(\mathcal{B}^c\times\mathcal{B}) = 0$, so we can use the separation result in Lemma C-3. Thus $(p^{\mathcal{B}},\tau^{\mathcal{B}})$ (see Lemma C-3 for notation) is a price-equilibrium pair in $\mathcal{B}$. Its revenue equals the revenue of (p,τ) in $\mathcal{B}$, and $V(y_0|\,p^{\mathcal{B}},\tau^{\mathcal{B}}) = V(y_0)$ and $V(y_1|\,p^{\mathcal{B}},\tau^{\mathcal{B}}) = V(y_0)$.

   For $\mathcal{B}^c$ we choose flows $\tau^{\mathcal{B}^c}(\mathcal{L}) = \mu(\pi_1(\mathcal{L}\cap\mathcal{D}))$ for all $\mathcal{L}\subset\mathcal{B}^c\times\mathcal{B}^c$. We choose prices $p^{\mathcal{B}^c}(x) = p_0$ for all $x\in\mathcal{B}^c$, where p_0 is as in Eq. (C-11). As we argued before this solution forms an price-equilibrium pair with revenue equal to $V(y_0)\cdot\mu_1\cdot(y_1-y_0)$.

   We can then paste both solutions (see Lemma C-3) to obtain a solution $(\tilde{p},\tilde{\tau})$ in the entire city. As before, it yields a strict revenue improvement.

   b) $\exists\, y \in (y_0,y_1]$, $\exists\, x > y_1$ such that $x \in \mathcal{IR}(y)$: Then the following points are well defined

$$\begin{aligned}
\overline{y}_1 &\triangleq \sup\{x \in [y_1,H] : \exists\, y \in [y_0,y_1] \text{ such that } x \in \mathcal{IR}(y)\}, \\
\underline{y}_1 &\triangleq \inf\{y \in [y_0,y_1] : \exists\, x \in [y_1,H] \text{ such that } x \in \mathcal{IR}(y)\}.
\end{aligned}$$

That is, $\overline{y}_1$ is largest point after y_1 for which some location in $[y_0, y_1]$ has drivers indifferent to travel to it. As for $\underline{y}_1$, it corresponds to the smallest point in $[y_0, y_1]$ that has drivers willing to travel to some location in $[y_1, H]$. Note that from the definition of $\overline{y}_1$ and Lemma A-6 we can deduce that there is no flow crossing $\overline{y}_1$ in any direction, that is, $\tau([-H, \overline{y}_1] \times [\overline{y}_1, H]) = 0$. Also, from Property 2 (which we prove at the end of the present proof) for any $z \in [\underline{y}_1, \overline{y}_1]$, $\overline{y}_1 \in \mathcal{IR}(z)$. This together with Lemma A-5 imply that for any $z \in [\underline{y}_1, \overline{y}_1]$, $V(z|\, p, \tau) = V(\overline{y}_1) - |\overline{y}_1 - z|$.

The idea is to again construct an strict objective improvement. First, define y^c to be such that $V(y_0) + (y^c - y_0) = V(\overline{y}_1)$, that is, $y^c = V(\overline{y}_1) - V(y_0) + y_0$. Next we argue that $y_c \in (y_0, \overline{y}_1)$. In fact, by the definition of $\overline{y}_1$ we must have $V(\overline{y}_1) > V(y_0)$ thus $y_c > y_0$. Also, if $y_c \geq \overline{y}_1$ then

$$V(y_0) + (y^c - y_0) \geq V(y_0) + (\overline{y}_1 - y_0) \Leftrightarrow V(\overline{y}_1) \geq V(y_0) + (\overline{y}_1 - y_0),$$

and since $V(\overline{y}_1) = V(y_1) + (\overline{y}_1 - y_1)$ we would have

$$V(y_1) + (\overline{y}_1 - y_1) \geq V(y_0) + (\overline{y}_1 - y_0) \Leftrightarrow V(y_1) - V(y_0) \geq y_1 - y_0,$$

which, since $y_1 > y_0$, implies that $V(y_1) > V(y_0)$, contradicting the definition of y_1. From this we can also infer that $y^c - y_0 = \overline{y}_1 - y_1$.

Second, let $h \triangleq \overline{y}_1 - y^c$ and for any set $\mathcal{L} \subseteq \mathcal{C} \times \mathcal{C}$ define the set

$$\mathcal{L}_h \triangleq \{(x + h, y + h) \in \mathcal{C} \times \mathcal{C} : (x, y) \in \mathcal{L}\}.$$

We now construct a new solution $(\tilde{p}, \tilde{\tau})$. Let $\mathcal{B} = [-H, y_0) \cup (\overline{y}_1, H]$, so that $\mathcal{B}^c = [y_0, \overline{y}_1]$. Following our previous scheme of proof we construct two price-equilibrium pairs one in $\mathcal{B}$ and another in $\mathcal{B}^c$, and then we paste them to create $(\tilde{p}, \tilde{\tau})$. As we did before we can use the separation result (see Lemma C-3) to obtain a solution $(p^{\mathcal{B}}, \tau^{\mathcal{B}})$ in $\mathcal{B}$ such that $V(y_0|\, p^{\mathcal{B}}, \tau^{\mathcal{B}}) = V(y_0)$ and $V(\overline{y}_1|\, p^{\mathcal{B}}, \tau^{\mathcal{B}}) = V(\overline{y}_1)$.

For $\mathcal{B}^c$ define the flow $\tau^{\mathcal{B}^c}$ for any $\mathcal{L} \subseteq \mathcal{B}^c \times \mathcal{B}^c$ by

$$\tau^{\mathcal{B}^c}(\mathcal{L}) = \tau\Big((\mathcal{L} \cap ([y_0, y^c] \times [y_0, \overline{y}_1]))_h\Big) + \mu(\pi_1(\mathcal{L} \cap ([y^c, \overline{y}_1] \times [y_0, \overline{y}_1]) \cap \mathcal{D})), \qquad \text{(C-12)}$$

We next show that this flow belongs to $\mathcal{F}_{\mathcal{B}^c}(\mu|_{\mathcal{B}^c})$ and that it is an equilibrium for some prices

$p^{\mathcal{B}^c}$ yet to be defined. Indeed, for any measurable subset K of $\mathcal{B}^c$ we have

$$
\begin{aligned}
\tau_1^{\mathcal{B}^c}(K) &= \tau\Big(((K\times\mathcal{B}^c)\cap([y_0,y^c]\times[y_0,\overline{y}_1]))_h\Big)+\mu(\pi_1((K\times\mathcal{B}^c)\cap([y^c,\overline{y}_1]\times[y_0,\overline{y}_1])\cap\mathcal{D}))\\
&= \tau\Big(((K\cap[y_0,y^c])\times[y_0,\overline{y}_1])_h\Big)+\mu(K\cap[y^c,\overline{y}_1])\\
&= \tau\Big(((K+h)\cap[y_0+h,y^c+h])\times[y_0+h,\overline{y}_1+h]\Big)+\mu(K\cap[y^c,\overline{y}_1])\\
&= \tau\Big(((K+h)\cap[y_1,\overline{y}_1])\times[y_1,\overline{y}_1+h]\Big)+\mu(K\cap[y^c,\overline{y}_1])\\
&\overset{(a)}{=} \tau\Big(((K+h)\cap[y_1,\overline{y}_1])\times\mathcal{C}\Big)+\mu(K\cap[y^c,\overline{y}_1])\\
&= \mu((K+h)\cap[y_1,\overline{y}_1])+\mu(K\cap[y^c,\overline{y}_1])\\
&= \mu((K\cap[y_0,y^c])+h)+\mu(K\cap[y^c,\overline{y}_1])\\
&\overset{(b)}{=} \mu(K\cap[y_0,y_c])+\mu(K\cap[y^c,\overline{y}_1])\\
&= \mu(K),
\end{aligned}
$$

where (a) holds because by construction in $[y_1,\overline{y}_1]$ the flow there can be transported only inside the same set and, therefore, $\tau([y_1,\overline{y}_1]\times[y_1,\overline{y}_1+h]^c)$ equals zero. Equality (b) comes from the fact that μ is invariant under translation (it is a multiple of the Lebesgue measure). Therefore, $\tau_1^{\mathcal{B}^c}$ coincides with $\mu|_{\mathcal{B}^c}$. Also, it is clear from the definition of $\tau^{\mathcal{B}^c}$ that $\tau_2^{\mathcal{B}^c}\ll\Gamma$. Hence, $\tau^{\mathcal{B}^c}$ belongs to $\mathcal{F}_{\mathcal{B}^c}(\mu|_{\mathcal{B}^c})$. Furthermore, Property 3 (which we prove at the end of the present proof) ensures that

$$
\frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(x)\le\frac{d\tau_2}{d\Gamma}(x+h)\ \Gamma-a.e.\ x\ \text{ in }[y_0,y^c],\ \text{ and }\ \frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(x)=\mu_1\ \Gamma-a.e.\ x\ \text{ in }[y^c,\overline{y}_1]. \tag{C-13}
$$

We choose the prices $p^{\mathcal{B}^c}$ as follows. In $[y^c,\overline{y}_1]$ we set constant prices equal to p_1 such that

$$
\alpha\cdot p_1\cdot\min\{1,\frac{\lambda_1\cdot\overline{F}(p_1)}{\mu_1}\}=V(\overline{y}_1),
$$

this price is well defined because $V(\overline{y}_1)\le\psi_1$. For locations in $[y_0,y^c]$ consider the set

$$
K\triangleq\Big\{x\in[y_0,y^c]:\frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(x)\le\frac{d\tau_2}{d\Gamma}(x+h)\Big\}, \tag{C-14}
$$

note from Eq. C-13 we have $\Gamma(K^c)=0$. We set prices for $x\in K$ to be such that

$$
U\Big(x,p^{\mathcal{B}^c}(x),\frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(x)\Big)=U\Big(x+h,p(x+h),s^\tau(x+h)\Big), \tag{C-15}
$$

such prices are well defined because the new Radon-Nikodym is smaller than the old one (shifted by h) in K. For $x\in K^c$ we set the prices equal to zero. Now we need to verify that this selection of prices and flows yields an equilibrium. That is, we need show that the set

$$
\mathcal{E}_{\mathcal{B}^c}=\Big\{(x,y)\in\mathcal{B}^c\times\mathcal{B}^c:\Pi(x,y,p^{\mathcal{B}^c}(y),\frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(y))=\operatorname*{ess\,sup}_{\mathcal{B}^c}\Pi\Big(x,\cdot,p^{\mathcal{B}^c}(\cdot),\frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(\cdot)\Big)\Big\},
$$

has $\tau^{\mathcal{B}^c}$ measure equal to $\mu(\mathcal{B}^c)$. First, from Property 3 we have

$$V(x|\,p^{\mathcal{B}^c},\tau^{\mathcal{B}^c}) = \operatorname*{ess\,sup}_{\mathcal{B}^c} \mathit{\Pi}\Big(x,\cdot,p^{\mathcal{B}^c}(\cdot),\frac{d\tau_2^{\mathcal{B}^c}}{d\mathit{\Gamma}}(\cdot)\Big) = \begin{cases} V(y_1)+(x-y_0) & \text{if } x\in[y_0,y^c] \\ V(\overline{y}_1) & \text{if } [y^c,\overline{y}_1]. \end{cases} \tag{C-16}$$

For the first term in Eq. (C-12) observe that $\tau((\mathcal{E}_{\mathcal{B}^c}\cap[y_0,y^c]\times[y_0,\overline{y}_1])_h)$ equals

$$\tau\Big(\Big\{(x,y)\in[y_1,\overline{y}_1]\times[y_1,\overline{y}_1] : \mathit{\Pi}(x-h,y-h,p^{\mathcal{B}^c}(y-h),\frac{d\tau_2^{\mathcal{B}^c}}{d\mathit{\Gamma}}(y-h)) = V(y_1)+(x-y_1)\Big\}\Big),$$

using that $\mathit{\Gamma}(K^c)=0$ and Eq. (C-26) one can verify that this expression equals

$$\tau\Big(\Big\{(x,y)\in[y_1,\overline{y}_1]\times[y_1,\overline{y}_1] : \mathit{\Pi}(x,y,p(y),s^{\tau}(y)) = V(x|\,p,\tau)\Big\}\Big).$$

In turn, from the definition of $\underline{y}_1$ and $\overline{y}_1$, and the fact that τ is an equilibrium flow this last expression equals $\mu([y_1,\overline{y}_1])$. For the second term in Eq. (C-12) we have the set $\mathcal{E}_{\mathcal{B}^c}\cap[y^c,\overline{y}_1]\times[y_0,\overline{y}_1]\cap\mathcal{D}$ equals

$$\Big\{(x,y)\in[y^c,\overline{y}_1]\times[y_0,\overline{y}_1] : \mathit{\Pi}(x,y,p^{\mathcal{B}^c}(y),\frac{d\tau_2^{\mathcal{B}^c}}{d\mathit{\Gamma}}(y)) = V(\overline{y}_1)\Big\}\cap\mathcal{D},$$

Thus the second term in Eq. (C-12) equals

$$\mu\Big(\Big\{x\in[y^c,\overline{y}_1] : U(x,p^{\mathcal{B}^c}(x),\frac{d\tau_2^{\mathcal{B}^c}}{d\mathit{\Gamma}}(x)) = V(\overline{y}_1)\Big\}\Big) = \mu([y^c,\overline{y}_1]) = \mu([y_0,y_1]),$$

where the first equality comes from Eq. (C-13) and the discussion that it follows it. The second equality comes from μ being invariant under translation and $y^c-y_0=\overline{y}_1-y_1$. Putting all these together yields

$$\tau^{\mathcal{B}^c}(\mathcal{E}_{\mathcal{B}^c}) = \mu([\overline{y}_1,y_1]) + \mu([y_0,y_1]) = \mu([y_0,\overline{y}_1]) = \mu(\mathcal{B}^c),$$

as required.

In order to create the new solution $(\tilde{p},\tilde{\tau})$ we just use Lemma C-3 to paste the two solutions we constructed in $\mathcal{B}$ and $\mathcal{B}^c$. Note that the pasting is allowed because $V(y_0|\,p^{\mathcal{B}^c},\tau^{\mathcal{B}^c}) = V(y_0)$ and $V(\overline{y}_1|\,p^{\mathcal{B}^c},\tau^{\mathcal{B}^c}) = V(\overline{y}_1)$.

To conclude the proof we show the objective improvement. It is enough prove that $\mathbf{Rev}_{[y_0,\overline{y}_1]}(\tilde{p},\tilde{\tau}) >$

$\mathbf{Rev}_{[y_0,\overline{y}_1]}(p,\tau)$,

$$\begin{aligned}\mathbf{Rev}_{[y_0,\overline{y}_1]}(p,\tau) &= \int_{[y_0,\overline{y}_1]} V(x)\, d\tau_2(x)\\ &\overset{(a)}{<} \int_{[y_0,\overline{y}_1]} V(y_0)\, d\tau_2(x)\\ &\overset{(b)}{=} \int_{[y_0,\overline{y}_1]} V(y_0)\, d\tau_2^{\mathcal{B}^c}(x)\\ &\overset{(c)}{\leq} \int_{[y_0,\overline{y}_1]} V(x|\, p^{\mathcal{B}^c}, \tau^{\mathcal{B}^c})\, d\tau_2^{\mathcal{B}^c}(x)\\ &= \mathbf{Rev}_{[y_0,\overline{y}_1]}(\tilde{p},\tilde{\tau}),\end{aligned}$$

where in (a) use Property 1. In (b) we use that under τ no flow leaves or enters $\mathcal{B}^c$ and, thus,

$$\tau_2^{\mathcal{B}^c}(\mathcal{B}^c) = \tau^{\mathcal{B}^c}(\mathcal{B}^c\times\mathcal{B}^c) = \mu(\mathcal{B}^c) = \tau(\mathcal{B}^c\times\mathcal{C}) = \tau(\mathcal{B}^c\times\mathcal{B}^c) = \tau(\mathcal{C}\times\mathcal{B}^c) = \tau_2(\mathcal{B}^c).$$

In (c) we simply use Eq. (C-16).

In what follows we provide a complete proof of the three properties that we use to obtain the result.

Property 1. $\tau_2((y_0,\hat{x}]) > 0$.

Proof of Property 1. First we show that $\exists\, h \in (0, \hat{x}-y_0)$ such that $\tau((y_0, y_0+h)\times[\hat{x},y_1]) = 0$. Suppose this is not true then for all $n\in\mathbb{N}$ large enough we have that $\tau((y_0,y_0+\frac{1}{n})\times[\hat{x},y_1]) > 0$, which thanks to Lemma A-6 implies that for all $n\in\mathbb{N}$ large enough there exists $(x_n,y_n)\in(y_0,y_0+\frac{1}{n})\times[\hat{x},y_1]$ such that $y_n\in\mathcal{IR}(x_n)$. Our envelope result (see Lemma A-5) ensures that $V(x_n) = V(y_n) - |y_n - x_n|$. Since $y_n\in[\hat{x},y_1]$ we must have $V(y_n)\leq V(y_0)$ for all $n\in\mathbb{N}$ large (when y_1 is not well defined we replaced by H and the argument still goes through). Furthermore, x_n converges to y_0 so the continuity of $V(\cdot)$ yields

$$V(y_0) = \lim_{n\to\infty} V(x_n) = \lim_{n\to\infty} V(y_n) - |y_n - x_n| \leq V(y_0) - \lim_{n\to\infty}(y_n - x_n) < V(y_0),$$

not possible. We conclude that $\exists\, h\in(0,\hat{x}-y_0)$ such that $\tau((y_0,y_0+h)\times[\hat{x},y_1]) = 0$. Note that the same must be true for some $h\in(0,(\hat{x}-y_0)\wedge\frac{(y_1-y_0)}{2})$. We fix h in this interval with the property we just proved.

Next, note we also have that $\tau((y_0,y_0+h)\times(y_1,H]) = 0$; otherwise, by Lemma A-6 we can find $(x,y)\in(y_0,y_0+h)\times(y_1,H]$ such that $y\in\mathcal{IR}(x)$, which implies that $y\in\mathcal{IR}(y_1)$. Using the envelope result delivers

$$V(y_1) = V(y) - |y - y_1|, \quad V(x) = V(y) - |y - x|.$$

Since $V(y_1) = V(y_0)$ we have $(y_1 - x) = V(y_0) - V(x)$, but our choice of h implies that $y_1 - x > h$ thus

$$h < (y_1 - x) = V(y_0) - V(x) \leq |y_0 - x| \leq h,$$

again a contradiction. The last inequality comes from the Lipschitz property (see Lemma 1). In summary, we have that there exists $h \in (0, (\hat{x} - y_0) \wedge \frac{(y_1 - y_0)}{2})$ such that $\tau((y_0, y_0 + h) \times [\hat{x}, H]) = 0$. To conclude the proof note the following

$$\begin{aligned}
0 &\overset{(a)}{<} \mu((y_0, y_0 + h)) \\
&= \tau((y_0, y_0 + h) \times \mathcal{C}) \\
&\overset{(b)}{=} \tau((y_0, y_0 + h) \times [y_0, H]) \\
&= \tau((y_0, y_0 + h) \times [y_0, \hat{x})) + \tau((y_0, y_0 + h) \times [\hat{x}, H]) \\
&= \tau((y_0, y_0 + h) \times [y_0, \hat{x})) \\
&\leq \tau_2([y_0, \hat{x}]) \\
&\overset{(c)}{=} \tau_2((y_0, \hat{x}]),
\end{aligned}$$

where (a) comes from the fact that the measure μ has full support in $\mathcal{C}$. The equality (b) holds because by construction no flow leaves $[y_0, H]$, and (c) is true because $\tau_2 \ll \mathit{\Gamma}$ and $\mathit{\Gamma}$ does not have atoms in $[y_0, \hat{x}]$. This concludes the proof of Property 1.

Property 2. Both $\overline{y}_1$ and $\underline{y}_1$ are achieved in the set where they are defined. Furthermore, for any $z \in [\underline{y}_1, \overline{y}_1]$, $\overline{y}_1 \in \mathcal{IR}(z)$.

Proof of Property 2. First we show both

$$\exists\, y_q \in [y_0, y_1] \text{ such that } \overline{y}_1 \in \mathcal{IR}(y_q) \quad \text{and} \quad \exists\, x_q \in [y_1, H] \text{ such that } x_q \in \mathcal{IR}(\underline{y}_1). \tag{C-17}$$

Let us begin with the first statement. Let x^n be a sequence in A converging to $\overline{y}_1$, where

$$A = \{x \in [y_1, H] : \exists\, y \in [y_0, y_1] \text{ such that } x \in \mathcal{IR}(y)\}.$$

Then there exists a sequence $\{y^n\} \subset [y_0, y_1]$ such that $x^n \in \mathcal{IR}(y^n)$. Note that since $\{y^n\} \subset [y_0, y_1]$ and $x^n \in [y_1, H]$, Lemma A-3 implies that $x^n \in \mathcal{IR}(y_1)$. Fix $\epsilon > 0$ and $\delta > 0$ then we can find $n_0(\delta)$ such that for all $n \geq n_0(\delta)$ we have $B(x_n, \delta/2) \subset B(\overline{y}_1, \delta)$. This implies that $V_{B(x_n,\delta/2)}(y_1) \leq V_{B(\overline{y}_1,\delta)}(y_1)$ for all $n \geq n_0(\delta)$. Fix $n \geq n_0(\delta)$, because $x^n \in \mathcal{IR}(y_1)$ we know that

$$\exists\, \delta_0(\epsilon, n) \text{ such that } \forall \hat{\delta} \leq \delta_0(\epsilon, n) \;\; V_{B(x_n,\hat{\delta})}(y_1) \geq V(y_1) - \epsilon.$$

Let $r_0 = \delta_0(\epsilon, n) \wedge \frac{\delta}{2}$ then for all $\hat{\delta} \leq r_0$ we have

$$V_{B(\overline{y}_1,\delta)}(y_1) \geq V_{B(x_n,\delta/2)}(y_1) \geq V_{B(x_n,\hat{\delta})}(y_1) \geq V(y_1) - \epsilon.$$

This shows that for any $\epsilon, \delta > 0$ we have $V_{B(\overline{y}_1,\delta)}(y_1) \geq V(y_1) - \epsilon$. That is, $\overline{y}_1 \in \mathcal{IR}(y_1)$.

Now we prove that $\underline{y}_1 \in A$ where

$$A = \{y \in [y_0, y_1] : \exists\, x \in [y_1, H] \text{ such that } x \in \mathcal{IR}(y)\}.$$

By the definition of $\underline{y}_1$ we can always construct a sequence $\{y^n\} \subset A$ converging to $\underline{y}_1$. From the definition of A there exists another sequence $\{x^n\} \subset [y_1, H]$ such that $x^n \in \mathcal{IR}(y^n)$ for all n. Fix $\epsilon > 0$ then we can always find $n_0(\epsilon)$ such that for all $n \geq n_0(\epsilon)$ we have $|y^n - \underline{y}_1| \leq \epsilon/3$. Fix $n \geq n_0(\epsilon)$ then since $x^n \in \mathcal{IR}(y^n)$ we have

$$\exists\, \delta_0(\epsilon, n) \text{ such that } \forall \delta \leq \delta_0(\epsilon, n) \;\; V_{B(x_n,\delta)}(y^n) \geq V(y^n) - \frac{\epsilon}{3}, \tag{C-18}$$

but from the Lipchitz property we can deduce that

$$V_{B(x_n,\delta)}(y^n) \leq V_{B(x_n,\delta)}(\underline{y}_1) + \frac{\epsilon}{3} \quad \text{and} \quad V(y^n) \geq V(\underline{y}_1) - \frac{\epsilon}{3}.$$

Replacing this in Eq. (C-18) yields

$$\exists\, \delta_0(\epsilon, n) \text{ such that } \forall \delta \leq \delta_0(\epsilon, n) \;\; V_{B(x_n,\delta)}(\underline{y}_1) \geq V(\underline{y}_1) - \epsilon,$$

that is, $x^n \in \mathcal{IR}(\overline{y}_1)$. This concludes the proof for Eq. (C-17).

Next, we show that for all $z \in [\underline{y}_1, \overline{y}_1]$, $\overline{y}_1 \in \mathcal{IR}(z)$. First, from our previos argument we know there exists y_q and x_q as in Eq. (C-17). Then Lemma A-3 implies $\overline{y}_1 \in \mathcal{IR}(z)$ for all $z \in [y_q, \overline{y}_1]$. Observe that this yields $\overline{y}_1 \in \mathcal{IR}(x_q)$ because $x_q \in [y_q, \overline{y}_1]$. Take $z \in [\underline{y}_1, y_q]$ then since $x_q \in \mathcal{IR}(\underline{y}_1)$ from Lemma A-3 we conclude that $x_q \in \mathcal{IR}(z)$. Using envelope result, Lemma 3, we have that $V(x_q) = V(z) + (x_q - z)$. Furthermore, fix $\epsilon > 0$ then since $\overline{y}_1 \in \mathcal{IR}(x_q)$ we have

$$\exists\, \delta_0(\epsilon) \text{ such that } \forall \delta \leq \delta_0(\epsilon) \;\; V_{B(\overline{y}_1,\delta)}(x_q) + \epsilon \geq V(x_q) = V(z) + (x_q - z). \tag{C-19}$$

Thus for any $\delta \leq \delta_0(\epsilon)$, the Lipchitz property and Eq. (C-19) yield

$$V_{B(\overline{y}_1,\delta)}(z) \geq V_{B(\overline{y}_1,\delta)}(x_q) - (x_q - z) \geq V(z) + (x_q - z) - (x_q - z) - \epsilon = V(z) - \epsilon,$$

which implies that $\overline{y}_1 \in \mathcal{IR}(z)$. This concludes the proof of Property 2.

Property 3. Both Eq. (C-13) and Eq. (C-16) hold.

Proof of Property 3. Let us star with Eq. (C-13). In order to prove the first part in Eq. (C-13) consider the following set

$$K = \Big\{x \in [y_0, y^c] : \frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(x) \leq \frac{d\tau_2}{d\Gamma}(x + h)\Big\}.$$

We want to show that $\Gamma(K^c) = 0$ (the complement is taken with respect to $[y_0, y^c]$). If this is not true then $\Gamma(K^c) > 0$ and we have

$$\tau_2^{\mathcal{B}^c}(K^c) = \int_{K^c} \frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(x)\, d\Gamma(x) > \int_{K^c} \frac{d\tau_2}{d\Gamma}(x+h)\, d\Gamma(x) = \tau_2(K^c + h). \tag{C-20}$$

However,

$$\begin{aligned}
\tau_2^{\mathcal{B}^c}(K^c) &= \tau\Big(([y_0, y^c] \times K^c)_h\Big) + \mu(\pi_1(([y^c, \overline{y}_1] \times K^c) \cap \mathcal{D})) \\
&= \tau\Big(([y_0, y^c] \times K^c)_h\Big) \\
&= \tau\Big([y_0 + h, y^c + h] \times (K^c + h)\Big) \\
&\leq \tau\Big(\mathcal{C} \times (K^c + h)\Big) \\
&= \tau_2(K^c + h).
\end{aligned}$$

This together with Eq. C-20 yield a contradiction. To prove the second part of Eq. (C-13) consider any $\mathcal{R} \subset [y^c, \overline{y}_1]$, and observe that

$$\tau_2^{\mathcal{B}^c}(\mathcal{R}) = \tau\Big([y_1, \overline{y}_1] \times (\mathcal{R} + h)\Big) + \mu(\mathcal{R}) = \mu(\mathcal{R}) = \int_{\mathcal{R}} \mu_1 d\Gamma(x),$$

where the second equality comes from $\mathcal{R}+h \subset [\overline{y}_1, \overline{y}_1+h]$ and the fact that $\tau([y_1, \overline{y}_1] \times [\overline{y}_1, \overline{y}_1+h]) = 0$.

Finally, we provide a proof for Eq. (C-16). Let

$$Z(x) \triangleq \min\{V(y_0) + (x - y_0), V(\overline{y}_1)\}.$$

We verify that for all $x \in \mathcal{B}^c$

$$Z(x) \geq U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w - x|, \quad \Gamma - a.e.\ w \text{ in } \mathcal{B}^c, \tag{C-21}$$

and that $Z(x)$ is the smallest with such property. First, fix $x \in [y^c, \overline{y}_1]$ so $Z(x) = V(\overline{y}_1)$. Note that from our choice of prices in $[y^c, \overline{y}_1]$ we have

$$Z(x) = V(\overline{y}_1) \geq V(\overline{y}_1) - |w - x| = U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w - x|, \quad \Gamma - a.e.\ w \text{ in } [y^c, \overline{y}_1].$$

So we only need to show the same inequality but this time for $[y_0, y^c]$. From the definition of $\underline{y}_1$ and $\overline{y}_1$, Lemma A-3 and Lemma A-5 we have that $V(\overline{y}_1) - |\overline{y}_1 - y_1|$ equals $V(y_1|\, p, \tau)$ and, therefore,

$$\begin{aligned}
V(\overline{y}_1) &\geq U(w, p(w), s^{\tau}(w)) - |w - y_1| + |\overline{y}_1 - y_1|, \quad \Gamma - a.e.\ w \text{ in } [y_1, \overline{y}_1] \\
&\geq U(w, p(w), s^{\tau}(w)), \quad \Gamma - a.e.\ w \text{ in } [y_1, \overline{y}_1].
\end{aligned}$$

We can use this together with the fact that $[y_0, y^c] + h = [y_1, \overline{y}_1]$ to obtain

$$
\begin{aligned}
Z(x) = V(\overline{y}_1) &\overset{(a)}{\geq} U\Big(w+h, p(w+h), s^\tau(w+h)\Big), \quad \Gamma - a.e. \ w \ \text{ in } [y_0, y^c] \\
&\geq U\Big(w+h, p(w+h), s^\tau(w+h)\Big) - |w-x|, \quad \Gamma - a.e. \ w \ \text{ in } [y_0, y^c] \\
&\overset{(b)}{=} U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w-x|, \quad \Gamma - a.e. \ w \ \text{ in } [y_0, y^c],
\end{aligned}
$$

Inequality (a) comes from the fact that Γ in the interval under consideration is invariant under a shift of h. Line (b) comes from Eq. (C-26). That is, for $x \in [y^c, \overline{y}_1]$ Eq. (C-21) is satisfied. It is left to verify that $Z(x)$ is the smallest value satisfying this equation. For any $\epsilon > 0$, since $x \in [y^c, \overline{y}_1]$ we have

$$
\begin{aligned}
0 &< \Gamma(B(x,\epsilon) \cap [y^c, \overline{y}_1]) \\
&= \Gamma\Big(w \in [y^c, \overline{y}_1] : V(\overline{y}_1) - |w-x| > V(\overline{y}_1) - \epsilon\Big) \\
&= \Gamma\Big(w \in [y^c, \overline{y}_1] : U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w-x| > V(\overline{y}_1) - \epsilon\Big),
\end{aligned}
$$

hence $V(\overline{y}_1)$ is the smallest value satisfying Eq. (C-21).

Now we show Eq. (C-21) for $x \in [y_0, y^c]$. Fix $x \in [y_0, y^c]$ so $Z(x) = V(y_0) + (x - y_0)$. Note that $V(y_0)$ equals $V(y_1)$, and from the definition of $\overline{y}_1$ and the envelope result we have that $V(y_1)$ equals $V(\overline{y}_1) - (\overline{y}_1 - y_1)$. Therefore,

$$
\begin{aligned}
Z(x) &= V(\overline{y}_1) - (\overline{y}_1 - y_1) + (x - y_0) \\
&\overset{(a)}{\geq} V(\overline{y}_1) - (w - x), \quad \Gamma - a.e. \ w \ \text{ in } [y^c, \overline{y}_1] \\
&\overset{(b)}{=} U\Big(w, p^{\mathcal{B}^c}(w), \frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big) - |w-x|, \quad \Gamma - a.e. \ w \ \text{ in } [y^c, \overline{y}_1],
\end{aligned}
$$

where (a) follows from $w \geq y_c$ and $y^c - y_0 = \overline{y}_1 - y_1$. Line (b) holds from our choice of prices in $[y^c, \overline{y}_1]$. Hence, $Z(x)$ upper bounds (almost surely) the desire quantity in $[y^c, \overline{y}_1]$, so we just need to prove the same bound for $[y_0, y^c]$. Note that from the definition of $\underline{y}_1$ and $\overline{y}_1$ we have that

$$
V(x+h) = V(y_1) + (x + h - y_1) = V(y_1) + (x - y_0) = Z(x),
$$

and thus

$$
\begin{aligned}
Z(x) &= V(x+h | \, p, \tau) \\
&\overset{(a)}{\geq} U(w, p(w), s^\tau(w)) - |w - (x+h)|, \quad \Gamma - a.e. \ w \ \text{ in } [y_1, \overline{y}_1] \\
&\overset{(b)}{=} U(w+h, p(w+h), s^\tau(w+h)) - |w+h-(x+h)|, \quad \Gamma - a.e. \ w \ \text{ in } [y_0, y^c] \\
&\overset{(c)}{=} U(w, p^{\mathcal{B}^c}(w), \frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(w)) - |w-x)|, \quad \Gamma - a.e. \ w \ \text{ in } [y_0, y^c],
\end{aligned}
$$

where (a) comes from the definition of $V(x+h|\, p,\tau)$, (b) from the invariance under translation of Γ. Line (c) follows from Eq. (C-26). Therefore, $Z(x)$ satisfies Eq. (C-21). To see why $Z(x)$ is the smallest value satisfying this equation observe that

$$\begin{aligned}
0 &< \Gamma(B(y^c,\epsilon)\cap[y^c,\overline{y}_1]) \\
&\overset{(a)}{=} \Gamma\Big(w\in[y^c,\overline{y}_1] : V(\overline{y}_1)-(w-x) > V(\overline{y}_1)-(\overline{y}_1-y_1)+(x-y_0)-\epsilon\Big) \\
&= \Gamma\Big(w\in[y^c,\overline{y}_1] : U\Big(w,p^{\mathcal{B}^c}(w),\frac{d\tau_2^{\mathcal{B}^c}}{d\Gamma}(w)\Big)-|w-x| > Z(x)-\epsilon\Big),
\end{aligned}$$

where in (a) we use that $y^c-y_0=\overline{y}_1-y_1$. This implies that $Z(x)$ is the smallest value satisfying Eq. (C-21), completing the proof.

□

Proof of Proposition 9. If $X_r=H$ there is nothing to prove, so assume $X_r<H$. Let (p,τ) be a feasible solution such that $V(\cdot|p,\tau)$ is non-decreasing. Due to Proposition 8 we can always restrict attention to this type of solution. We proceed by contradiction. Assume that there exists $\tilde{x}\in(X_r,H]$ such that

$$V(\tilde{x}) < \min\{V(X_r)+(\tilde{x}-X_r),\psi_1\} \triangleq Z(\tilde{x}). \tag{C-22}$$

First, we construct an interval $\tilde{I}$ such that $\tau_2(\tilde{I})>0$ and $V(x)<Z(x)$ for all $x\in\tilde{I}$. Then, we show that $Z(x)$ can be achieved in a feasible manner by appropriately creating a price-equilibrium pair $(\tilde{p},\tilde{\tau})$ that mimics the flow generated by τ in $(X_r,H]$. The final step of the proof is to use the interval $\tilde{I}$ and the flow $\tilde{\tau}$ to show an strict objective improvement.

Interval construction. From Eq. (C-22) and the continuity of $V(\cdot)$ we can deduce the existence of an interval $[\tilde{a},\tilde{b}]\subset(X_r,H]$ such $V(x)<Z(x)$ for all $x\in[\tilde{a},\tilde{b}]$. Furthermore, the Lipchitz property (see Lemma 1) and Lemma 6 imply that $V(x)<Z(x)$ for all $x\in[\tilde{a},\tilde{c}]$ where $\tilde{c}$ is the minimum between H and the value c such that $V(\tilde{a})+(c-\tilde{a})=\psi_1$. Also, Proposition 8 and Lemma A-6 together with Lemma 3 imply that $\tau([\tilde{a},\tilde{c}]\times\mathcal{C})=\tau([\tilde{a},\tilde{c}]\times[\tilde{a},\tilde{c}])$. Putting all of this together we conclude that there exists an interval $\tilde{I}=(\tilde{a},\tilde{c})$ such that $\tau_2(\tilde{I})>0$ and $V(x)<Z(x)$ for all $x\in\tilde{I}$.

Flow mimicking. Define the collection of intervals

$$\mathcal{I}\triangleq\{I\subset(X_r,H] : I=[a,b], a<b, b\in\mathcal{IR}(a), a \text{ is minimal and } b \text{ is maximal}\}.$$

There are two cases: $\mathcal{I}=\emptyset$ and $\mathcal{I}\neq\emptyset$. We only do the latter because its treatment contains the former.

Suppose $\mathcal{I} \neq \emptyset$, then there exists $X_r < a < b$ such that $b \in \mathcal{IR}(a)$, where a and b are minimal and maximal with this property, respectively. We first look at some properties of the equilibrium in each element of $\mathcal{I}$ and then we look at its complement.

Note that from the minimality of a we have that for any $x < a$, $a \notin \mathcal{IR}(x)$. Similarly, for any $x > b$ we have $x \notin \mathcal{IR}(b)$. This, together with Proposition 8 and Lemma A-6 imply that $[a, b]$ is a flow-separated region, that is, there is no flow coming in nor flow going out of $[a, b]$, $\tau([a, b] \times [a, b]^c) = 0$ and $\tau([a, b]^c \times [a, b]) = 0$. Observe that our flow separation result in Lemma C-3 implies that in each interval $I \in \mathcal{I}$ we have an equilibrium. Furthermore, from Lemma A-5 we must have

$$V(x) = V(a) + (x - a), \quad \forall x \in [a, b].$$

From the previous discussion we infer that the elements in the collection $\mathcal{I}$ are disjoint intervals and, since V is non-decreasing, the collection is at most countable.

For any a, b such that $[a, b] \in \mathcal{I}$ we define

$$t(a) \triangleq V(a) - V(X_r) + X_r, \quad \text{and} \quad t(b) \triangleq V(b) - V(X_r) + X_r.$$

Note that since V is non-decreasing we have $V(a) \geq V(X_r)$ and, therefore, $t(b) > t(a) \geq X_r$. Also,

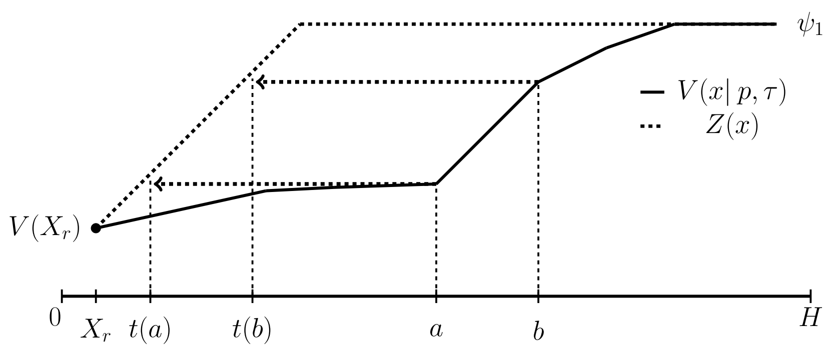

Figure 12: Graphical representation of $t(a)$ and $t(b)$.

for any such b we have $t(b) < Y_r$. The points $t(a), t(b)$ are the corresponding points to a, b in the interval $[X_r, Y_r]$ (see Figure 12). Furthermore, $t(\cdot)$ is a non-decreasing mapping.

We denote by $\mathcal{I}^c$ the collection of intervals whose elements are the intervals that do not belong to $\mathcal{I}$. Observe that the elements in $\mathcal{I}$ and $\mathcal{I}^c$ alternate in a consecutive manner. That is, if we have an interval $(c, d) \in \mathcal{I}^c$ then it can only be followed by and interval $[a, b] \in \mathcal{I}$ with $a = d$. In the case that $I = (c, d) \in \mathcal{I}^c$ is not followed by an interval in $\mathcal{I}$ then I equals $(c, H]$. Define the disjoint unions

$$\mathcal{K} \triangleq \bigcup_{I \in \mathcal{I}} I \text{ and } \mathcal{K}^c \triangleq \bigcup_{I \in \mathcal{I}^c} I.$$

Note that $(X_r, H] = \mathcal{K} \cup \mathcal{K}^c$ up to a set of Γ measure zero. Also, for each interval $I \in \mathcal{I}^c$ we must have that for all measurable sets $A \subset I$, $\tau(A \times A) = \mu(A) = \tau_2(A)$; otherwise, by Lemma A-6 we would get a contradiction with the definition of $\mathcal{I}$. In turn, this implies that $\frac{d\tau_2}{d\Gamma}(x) = \mu_1$, $\Gamma - a.e.\ x$ in $\mathcal{K}^c$.

We denote by $\mathcal{I}_t$ the collection of intervals $\{[t(a), t(b)]\}_{[a,b] \in \mathcal{I}}$, and $\mathcal{I}_t^c$ is defined in analogous manner. Also, $\mathcal{K}_t$ and $\mathcal{K}_t^c$ are defined similarly to $\mathcal{K}$ and $\mathcal{K}^c$ replacing $\mathcal{I}$ with $\mathcal{I}_t$ and $\mathcal{I}^c$ with $\mathcal{I}_t^c$, respectively.

The idea now is to construct a solution $(\tilde{p}, \tilde{\tau})$ in $(X_r, H]$ and then paste it with the old solution (p, τ) restricted to $[-H, X_r)$. To construct $(\tilde{p}, \tilde{\tau})$ we will make use of the collections $\mathcal{I}_t$ and $\mathcal{I}_t^c$. For each element in these collections we will create a price-equilibrium. For intervals $[t(a), t(b)] \in \mathcal{I}_t$ the idea is that the solution $(\tilde{p}, \tilde{\tau})$ has the same equilibrium than (p, τ) in $[a, b]$. For the interval in $\mathcal{I}_t^c$ we choose prices such that no drivers will have an incentive to move. Finally, using Lemma C-3 we will paste the equilibria generated in all the intervals.

First, we show how to construct prices and an equilibrium in some $[t(a), t(b)]$. Fix $[a, b] = I \in \mathcal{I}$ and denote the mimicking set $[t(a), t(b)]$ by I_t. Choose prices $p^{I_t}(x)$ equal to $p(x + (a - t(a)))$ for all $x \in I_t$. For the flows, we define τ^{I_t} for any $\mathcal{L} \subseteq I_t \times I_t$ by

$$\tau^{I_t}(\mathcal{L}) = \tau\Big(\mathcal{L} + (a - t(a), a - t(a))\Big),$$

that is, τ^{I_t} mimics τ in $I \times I$. It can be shown that (see Property 1 at the end of this proof) (p^{I_t}, τ^{I_t}) forms a price-equilibrium pair in I_t such that $\tau^{I_t} \in \mathcal{F}_{I_t}(\mu|_{I_t})$. Also, $V(x|\, p^{I_t}, \tau^{I_t})$ equals $V(x + a - t(a)|\, p, \tau)$ for all $x \in I_t$, and

$$\frac{d\tau_2^{I_t}}{d\Gamma}(x) = \frac{d\tau_2}{d\Gamma}(x + a - t(a)), \quad \Gamma - a.e.\ x \text{ in } I_t. \tag{C-23}$$

Furthermore, because $I \in \mathcal{I}$ we have

$$V(x|\, p^{I_t}, \tau^{I_t}) = V(x + a - t(a)|\, p, \tau) = V(a) + (x - t(a)) = V(X_r) + (x - X_r) = Z(x), \quad \forall x \in I_t,$$

that is, for all intervals I_t the associated solution (p^{I_t}, τ^{I_t}) achieves the upper bound $Z(x)$.

Second, we show how to set the prices and construct an equilibrium everywhere else. Consider any two consecutive sets in $\mathcal{I}$, $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$. The corresponding mimicking sets are $[t(a_1), t(b_1)]$ and $[t(a_2), t(b_2)]$. We need to set prices and define the flow in the interval $J_t = (t(b_1), t(a_2))$. We choose the prices p^{J_t} to be such that

$$U\Big(x, p^{J_t}(x), \mu_1\Big) = Z(x), \quad \forall x \in J_t.$$

Since $Z(x) \leq \psi_1$ these prices are guaranteed to exist. We define the measure τ^{J_t} for any measurable set $\mathcal{L} \subseteq J_t \times J_t$ by

$$\tau^{J_t}(\mathcal{L}) = \mu(\pi_1(\mathcal{L} \cap \mathcal{D})).$$

This measure has $d\tau_2^{J_t}/d\mathit{\Gamma} = \mu_1$, $\mathit{\Gamma} - a.e$ in J_t. It can be shown that (see Property 2 at the end of this proof) (p^{J_t}, τ^{J_t}) forms a price-equilibrium pair in J_t such that $\tau^{J_t} \in \mathcal{F}_{J_t}(\mu|_{J_t})$ and $V(x|p^{J_t}, \tau^{J_t})$ equals $Z(x)$ for all $x \in J_t$.

Third, the solutions $\{(p^{I_t}, \tau^{I_t})\}_{I_t \in \mathcal{I}_t}$ and $\{(p^{J_t}, \tau^{J_t})\}_{J_t \in \mathcal{I}_t^c}$ cover the whole interval $(X_r, H]$. Moreover they are defined in disjoint interval, and are such that the respective $V(\cdot)$ functions coincide at the boundaries of the interval (these functions coincide with $Z(\cdot)$). Thus, we can apply Lemma C-3 to paste all these solutions and obtain a new solution $(\tilde{p}, \tilde{\tau})$ in $(X_r, H]$. As mentioned before we can use the same lemma to paste this solution with the old solution restricted to $[-H, X_r]$. This would yield a solution in the entire city.

Objective improvement. Consider the revenue under (p, τ) in $(X_r, H]$, it easy to observe that

$$\begin{aligned}
\mathbf{Rev}_{(X_r,H]}(p,\tau) &= \int_{(X_r,H]} V(x|p,\tau) \cdot s^{\tau}(x)\, d\mathit{\Gamma}(x) \\
&= \int_{\mathcal{K}} V(x|p,\tau) \cdot s^{\tau}(x)\, d\mathit{\Gamma}(x) + \int_{\mathcal{K}^c} V(x|p,\tau) \cdot s^{\tau}(x)\, d\mathit{\Gamma}(x) \\
&= \underbrace{\sum_{I \in \mathcal{I}} \int_I V(x|p,\tau) \cdot s^{\tau}(x)\, d\mathit{\Gamma}(x)}_{=(a)} + \underbrace{\int_{\mathcal{K}^c} V(x|p,\tau) \cdot s^{\tau}(x)\, d\mathit{\Gamma}(x)}_{=(b)}.
\end{aligned}$$

Let us develop the integral of the term (a). Let I be equal to $[a, b]$ and I_t equal to $[t(a), t(b)]$ then

$$\begin{aligned}
\int_{[a,b]} V(x|p,\tau) \cdot s^{\tau}(x)\, d\mathit{\Gamma}(x) &= \int_{[t(a),t(b)]} V(x+a-t(a)|p,\tau) \cdot s^{\tau}(x+a-t(a))\, d\mathit{\Gamma}(x) \\
&= \int_{[t(a),t(b)]} V(x|p^{I_t},\tau^{I_t}) \cdot s^{\tau^{I_t}}(x)\, d\mathit{\Gamma}(x),
\end{aligned}$$

where in the first line we use the invariance under translation of $\mathit{\Gamma}$, and in the second line we use that $V(x|\, p^{I_t}, \tau^{I_t})$ equals $V(x + a - t(a)|\, p, \tau)$ for all $x \in I_t$ and Eq. (C-23). Thus,

$$\begin{aligned}
\mathbf{Rev}_{(X_r,H]}(p,\tau) &= \sum_{I_t \in \mathcal{I}_t} \int_{I_t} V(x|p^{I_t},\tau^{I_t}) \cdot s^{\tau^{I_t}}(x)\, d\mathit{\Gamma}(x) + (b) \\
&= \int_{\mathcal{K}_t} Z(x) \cdot s^{\tilde{\tau}}(x)\, d\mathit{\Gamma}(x) + (b).
\end{aligned}$$

Thus, to conclude the proof we only need to show that

$$(b) = \int_{\mathcal{K}^c} V(x|p,\tau) \cdot s^{\tau}(x)\, d\mathit{\Gamma}(x) < \int_{\mathcal{K}_t^c} Z(x) \cdot s^{\tilde{\tau}}(x)\, d\mathit{\Gamma}(x). \tag{C-24}$$

Define the following functions

$$V_e(x) = \begin{cases} V(x|p,\tau) & \text{if } x \in \mathcal{K}^c, \\ V(a|p,\tau) & \text{if } x \in [a,b], \text{ some } [a,b] \in \mathcal{I}, \end{cases}$$

$$Z_e(x) = \begin{cases} Z(x) & \text{if } x \in \mathcal{K}_t^c, \\ Z(t(a)) & \text{if } x \in [t(a),t(b)], \text{ some } [t(a),t(b)] \in \mathcal{I}_t. \end{cases}$$

We verify that $V_e(x) \leq Z_e(x)$ for all $x \in (X_r, H]$, and the we use this inequality to prove the objective improvement. Let $x \in \mathcal{K}^c$ then there exists an interval $(c,d) \in \mathcal{I}^c$ with $x \in (c,d)$. If $x \in \mathcal{K}_t^c$ then the upper bound is trivial. If $x \notin \mathcal{K}_t^c$ then $x \in [t(a),t(b)]$ for some $[t(a),t(b)] \in \mathcal{I}_t$. We must have that $a \geq d$; otherwise, since $(c,d) \in \mathcal{I}$, it must be the case that $b \leq c$. In turn, this implies that $[t(a),t(b)] \cap (c,d) = \emptyset$ which contradiction our current assumption. Therefore,

$$V_e(x) = V(x|p,\tau) \leq V(d|p,\tau) \leq V(a|p,\tau) = Z(t(a)) = Z_e(x).$$

Let $x \in [a,b]$ for some $[a,b] \in \mathcal{I}$. If $x \in \mathcal{K}_t^c$ then $t(b) < x$ because otherwise we would have that $t(a) \leq a \leq a \leq t(b)$, that is, $x \in [t(a),t(b)] \in \mathcal{I}_t$. Under our current assumption this is not possible. Then,

$$V_e(x) = V(a|p,\tau) < V(b|p,\tau) = Z(t(b)) \leq Z(x) = Z_e(x), \tag{C-25}$$

that is, when $x \in \mathcal{K} \cap \mathcal{K}_t^c$ we have $V_e(x) < Z_e(x)$. If $x \in [t(\hat{a}),t(\hat{b})]$ for some $[t(\hat{a}),t(\hat{b})] \in \mathcal{I}_t$. Using similar arguments as before we can show that $\hat{a} \geq a$ and, therefore,

$$V_e(x) = V(a|p,\tau) = Z(t(a)) \leq Z(t(\hat{a})) = Z_e(x).$$

Now, recall that in the **Interval construction** part of the proof we defined an interval $\tilde{I} = [\tilde{a},\tilde{c}]$ in which the function $V(\cdot|p,\tau)$ is uniformly strictly bounded by $Z(\cdot)$. Now we relate this interval to $\mathcal{K}_t^c$ by showing that there exists $\epsilon > 0$ such that $(\tilde{c}-\epsilon,\tilde{c}) \subseteq I_t^c$ with $I_t^c \in \mathcal{I}_t^c$. The idea is to use that $(\tilde{c}-\epsilon,\tilde{c}) \subset \tilde{I}$ and $(\tilde{c}-\epsilon,\tilde{c}) \subset \mathcal{K}_t^c$ together with Eq. (C-25) to show an strict objective improvement.

Note that if $\tilde{c} = H$ then

$$\begin{aligned}\sup_{[t(a),t(b)]\in\mathcal{I}_t} t(b) &\overset{(1)}{\leq} t(\tilde{c}) \\ &= V(\tilde{c}) - V(X_r) + X_r \\ &= (V(\tilde{c}) - V(\tilde{a})) + (V(\tilde{a}) - V(X_r)) + X_r \\ &\overset{(2)}{<} (V(\tilde{c}) - V(\tilde{a})) + (Z(\tilde{a}) - Z(X_r)) + X_r \\ &\overset{(3)}{\leq} (\tilde{c} - \tilde{a}) + (\tilde{a} - X_r) + X_r \\ &= \tilde{c},\end{aligned}$$

where (1) comes from the fact that $t(\cdot)$ is non-decreasing and $\tilde{c} = H$, line (2) follows from the $V(\tilde{a}) < Z(\tilde{a})$ and $V(X_r) = Z(X_r)$. Inequality, (3) holds because both V and Z are $1-$Lipschitz functions. In the case that $\tilde{c} < H$ we have $V(\tilde{a}) + (\tilde{c} - \tilde{a}) = \psi_1$. Also, we always have that $t(b) \leq Y_r$ where Y_r is such that $V(X_r) + (Y_r - X_r) = \psi_1$. From this we deduce that $Y_r < \tilde{c}$ and, therefore, we have that $\sup_{[t(a),t(b)]\in\mathcal{I}_t} t(b) < \tilde{c}$. Either way we can always find $\epsilon \in (0, \tilde{c} - \tilde{a})$ such that the interval $(\tilde{c} - \epsilon, \tilde{c})$ does not intersect with any interval in $\mathcal{I}_t$. Hence, since $\mathcal{I}_t^c$ are all the intervals that do not belong to $\mathcal{I}_t$ we must have that $(\tilde{c} - \epsilon, \tilde{c}) \subseteq I_t^c$ for some $I_t^c \in \mathcal{I}_t^c$.

Because $(\tilde{c} - \epsilon, \tilde{c})$ is a subset of both $\mathcal{K}_t^c$ and $(\tilde{a}, \tilde{c})$, for $x \in (\tilde{c} - \epsilon, \tilde{c}) \cap \mathcal{K}^c$ we have $V_e(x) < Z_e(x)$. Also, for $x \in (\tilde{c} - \epsilon, \tilde{c}) \cap \mathcal{K}$ from equation Eq. (C-25) we have $V_e(x) < Z_e(x)$. That is, $V_e(x) < Z_e(x)$ for all $x \in (\tilde{c} - \epsilon, \tilde{c})$ and, therefore,

$$\begin{aligned}
\int_{\mathcal{K}^c} V(x|p,\tau) \cdot s^{\tau}(x)\, d\Gamma(x) &= \int_{(X_r,H]} V_e(x|p,\tau) \cdot \mu_1 d\Gamma(x) - \sum_{[a,b]\in\mathcal{I}} \int_{[a,b]} V(a|p,\tau) \cdot \mu_1 d\Gamma(x) \\
&< \int_{(X_r,H]} Z_e(x) \cdot \mu_1 d\Gamma(x) - \sum_{[a,b]\in\mathcal{I}} \int_{[a,b]} V(a|p,\tau) \cdot \mu_1 d\Gamma(x) \\
&= \int_{(X_r,H]} Z_e(x) \cdot \mu_1 d\Gamma(x) - \sum_{[a,b]\in\mathcal{I}} V(a|p,\tau)\mu([a,b]) \\
&= \int_{(X_r,H]} Z_e(x) \cdot \mu_1 d\Gamma(x) - \sum_{[t(a),t(b)]\in\mathcal{I}_t} Z(t(a))\mu([t(a),t(b)]) \\
&= \int_{\mathcal{K}_t^c} Z(x) \cdot \mu_1 d\Gamma(x),
\end{aligned}$$

which proves Eq. (C-24).

We provide a proof for both Property 1 and Property 2.

Property 1. (p^{I_t}, τ^{I_t}) forms a price-equilibrium pair in I_t such that $\tau^{I_t} \in \mathcal{F}_{I_t}(\mu|_{I_t})$. Also, $V(x|\, p^{I_t}, \tau^{I_t})$ equals $V(x + a - t(a)|\, p, \tau)$ for all $x \in I_t$, and

$$\frac{d\tau_2^{I_t}}{d\Gamma}(x) = \frac{d\tau_2}{d\Gamma}(x + a - t(a)), \quad \Gamma - a.e.\ x \text{ in } I_t.$$

Proof of Property 1. We first show that $\tau^{I_t} \in \mathcal{F}_{I_t}(\mu|_{I_t})$. It is clear that $\tau^{I_t} \in \mathcal{M}(I_t \times I_t)$, and that $\tau_2^{I_t} \ll \Gamma$. To see why $\tau_1^{I_t}$ coincides with μ_{I_t} consider a set $K \subset I_t$ then

$$\begin{aligned}
\tau_1^{I_t}(K) &= \tau_1^{I_t}(K \times I_t) \\
&= \tau((K + a - t(a)) \times (I_t + a - t(a))) \\
&= \tau((K + a - t(a)) \times [a, b]) \\
&= \tau((K + a - t(a)) \times \mathcal{C}) \\
&= \mu(K + a - t(a)) \\
&= \mu(K),
\end{aligned}$$

where the fourth line holds because the set $K + a - t(a)$ is contained in $[a, b]$, and we know there is no flow leaving this interval. Next, using a similar argument we show the property for $d\tau_2^{I_t}/d\Gamma$, let K be a measurable subset of I_t then

$$\begin{aligned}
\int_K \frac{d\tau_2^{I_t}}{d\Gamma}(x)\, d\Gamma(x) &= \tau^{I_t}(I_t \times K) \\
&= \tau([a, b] \times (K + a - t(a))) \\
&= \int_{(K+a-t(a))} \frac{d\tau_2}{d\Gamma}(x)\, d\Gamma(x) \\
&= \int_K \frac{d\tau_2}{d\Gamma}(x + a - t(a))\, d\Gamma(x).
\end{aligned}$$

Using this last property and the prices definition is easy to see that

$$\begin{aligned}
V(x|\, p^{I_t}, \tau^{I_t}) &= \inf\{u \in \mathbb{R} : \Gamma(y \in I_t : U(y, p^{I_t}(y), \frac{d\tau_2^{I_t}}{d\Gamma}(y)) - |y - x| > u) = 0\} \\
&= \inf\{u \in \mathbb{R} : \Gamma(y \in I_t : U(y, p(y + a - t(a)), \frac{d\tau_2}{d\Gamma}(y + a - t(a))) - |y - x| > u) = 0\} \\
&= \inf\{u \in \mathbb{R} : \Gamma(y \in I : U(y, p(y), \frac{d\tau_2}{d\Gamma}(y)) - |y - (x + a - t(a))| > u) = 0\} \\
&= V_I(x + a - t(a)|\, p, \tau),
\end{aligned}$$

but from out flow separation result (see Lemma C-3) we have that $V_I(x + a - t(a)|\, p, \tau) = V(x + a - t(a)|\, p, \tau)$. Using this same approach, the definition of τ^{I_t} and the fact that τ is an equilibrium in $[a, b]$ it is easy to verify the equilibrium condition.

Property 2. (p^{J_t}, τ^{J_t}) forms a price-equilibrium pair in J_t such that $\tau^{J_t} \in \mathcal{F}_{J_t}(\mu|_{J_t})$ and $V(x|p^{J_t}, \tau^{J_t})$ equals $Z(x)$ for all $x \in J_t$.

Proof of Property 2. From the definition of τ^{J_t} it is clear that $\tau^{J_t} \in \mathcal{F}_{J_t}(\mu|_{J_t})$. Also, $d\tau_2^{J_t}/d\Gamma = \mu_1$, $\Gamma - a.e$ in J_t. To see why $V(x|\, p^{J_t}, \tau^{J_t})$ equals $Z(x)$ for all $x \in J_t$, note that for fixed $x \in J_t$

$$\Gamma(y \in J_t : U(y, p^{J_t}(y), \frac{d\tau_2^{J_t}}{d\Gamma}(y)) - |y - x| > Z(x)) = \Gamma(y \in J_t : Z(y) - |x - y| > Z(x)) = 0,$$

where in the first equality we use the definition of $p^{J(t)}$ together with $d\tau_2^{J_t}/d\Gamma = \mu_1$, $\Gamma-a.e$ in J_t. In the second equality we use the Lipschitz property of the function $Z(\cdot)$. That is, $Z(x) \geq V(x|p^{J_t}, \tau^{J_t})$. This upper bound ($\Gamma$−a.e) is tight. Let $\epsilon > 0$ then

$$\begin{aligned}
0 &< \Gamma(B(x,\epsilon/2) \cap J_t) \\
&\leq \Gamma(y \in B(x,\epsilon/2) \cap J_t) : \epsilon > |x-y| + (Z(x) - Z(y))) \\
&= \Gamma(y \in B(x,\epsilon/2) \cap J_t) : Z(y) - |y-x| > Z(x) - \epsilon) \\
&= \Gamma(y \in B(x,\epsilon/2) \cap J_t) : U(y, p^{J_t}(y), \frac{d\tau_2^{J_t}}{d\Gamma}(y)) - |y-x| > Z(x) - \epsilon),
\end{aligned}$$

thus $Z(x)$ is the smallest upper bound (Γ−a.e) and we have $Z(x) = V(x|\, p^{J_t}, \tau^{J_t})$. It is not hard to verify that the equilibrium condition reduces to

$$\tau^{J_t}((x,y) \in J_t \times J_t : Z(y) - |y-x| = Z(x)) = \mu(J_t),$$

and by the definition of τ^{J_t} this is immediately satisfied. □

Proof of Theorem 2. The result follows directly from Proposition 9, and the fact that $[X_l, X_r]$ is an attraction region where $V(\cdot)$ is pinned down. □

Proof of Theorem 3. We separate the proof in several steps. First, we argue that there are at most three attraction regions in the any optimal solution. Then we show that any optimal solution does not have drivers moving to the interval $[W_r, X_r]$ and $[X_l, W_l]$; otherwise, the platform can incentivize the movement of a positive fraction of drivers outside of the center and make strictly larger revenue. After this we put into practice Theorem 1 which prescribes what are the optimal prices and post-relocation supply in each attraction region. In the final main step of the proof we argue that the optimal solution has to be symmetric. We present the proof of two properties that we will use during the main arguments, Property 1 and Property 2, after the main proof.

Attraction regions identification: Lemma 5 establishes that at an optimal solution the attraction region of the origin is well defined with $X_l < 0 < X_r$. So Our first attraction region is the interval $[X_l, X_r]$.

The second and third attraction regions correspond to the intervals $[Y_l, X_l]$ and $[X_r, Y_r]$ with Y_l and Y_r being sink locations. WLOG consider only the right interval, if $Y_r = X_r$ we do not identify any attraction region to the right of X_r. Assume that $X_r < Y_r$, we will show that $A(Y_r) = [X_r, Y_r]$ and $Y_r \notin A(z)$ for any $z \neq Y_r$. In order, to show this we first show that $Y_r \in \mathcal{IR}(X_r|\, p, \tau)$. From Theorem 2 we know that $V(x)$ equals $V(X_r) + (x - X_r)$ for all $x \in [X_r, Y_r]$. Fix $\epsilon > 0$ and

$\delta_0 \in (0, Y_r - X_r)$ then for any $\delta \leq \delta_0$ define the set

$$K^\delta \triangleq \{y \in B(Y_r, \delta) \cap [X_r, Y_r] : U(y) = V(y)\}.$$

Since $\mu((Y_r - \delta, Y_r]) > 0$ and $\tau((Y_r - \delta, Y_r] \times (\mathcal{C} \setminus (Y_r - \delta, Y_r])) = 0$ (otherwise we would obtain a contradiction with Theorem 2), we must have that $\tau_2((Y_r - \delta, Y_r]) > 0$. This together with Lemma A-2 and $\tau_2 \ll \mathit{\Gamma}$ imply that $\mathit{\Gamma}(K^\delta) > 0$. Hence,

$$\begin{aligned}
0 &< \mathit{\Gamma}(K^\delta) \\
&= \mathit{\Gamma}(y \in K^\delta : \epsilon > 0) \\
&= \mathit{\Gamma}(y \in K^\delta : V(X_r) > V(X_r) - \epsilon) \\
&= \mathit{\Gamma}(y \in K^\delta : V(y) - |y - X_r| > V(X_r) - \epsilon) \\
&= \mathit{\Gamma}(y \in K^\delta : U(y) - |y - X_r| > V(X_r) - \epsilon) \\
&\leq \mathit{\Gamma}(y \in B(Y_r, \delta) : U(y) - |y - X_r| > V(X_r) - \epsilon)
\end{aligned}$$

This implies that $V_{B(Y_r,\delta)}(X_r) \geq V(X_r) - \epsilon$. By the choice of ϵ and δ we conclude that $\lim_{\delta \downarrow 0} V_{B(Y_r,\delta)}(X_r)$ is $V(X_r)$. In other words, $Y_r \in \mathcal{IR}(X_r | p, \tau)$. Now, Y_r cannot belong to any other attraction region; otherwise, by the Lemma A-5 the value function would not be as in Theorem 2. Therefore, Y_r is a sink location and $[X_r, Y_r] \subseteq A(Y_r)$. Now if there existed $x \in A(Y_r)$ but $x \notin [X_r, Y_r]$ then we again the value function would not be as in Theorem 2. In conclusion, $A(Y_r) = [X_r, Y_r]$ and $Y_r \notin A(z)$ for any $z \neq Y_r$.

No supply in $[W_r, X_r]$: Next we argue that at an optimal solution (p, τ) we must have that $\tau_2([W_r, X_r]) = 0$, the same is true for the left side. Suppose by contradiction that $\tau_2([W_r, X_r]) > 0$ and denote this amount of supply by q_r, we construct a new solution $(\tilde{p}, \tilde{\tau})$ that yields an strict objective improvement. Observe that,

$$0 < q_r = \tau(\mathcal{C} \times [W_r, X_r]) = \tau([W_r, X_r] \times [W_r, X_r]) \leq \mu([W_r, X_r]) = \mu_1 \cdot (X_r - W_r).$$

That is, from the total amount of initial supply in $[W_r, X_r]$ we have that q_r units stay within $[W_r, X_r]$ and a total of $\mu_1 \cdot (X_r - W_r) - q_r$ units travel to $[0, W_r]$. Note that for this q_r units of mass their V is bounded by ψ_1 and, therefore, what the platform can make from them is strictly bounded by $\psi_1 \cdot q_r$ (times a scaling factor). Let $\tilde{X}_r \in [W_r, X_r)$ be such that $q_r = \mu_1 \cdot (X_r - \tilde{X}_r)$. In the new solution, we will modify the attraction region $[X_l, X_r]$ to be $[X_l, \tilde{X}_r]$. We will maintain the same prices and post-relocation supply in the origin's attraction region. However, to the right side of $\tilde{X}_r$ we will set new prices that will be consistent with a new value function and flows that upper bound those of the old solution, see Figure 13.

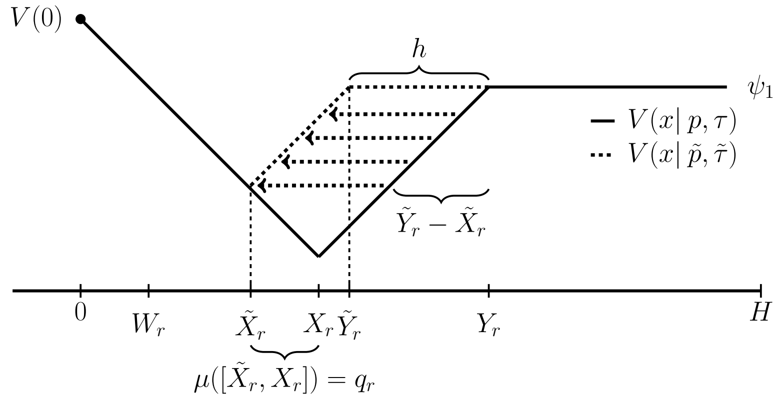

Figure 13: **No supply in** $[W_r, X_r]$. The new solution moves the right end of the attraction region from X_r to $\tilde{X}_r$, so now a mass q_r of drivers can travel towards the periphery. From this mass the platform now makes ψ_1 instead of $V(x)$ with $V(x) < \psi_1$.

We begin our construction of $(\tilde{p}, \tilde{\tau})$ with the interval $I_r^1 = [\tilde{X}_r, \tilde{Y}_r]$, where $\tilde{Y}_r$ is such that $\psi_1 = V(\tilde{X}_r) + (\tilde{Y}_r - \tilde{X}_r)$. Let $h \triangleq 2 \cdot (X_r - \tilde{X}_r)$, we define flows for any $\mathcal{L} \subseteq I_r^1 \times I_r^1$ by

$$\tau^{I_r^1}(\mathcal{L}) = \tau\Big(\mathcal{L} + (h, h)\Big).$$

Consider the set

$$K \triangleq \{x \in I_r^1 : \frac{d\tau_2^{I_r^1}}{d\Gamma}(x) \leq \frac{d\tau_2}{d\Gamma}(x+h)\}.$$

We set prices to be such that

$$U\Big(x, p^{I_r^1}(x), \frac{d\tau_2^{I_r^1}}{d\Gamma}(x)\Big) = U\Big(x+h, p(x+h), s^{\tau}(x+h)\Big), \quad \forall x \in K, \tag{C-26}$$

otherwise, we set the prices equal to zero. We will prove at the end of the present proof, in Property 1, that $(p^{I_r^1}, \tau^{I_r^1})$ forms a price-equilibrium pair in I_r^1 such that $V(x|p^{I_r^1}, \tau^{I_r^1})$ equals $V(\tilde{X}_r)+(x-\tilde{X}_r)$ and $\Gamma(K^c) = 0$.

In the interval $I_r^2 = (\tilde{Y}_r, H]$ we can achieve the optimal solution when there is no demand shock. As in the optimal solution in the pre-demand shock environment (see Proposition 6) we set prices equal to ρ_1 and the flows are such that $d\tau^{I_r^2}/d\Gamma$ equals μ_1, $\Gamma - a.e$ in I_r^2.

The interval $I_r^0 = [X_l, \tilde{X}_r]$ is more involved. Observe that all the initial flow to the right of the origin that we have to allocate in $[0, \tilde{X}_r]$ equals $\mu_1 \cdot X_r - q_r$. This is exactly the same amount of drivers in $[0, X_r]$ that travels to $[0, W_r]$ according to τ. Our new solution will generate the same post-relocation supply than τ in $[0, W_r]$ but this time only using drivers from $[0, \tilde{X}_r]$.

We use the same prices, that is $p^{I_r^0}(x) = p(x)$ for all $x \in [X_l, \tilde{X}_r]$. For the flows we define them through two measures: the flow that goes from $[X_l, 0]$ to $[X_l, 0]$ and the flow that goes from $[0, \tilde{X}_r]$

to $[0, \tilde{X}_r]$. For the first flow we use $\tau^\ell = \tau|_{[X_l,0]}$, for the second measure τ^r we will use a monotone coupling as in the proof of Theorem 1 (see e.g, Santambrogio (2015) for details). Define the initial supply to the right measure μ^r to be equal to $\mu|_{[0,\tilde{X}_r]}$, and the final supply S^r to be

$$S^r(\mathcal{B}) \triangleq \tau([0, X_r] \times \mathcal{B}), \quad \text{for any measurable set } \mathcal{B} \subseteq [0, \tilde{X}_r].$$

Note that $S^r([0, W_r])$ equals $\mu^r([0, \tilde{X}_r])$. Given this we define τ^r by

$$\tau^r(\mathcal{L}) \triangleq (F_{\mu^r}^{[-1]}, F_{S^r}^{[-1]})_{\#} m(\mathcal{L}), \quad \text{for any measurable set } \mathcal{L} \subseteq [0, \tilde{X}_r] \times [0, \tilde{X}_r],$$

where # correspond to the push-forward operator. For any measure ν defined in $[0, \tilde{X}_r]$ we define its cumulative function and pseudo-inverse by

$$F_\nu(y) \triangleq \nu([0, y]), \quad \forall\, y \geq 0 \quad \text{and} \quad F_\nu^{[-1]}(t) \triangleq \inf\{y \geq 0 : F_\nu(y) \geq t\}, \quad \forall t \in [0, \mu^r([0, \tilde{X}_r])].$$

Effectively, τ^r transports the initial mass in $[0, \tilde{X}_r]$ to the final supply distribution (considering only drivers that come from the right) in $[0, W_r]$ as prescribed by τ. The final flow measure $\tau^{I_r^0}$ correspond to $\tau^\ell + \tau^r|_{[0,\tilde{X}_r]}$. In Property 2 below we show that $(p^{I_r^0}, \tau^{I_r^0})$ is a price-equilibrium pair such that

$$\mathbf{Rev}_{[X_l,W_r]}(p^{I_r^0}, \tau^{I_r^0}) = \mathbf{Rev}_{[X_l,W_r]}(p, \tau).$$

The solution $(\tilde{p}, \tilde{\tau})$ is constructed by pasting (see Lemma C-3) the old solution is $[-H, X_l)$ with the new solution in I^0, I_r^1 and I_r^2. The pasting is possible because the equilibrium utility function coincide in the boundaries of these intervals. This new solution preserves the platform's revenue in $[-H, W_r] \cup [Y_r, H]$ but it strictly improves it in $[W_r, Y_r]$. Indeed, note that

$$q_r = \int_{[X_r,Y_r]} s^\tau(x)\,dx - \int_{[\tilde{X}_r,\tilde{Y}_r]} s^{\tilde{\tau}}(x)\,dx = \int_{[\tilde{X}_r,\tilde{Y}_r]} \underbrace{(s^\tau(x+h) - s^{\tilde{\tau}}(x))}_{\geq 0 \ \Gamma-a.e}\,dx + \int_{[X_r,X_r+(X_r-\tilde{X}_r)]} s^\tau(x)\,dx, \tag{C-27}$$

thus

$$\begin{aligned}
\frac{1}{\gamma} \cdot \mathbf{Rev}_{[W_r,Y_r]}(\tilde{p}, \tilde{\tau}) &= \int_{[W_r,\tilde{X}_r]} V(x|\tilde{p}, \tilde{\tau}) \cdot s^{\tilde{\tau}}(x)\,dx + \int_{[\tilde{X}_r,Y_r]} V(x|\tilde{p}, \tilde{\tau}) \cdot s^{\tilde{\tau}}(x)\,dx \\
&\overset{(a)}{=} \int_{[\tilde{X}_r,Y_r]} V(x|\tilde{p}, \tilde{\tau}) \cdot s^{\tilde{\tau}}(x)\,dx \\
&\overset{(b)}{=} \int_{[\tilde{X}_r,\tilde{Y}_r]} V(x|\tilde{p}, \tilde{\tau}) \cdot s^{\tilde{\tau}}(x)\,dx + \psi_1 \cdot 2 \cdot q_r \\
&\overset{(c)}{>} \int_{[\tilde{X}_r,\tilde{Y}_r]} V(x|\tilde{p}, \tilde{\tau}) \cdot s^{\tilde{\tau}}(x)\,dx + \psi_1 \cdot q_r + \int_{[W_r,X_r]} V(x) \cdot s^\tau(x)\,dx \\
&\overset{(d)}{\geq} \int_{[\tilde{X}_r,\tilde{Y}_r]} V(x|\tilde{p}, \tilde{\tau}) \cdot s^\tau(x+h)\,dx + \int_{[W_r,X_r+(X_r-\tilde{X}_r)]} V(x) \cdot s^\tau(x)\,dx \\
&\overset{(e)}{=} \int_{[W_r,Y_r]} V(x) \cdot s^\tau(x)\,dx = \frac{1}{\gamma} \cdot \mathbf{Rev}_{[W_r,Y_r]}(p, \tau),
\end{aligned}$$

where (a) follows because $\tilde{\tau}$ does not put mass in $[W_r, \tilde{X}_r]$, (b) because $Y_r - \tilde{Y}_r$ equals $2 \cdot (X_r - \tilde{X}_r)$. Using the fact that $\tau_2([W_r, X_r]) = q_r$ we obtain (c), while (d) follows from Eq. (C-27) and (e) from $V(x|\,\tilde{p}, \tilde{\tau})$ being equal to $V(x+h)$ for all $x \in [\tilde{X}_r, \tilde{Y}_r]$.

In conclusion, any optimal solution must satisfy both $\tau_2([W_r, X_r]) = 0$ and $\tau_2([X_l, W_l]) = 0$.

Using Theorem 1: All the conditions in Theorem 1 are met. So, for any of the three attraction regions if (p, τ) is not already as in the statement of the theorem we can find at least a weak improvement. That is, we can restrict to solution as in Theorem 1. Therefore, the prices are as stated in the present theorem, and there exists $\beta_c^l \in [W_l, 0]$, $\beta_c^r \in [0, W_r]$, $\beta_p^l \in [Y_l, X_l]$ and $\beta_p^r \in [X_r, Y_r]$ such that

$$s^\tau(x) = \begin{cases} 0 & \text{if } x \in (\beta_c^r, \beta_p^r) \cup (\beta_p^l, \beta_c^l), \\ \psi_x^{-1}(V(x|\,p,\tau)) & \text{otherwise,} \end{cases}$$

with

$$\int_{\beta_c^l}^{\beta_c^r} \psi_x^{-1}(V(x|\,p,\tau))\, d\Gamma(x) = \mu_1 \cdot (X_r - X_l)$$

and

$$\int_{\beta_p^r}^{Y_r} \psi_x^{-1}(V(x|\,p,\tau))\, d\Gamma(x) = \mu_1 \cdot (Y_r - X_r), \quad \int_{Y_l}^{\beta_p^l} \psi_x^{-1}(V(x|\,p,\tau))\, d\Gamma(x) = \mu_1 \cdot (X_l - Y_l).$$

Note that the fact that $\beta_c^l \in [W_l, 0]$ and $\beta_c^r \in [0, W_r]$, does not come directly from Theorem 1 but rather is a consequence of that any optimal solution must satisfy both $\tau_2([W_r, X_r]) = 0$ and $\tau_2([X_l, W_l]) = 0$. Also, observe that Theorem 1 only gives us a solution in each attraction but above we have stated the solution for the entire city. The only missing interval are $[-H, Y_l]$ and $[Y_r, H]$. In this intervals, as in the pre-shock environment, the solution set prices equal to ρ_1 and the supply at every location is μ_1, in turn, the V equals ψ_1 in this region. This gives a complete solution to the platform's problem up to three values: $V(0), X_l, X_r$.

Symmetry: In the last main step of the proof we argue that the solution is symmetric. After proving this, the solution will take the exact form in the statement of the present theorem.

Note that given a value for $V(0)$ and an central attraction region characterize by X_l and X_r we can characterize the optimal solution as we did in **Using Theorem 1**. So fix these three values and the optimal solution associated to them. We now proceed to construct a new solution that yields a strict objective improvement when the solution is not symmetric. WLOG assume that $|X_l| > X_r$ and let $\delta = (|X_l| - X_r)/2$. Consider the solution $(\tilde{p}, \tilde{\tau})$ associated to the values

$$\tilde{V}(0) = V(0), \quad \tilde{X}_l = X_l + \delta, \quad \tilde{X}_r = X_r + \delta.$$

Note that with this values we have $|\tilde{X}_l|, \tilde{W}_r \geq W_r$ and $\tilde{Y}_i = Y_i + 2 \cdot \delta$ for $i \in \{l, r\}$. We next show that this new solution yields a weak objective improvement in the center, and a strict objective improvement in the periphery.

Note that given $\tilde{V}(0), \tilde{X}_l$ and $\tilde{X}_r$ Theorem 2 characterizes $V(\cdot|\, \tilde{p}, \tilde{\tau})$. It has the same shape than $V(\cdot|p, \tau)$ except that now the dip in $[\tilde{Y}_l, W_l]$ is smaller, while the dip in $[W_r, Y_r]$ is larger. See Figure 14 for a graphical representation. Consider first the solution in the center, $[\tilde{X}_l, \tilde{X}_r]$. This interval

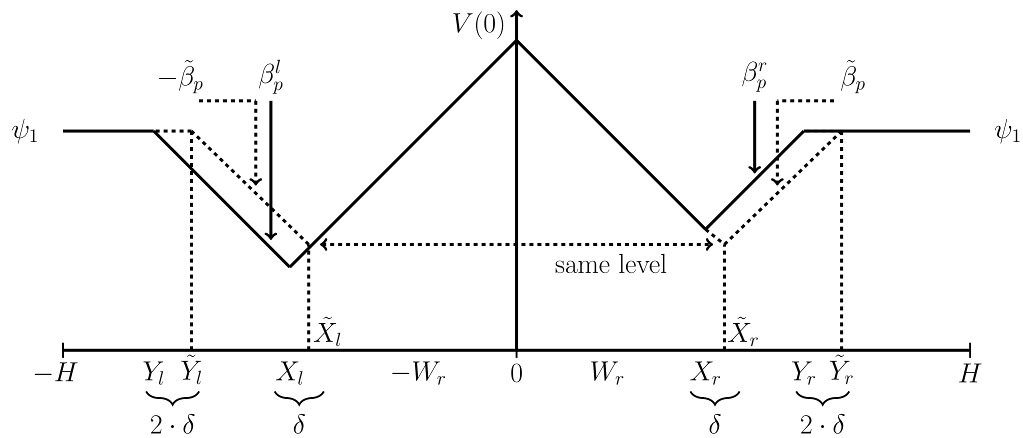

Figure 14: **Symmetry** argument.

contains the same amount of drivers that the old attraction region. The difference is that it lost a mass of $\mu_1 \cdot \delta$ drivers to the left and gain the same mass to the right. As in the discussion that follows Theorem 1 the optimal solution in $[\tilde{X}_l, \tilde{X}_r]$ can be obtained using a knapsack argument. This new attraction region is symmetric, $|\tilde{X}_l| = \tilde{X}_r$, with equal mass of drivers at both sides of the origin. Therefore the knapsack solution must be symmetric, with $\tilde{\beta}_c \in [0, W_r]$ such that

$$s^{\tilde{\tau}}(x) = \psi_x^{-1}(V(x|\, \tilde{p}, \tilde{\tau})) = \psi_x^{-1}(V(x|\, p, \tau)), \quad \forall x \in [-\tilde{\beta}_c, \tilde{\beta}_c],$$

and equals zero otherwise, and

$$\int_{-\tilde{\beta}_c}^{\tilde{\beta}_c} \psi_x^{-1}(V(x|\, \tilde{p}, \tilde{\tau})\, d\Gamma(x) = \mu_1 \cdot (\tilde{X}_r - \tilde{X}_l) = \mu_1 \cdot (X_r - X_l).$$

Note that $\tilde{\beta}_c \in [0, W_r]$ is a consequence of the having $\beta_c^l \in [W_l, 0]$ and $\beta_c^r \in [0, W_r]$ in the old solution. Theorem 1 prescribes how to formally implement this solution through prices and flows. We omit the details of how to construct the flows, but we note that the optimal prices are given $\tilde{p}(x) = \rho_x^{loc}(s^{\tilde{\tau}}(x))$. In the case that $\tilde{\beta} = 0$ then $s^{\tilde{\tau}}(0) = \mu_1 \cdot (X_r - X_l)$ and $\tilde{p}(0)$ is such that $U(0, p(0), s^{\tilde{\tau}}(0)) = V(0)$. The platform's revenue in the new center is then

$$\frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l, \tilde{X}_r]}(\tilde{p}, \tilde{\tau}) = \int_{\tilde{X}_l}^{\tilde{X}_r} V(x|\, \tilde{p}, \tilde{\tau}) \cdot s^{\tilde{\tau}}(x)\, dx = \int_{-\tilde{\beta}_c}^{\tilde{\beta}_c} V(x) \cdot \psi_x^{-1}(V(x))\, dx.$$

This expression is an upper bound for the platform's revenue under (p, τ) in $[X_l, X_r]$. In fact, WLOG assume $\beta_c^r \geq |\beta_c^l|$ which implies that $\tilde{\beta}_c \in [|\beta_c^l|, \beta_c^r]$ and we must have

$$\begin{aligned}
\frac{1}{\gamma} \cdot \mathbf{Rev}_{[X_l, X_r]}(p, \tau) &= \int_{\beta_c^l}^{\beta_c^r} V(x) \cdot \psi_x^{-1}(V(x))\, dx \\
&= \int_{\beta_c^l}^{|\beta_c^l|} V(x) \cdot \psi_x^{-1}(V(x))\, dx + \int_{|\beta_c^l|}^{\beta_c^r} V(x) \cdot \psi_x^{-1}(V(x))\, dx \\
&= \frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l, \tilde{X}_r]}(\tilde{p}, \tilde{\tau}) - 2 \cdot \int_{|\beta_c^l|}^{\tilde{\beta}_c} V(x) \cdot \psi_x^{-1}(V(x))\, dx + \int_{|\beta_c^l|}^{\beta_c^r} V(x) \cdot \psi_x^{-1}(V(x))\, dx \\
&= \frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l, \tilde{X}_r]}(\tilde{p}, \tilde{\tau}) - \int_{|\beta_c^l|}^{\tilde{\beta}_c} V(x) \cdot \psi_x^{-1}(V(x))\, dx + \int_{\tilde{\beta}_c}^{\beta_c^r} V(x) \cdot \psi_x^{-1}(V(x))\, dx \\
&\leq \frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l, \tilde{X}_r]}(\tilde{p}, \tilde{\tau}) + V(\tilde{\beta}_c) \cdot \Big(- \int_{|\beta_c^l|}^{\tilde{\beta}_c} \psi_x^{-1}(V(x))\, dx + \int_{\tilde{\beta}_c}^{\beta_c^r} \psi_x^{-1}(V(x))\, dx \Big) \\
&= \frac{1}{\gamma} \cdot \mathbf{Rev}_{[\tilde{X}_l, \tilde{X}_r]}(\tilde{p}, \tilde{\tau}).
\end{aligned}$$

That is, the new solution in the center is a weakly improvement over the old solution.

Now let us consider the periphery. Since $|\tilde{X}_l| = \tilde{X}_r$ both right and left periphery are symmetric. Thus the optimal solution as given by Theorem 1 is the symmetric at both sides. The post-relocation supply is characterize by $\tilde{\beta}_p \in [\tilde{X}_r, \tilde{Y}_r]$ such that

$$s^{\tilde{\tau}}(x) = \psi_x^{-1}(V(x|\, \tilde{p}, \tilde{\tau})) = \psi_x^{-1}(V(X_r) + (x - X_r) - 2 \cdot \delta), \quad \forall x \in [\tilde{\beta}_p, \tilde{Y}_r],$$

and equals zero otherwise, and

$$\int_{\tilde{\beta}_p}^{\tilde{Y}_r} \psi_x^{-1}(V(x|\, \tilde{p}, \tilde{\tau})\, d\Gamma(x) = \mu_1 \cdot (\tilde{Y}_r - \tilde{X}_r) = \mu_1 \cdot (Y_r - X_r) + \mu_1 \cdot \delta.$$

The optimal prices are $\tilde{p}(x) = \rho_x^{loc}(s^{\tilde{\tau}}(x))$. As before we omit the characterization of the equilibrium flow as their existence is guaranteed by Theorem 1. The platforms revenue in the periphery is

$$\frac{1}{\gamma} \cdot \mathbf{Rev}_{[-H, \tilde{X}_l] \cup [\tilde{X}_r, H]}(\tilde{p}, \tilde{\tau}) = 2 \cdot \int_{\tilde{\beta}_p}^{\tilde{Y}_r} V(x|\, \tilde{p}, \tilde{\tau}) \cdot \psi^{-1}(V(x|\, \tilde{p}, \tilde{\tau}))\, dx + 2 \cdot \psi_1 \cdot \mu_1 \cdot (H - \tilde{Y}_r),$$

where we have dropped the subindex x from ψ_x^{-1} to stress the fact that in this part of the city this subindex does not change the congestion function. We need to compare this revenue with the revenue of the old solution in the periphery. Not that since $|X_l| > X_r$ we must have

$$Y_r - \beta_p^r < \tilde{Y}_r - \tilde{\beta}_p < \beta_p^l - Y_l.$$

Thus,

$$\begin{aligned}
\frac{1}{\gamma}\cdot \mathbf{Rev}_{[-H,X_l]\cup[X_r,H]}(p,\tau) &= \int_{Y_l}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx + \int_{\beta_p^r}^{Y_r} V(x)\cdot\psi^{-1}(V(x))\,dx \\
&+ \psi_1\cdot\mu_1\cdot(H - Y_r + Y_l + H) \\
&= \int_{Y_l+(Y_r-\beta_p^r)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx + 2\cdot\int_{\beta_p^r}^{Y_r} V(x)\cdot\psi^{-1}(V(x))\,dx \\
&+ \psi_1\cdot\mu_1\cdot(H - Y_r + Y_l + H) \\
&= \int_{Y_l+(Y_r-\beta_p^r)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx + 2\cdot\int_{\beta_p^r+2\cdot\delta}^{\tilde{Y}_r} V(x|\,\tilde{p},\tilde{\tau})\cdot\psi^{-1}(V(x|\,\tilde{p},\tilde{\tau}))\,dx \\
&+ 2\cdot\psi_1\cdot\mu_1\cdot(H-\tilde{Y}_r) \\
&= \underbrace{\int_{Y_l+(Y_r-\beta_p^r)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx - 2\cdot\int_{\tilde{\beta}_p}^{\beta_p^r+2\cdot\delta} V(x|\,\tilde{p},\tilde{\tau})\cdot\psi^{-1}(V(x|\,\tilde{p},\tilde{\tau}))\,dx}_{(a)} \\
&+ \frac{1}{\gamma}\cdot\mathbf{Rev}_{[-H,\tilde{X}_l]\cup[\tilde{X}_r,H]}(\tilde{p},\tilde{\tau}),
\end{aligned}$$

So if we show that the term (a) is strictly negative we will be done. Not that

$$\begin{aligned}
(a) &= \int_{Y_l+(Y_r-\beta_p^r)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx - 2\cdot\int_{Y_l+(Y_r-\beta_p^r)}^{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)} V(x)\cdot\psi^{-1}(V(x))\,dx \\
&= \int_{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)}^{\beta_p^l} V(x)\cdot\psi^{-1}(V(x))\,dx - \int_{Y_l+(Y_r-\beta_p^r)}^{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)} V(x)\cdot\psi^{-1}(V(x))\,dx \\
&< V(Y_l+(\tilde{Y}_r-\tilde{\beta}_p))\cdot\Big(\int_{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)}^{\beta_p^l} \psi^{-1}(V(x))\,dx - \int_{Y_l+(Y_r-\beta_p^r)}^{Y_l+(\tilde{Y}_r-\tilde{\beta}_p)} \psi^{-1}(V(x))\,dx\Big) \\
&= 0.
\end{aligned}$$

In conclusion, we have constructed a new symmetric solution that yields an strict revenue improvement over the old solution. Therefore, any optimal solution ought to be symmetric.

Property 1. $(p^{I_r^1},\tau^{I_r^1})$ forms a price-equilibrium pair in I_r^1 such that $V(x|p^{I_r^1},\tau^{I_r^1})$ equals $V(\tilde{X}_r)+(x-\tilde{X}_r)$ and $\Gamma(K^c)=0$.

Proof of Property 1. We first show that $\tau^{I_r^1}\in\mathcal{F}_{I_r^1}(\mu|_{I_r^1})$. It is clear that $\tau^{I_r^1}\in\mathcal{M}(I_r^1\times I_r^1)$, and that $\tau_2^{I_r^1} \ll \Gamma$. To see why $\tau_1^{I_r^1}$ coincides with $\mu_{I_r^1}$ consider a set $I \subset I_r^1$ then

$$\begin{aligned}
\tau_1^{I_r^1}(K) &= \tau_1^{I_r^1}(K \times I_r^1) \\
&= \tau((I+h) \times (I_r^1 + h)) \\
&= \tau((I+h) \times [\tilde{X}_r + h, Y_r]) \\
&= \tau((I+h) \times \mathcal{C}) \\
&= \mu(I+h) \\
&= \mu(I),
\end{aligned}$$

where the fourth line holds because the set $I+h$ is contain in $[\tilde{X}_r + h, Y_r]$, and we know there is no flow leaving this interval. Next, using a similar argument we show the property for $d\tau_2^{I_r^1}/d\Gamma$, let I be a measurable subset of I_r^1 then

$$\begin{aligned}
\int_I \frac{d\tau_2^{I_r^1}}{d\Gamma}(x)\, d\Gamma(x) &= \tau^{I_r^1}(I_r^1 \times I) \\
&= \tau([\tilde{X}_r + h, Y_r] \times (I+h)) \\
&\leq \tau([X_r, Y_r] \times (I+h)) \\
&= \int_{(I+h)} \frac{d\tau_2}{d\Gamma}(x)\, d\Gamma(x) \\
&= \int_I \frac{d\tau_2}{d\Gamma}(x+h)\, d\Gamma(x),
\end{aligned}$$

that is, $\Gamma(K^c) = 0$. As for the equilibrium utility function let $x \in [\tilde{X}_r, \tilde{Y}_r)$ we have

$$\begin{aligned}
V(x|\, p^{I_r^1}, \tau^{I_r^1}) &= \inf\{u \in \mathbb{R} : \Gamma(y \in I_r^1 : U(y, p^{I_r^1}(y), \frac{d\tau_2^{I_r^1}}{d\Gamma}(y)) - |y-x| > u) = 0\} \\
&= \inf\{u \in \mathbb{R} : \Gamma(y \in I_r^1 : U(y, p(y+h), \frac{d\tau_2}{d\Gamma}(y+h)) - |y-x| > u) = 0\} \\
&= \inf\{u \in \mathbb{R} : \Gamma(y \in [\tilde{X}_r + h, Y_r] : U(y, p(y), \frac{d\tau_2}{d\Gamma}(y)) - |y-(x+h)| > u) = 0\} \\
&\leq V(x+h|\, p, \tau).
\end{aligned}$$

Actually this upper bound is tight. Indeed, Fix any $\epsilon > 0$ and consider $\delta > 0$ small enough such that $(x+h) \notin B(Y_r, \delta)$. We have $\tau_2(\{y \in B(y,\delta) \cap [\tilde{X}_r + h, Y_r] : U(y) = V(y)\}) > 0$ which implies that $\Gamma(\{y \in B(Y_r, \delta) \cap [\tilde{X}_r + h, Y_r] : U(y) = V(y)\}) > 0$ and, therefore,

$$\begin{aligned}
0 &< \Gamma(\{y \in B(Y_r,\delta) \cap [\tilde{X}_r + h, Y_r] : U(y) = V(y), \epsilon + y - (x+h) > |y-(x+h)|\}) \\
&= \Gamma(\{y \in B(Y_r,\delta) \cap [\tilde{X}_r + h, Y_r] : U(y) = V(y), U(y) - |y-(x+h)| > V(x+h) - \epsilon\}) \\
&\leq \Gamma(\{y \in [\tilde{X}_r + h, Y_r] : U(y) - |y-(x+h)| > V(x+h) - \epsilon\}) \\
&= \Gamma(\{y \in I_r^1 : U(y, p^{I_r^1}(y), \frac{d\tau_2^{I_r^1}}{d\Gamma}(y)) - |y-x| > V(x+h) - \epsilon\}),
\end{aligned}$$

therefore $V(x|\, p^{I_r^1}, \tau^{I_r^1})$ equals $V(x+h)$ for all $x \in [\tilde{X}_r, \tilde{Y}_r)$, and by continuity for all $x \in I_r^1$. Since $V(x+h)$ equals $V(\tilde{X}_r) + (x - \tilde{X}_r)$ we obtain the desired result.

Now we need to verify that this selection of prices and flows yields an equilibrium. That is, we need show that the set

$$\mathcal{E}_{I_r^1} = \Big\{(x,y) \in I_r^1 \times I_r^1 : \Pi(x, y, p^{I_r^1}(y), \frac{d\tau_2^{I_r^1}}{d\Gamma}(y)) = V(x|\, p^{I_r^1}, \tau^{I_r^1})\Big\},$$

has $\tau^{I_r^1}$ measure equal to $\mu(I_r^1)$. Observe that $\tau(\mathcal{E}_{I_r^1})$ equals

$$\tau\Big(\Big\{(x,y) \in [\tilde{X}_r + h, Y_r] \times [\tilde{X}_r + h, Y_r] : \Pi(x-h, y-h, p^{I_r^1}(y-h), \frac{d\tau_2^{I_r^1}}{d\Gamma}(y-h)) = V(x)\Big\}\Big),$$

using that $\Gamma(K^c) = 0$ and the way we chose the prices one can verify that this expression equals

$$\tau\Big(\Big\{(x,y) \in [\tilde{X}_r + h, Y_r] \times [\tilde{X}_r + h, Y_r] : \Pi(x, y, p(y), s^\tau(y)) = V(x|\, p, \tau)\Big\}\Big).$$

There is no τ flow of drivers leaving $[\tilde{X}_r + h, Y_r]$ so the fact that τ is an equilibrium flow implies that this last expression equals $\mu([\tilde{X}_r + h, Y_r])$, which equals $\mu(I_r^1)$.

Property 2. $(p^{I_r^0}, \tau^{I_r^0})$ is a price-equilibrium pair such that

$$\mathbf{Rev}_{[X_l, W_r]}(p^{I_r^0}, \tau^{I_r^0}) = \mathbf{Rev}_{[X_l, W_r]}(p, \tau).$$

Proof of Property 2. First a couple of observations, note that for any $y \in [0, \tilde{X}_r]$ and the set $[0, y]$ then

$$\begin{aligned}
\tau_1^r([0,y]) &= \tau^r([0,y] \times [0, \tilde{X}_r]) \\
&= m\Big(t \in [0, \mu^r([0, \tilde{X}_r])] : F_{\mu^r}^{[-1]}(t) \in [0, y]\Big) \\
&= m\Big(t \in [0, \mu^r([0, \tilde{X}_r])] : 0 \leq t \leq F_{\mu^r}(y)\Big) \\
&= F_{\mu^r}(y),
\end{aligned}$$

and the same argument holds for τ_2^r and S^r, this characterizes the first and second marginals of τ^r. Furthermore, it's not difficult to see that for $y_1, y_2 \in [0, \tilde{X}_r]$ we have

$$\tau^r([0,y_1] \times [0,y_2]) = m\Big(t \in [0, \mu^r([0, \tilde{X}_r])] : t \leq F_{\mu^r}(y_1),\ t \leq F_{S^r}(y_2)\Big) = F_{\mu^r}(y_1) \wedge F_{S^r}(y_2). \quad \text{(C-28)}$$

Next, we show that $\tau^{I_r^0} \in \mathcal{F}_{I_r^0}(\mu|_{I_r^0})$ is an equilibrium in I_r^0. In order to do so we first show that$\tau^{I_r^0} \in \mathcal{F}_{I_r^0}(\mu|_{I_r^0})$. Second, we compute the supply density of $\tau_2^{I_r^0}$ and corroborate they coincide with s^τ. Third, we compute $V_{I_r^0}(\cdot|\, p^{I_r^0}, \tau^{I_r^0})$ and verify is coincides with $V(\cdot|\, p, \tau)$ in I_r^0. Finally, we check the equilibrium condition.

Clearly $\tau^{I_r^0}$ is a non-negative measure in $I_r^0 \times I_r^0$ because is the sum of non-negative measures. Now we check that $\tau_1^{I_r^0} = \mu|_{I_r^0}$. Consider a measurable set $\mathcal{B} \subseteq I_r^0$ then

$$\begin{aligned}
\tau_1^{I_r^0}(\mathcal{B}) &= \tau((\mathcal{B} \cap [X_l, 0]) \times [X_l, 0]) + \tau^r((\mathcal{B} \cap [0, \tilde{X}_r]) \times [0, \tilde{X}_r]) \\
&= \tau((\mathcal{B} \cap [X_l, 0]) \times \mathcal{C}) + \mu^r(\mathcal{B} \cap [0, \tilde{X}_r]) \\
&= \mu(\mathcal{B} \cap [X_l, 0]) + \mu(\mathcal{B} \cap [0, \tilde{X}_r]) \\
&= \mu|_{I_r^0}(\mathcal{B})
\end{aligned}$$

and thus we also have $\tau_1^{I_r^0} \ll \Gamma$. For the second marginal of $\tau^{I_r^0}$ we have

$$\begin{aligned}
\tau_2^{I_r^0}(\mathcal{B}) &= \tau([X_l, 0] \times (\mathcal{B} \cap [X_l, 0])) + \tau^r([0, \tilde{X}_r] \times (\mathcal{B} \cap [0, \tilde{X}_r])) \\
&= \tau([X_l, 0] \times (\mathcal{B} \cap [X_l, 0])) + S^r(\mathcal{B} \cap [0, \tilde{X}_r]) \\
&= \tau([X_l, 0] \times (\mathcal{B} \cap [X_l, 0])) + \tau([0, X_r] \times (\mathcal{B} \cap [0, \tilde{X}_r])) \\
&= \tau_2(\mathcal{B} \cap [X_l, 0)) + \tau_2(\mathcal{B} \cap (0, \tilde{X}_r]) + \tau_2(\mathcal{B} \cap \{0\}) \\
&= \tau_2|_{I_r^0}(\mathcal{B}),
\end{aligned}$$

and thus $\tau_2^{I_r^0} \ll \Gamma$. We conclude that $\tau^{I_r^0} \in \mathcal{F}_{I_r^0}(\mu|_{I_r^0})$. From this we can also conclude that

$$\frac{d\tau_2^{I_r^0}}{d\Gamma}(x) = s^\tau(x), \quad \Gamma - a.e. \ x \ \text{ in } \ I_r^0.$$

Next we compute the equilibrium utilities. We show that $V(x|\, p^{I_r^0}, \tau^{I_r^0})$ equals $V(x|\, p, \tau)$ for all $x \in I_r^0$. Observe that $\Gamma - a.e. \ y$ in I_r^0

$$U(y, p^{I_r^0}(y), s^{\tau^{I_r^0}}(y)) = U(y, p(y), s^\tau(y)),$$

and, therefore, $V(x|\, p, \tau) \geq V(x|\, p^{I_r^0}, \tau^{I_r^0})$. Using the same argument that we used for the proof of Property 1 we can argue that this upper bound is tight, that is, $V(x|\, p, \tau) = V(x|\, p^{I_r^0}, \tau^{I_r^0})$.

Now the equilibrium condition. Consider the equilibrium set

$$\mathcal{E}_{I_r^0} \triangleq \Big\{(x, y) \in I_r^0 \times I_r^0 : U(y, p^{I_r^0}(y), s^{\tau^{I_r^0}}(y)) - |y - x| = V(x|\, p^{I_r^0}, \tau^{I_r^0})\Big\},$$

we need to verify that $\tau^{I_r^0}(\mathcal{E}_{I_r^0})$ equals $\mu(I_r^0)$. First, for $\tau^l(\mathcal{E}_{I_r^0})$ we have

$$\begin{aligned}
\tau^l(\mathcal{E}_{I_r^0}) &= \tau\Big(\Big\{(x, y) \in [X_l, 0] \times [X_l, 0] : U(y, p(y), s^\tau(y)) - |y - x| = V(x|\, p, \tau)\Big\}\Big) \\
&= \tau([X_l, 0] \times [X_l, 0]) \\
&= \tau([X_l, 0] \times \mathcal{C}) \\
&= \mu([X_l, 0])
\end{aligned}$$

where we have used our choice of prices, the relation between $d\tau_2^{I_r^0}/d\Gamma$ and s^τ, and the fact that τ is an equilibrium flow that does not setnd flow out of $[X_l, 0]$. For $\tau^r|_{[0,\tilde{X}_r]}$, note that its second marginal is S^r and, therefore, Lemma A-2 implies that

$$\tau^r|_{[0,\tilde{X}_r]}(\mathcal{E}_{I_r^0}) = \tau^r\Big(\Big\{(x,y) \in [0,\tilde{X}_r] \times [0,\tilde{X}_r] : V(y|\, p,\tau) - |y-x| = V(x|\, p,\tau)\Big\}\Big),$$

and because $V(z|\, p,\tau)$ equals $V(0) - z$ for any $z \in [0,\tilde{X}_r]$ we have

$$\begin{aligned}
\tau^r|_{[0,\tilde{X}_r]}(\mathcal{E}_{I_r^0}) &= \tau^r\Big(\Big\{(x,y) \in [0,\tilde{X}_r] \times [0,\tilde{X}_r] : -y - |y-x| = -x\Big\}\Big) \\
&= \tau^r\Big(\Big\{(x,y) \in [0,\tilde{X}_r] \times [0,\tilde{X}_r] : x \geq y\Big\}\Big) \\
&= \mu^r([0,\tilde{X}_r]) - \tau^r\Big(\Big\{(x,y) \in [0,\tilde{X}_r] \times [0,\tilde{X}_r] : x < y\Big\}\Big),
\end{aligned}$$

but

$$\begin{aligned}
\tau^r\Big(\Big\{(x,y) \in [0,\tilde{X}_r] \times [0,\tilde{X}_r] : x < y\Big\}\Big) &\leq \sum_{q \in \cap [0,\tilde{X}_r]} \tau^r([0,q] \times (q,\tilde{X}_r]) \\
&= \sum_{q \in \mathbb{Q} \cap [0,\tilde{X}_r]} \tau^r([0,q] \times [0,\tilde{X}_r]) - \tau^r([0,q] \times [0,q]) \\
&= \sum_{q \in \mathbb{Q} \cap [0,\tilde{X}_r]} \mu^r([0,q]) \wedge S^r([0,\tilde{X}_r]) - \mu^r([0,q]) \wedge S^r([0,q]) \\
&= \sum_{q \in \mathbb{Q} \cap [0,\tilde{X}_r]} \mu^r([0,q]) \wedge S^r([0,\tilde{X}_r]) - \mu^r([0,q]) \wedge S^r([0,q]) = 0,
\end{aligned}$$

where in the last line we used that $\mu^r([0,q]) \leq S^r([0,q])$. Adding up $\tau^l(\mathcal{E}_{I_r^0})$ with $\tau^r|_{[0,\tilde{X}_r]}(\mathcal{E}_{I_r^0})$, yields that $\tau^{I_r^0}(\mathcal{E}_{I_r^0})$ equals $\mu(I_r^0)$, and the equilibrium condition is satisfied. Finally, the revenue condition in the statement of the Property is immediately satisfied as $d\tau_2^{I_r^0}/d\Gamma$ coincide with s^τ in I_0^r, and the same is true for the equilibrium utilities.

$\square$