How Often Should You Reward Your Salesforce? Multi-Period

Incentives and Effort Dynamics

Kinshuk Jerath
kj2323@gsb.columbia.edu
Columbia Business School
3022 Broadway
New York, NY 10027

Fei Long
FeiLong18@gsb.columbia.edu
Columbia Business School
3022 Broadway
New York, NY 10027

April 2018
HOW OFTEN SHOULD YOU REWARD YOUR SALESFORCE?
MULTI-PERIOD INCENTIVES AND EFFORT DYNAMICS

ABSTRACT

We study multi-period salesforce incentive contracting when sales agents can decide their effort levels dynamically. Focusing on a two-period scenario with a risk-neutral agent with limited liability, we analyze a central question faced by firms: Should salespeople be evaluated and granted compensation over a short or a long time horizon (in our case, per period or at the end of two periods), and what should be the optimal compensation contract? While it is elementary that a fully flexible two-period contract will weakly dominate a period-by-period contract, we find that a two-period contract can strictly dominate a period-by-period contract, because it allows reward to be made contingent on a more extreme sales outcome, even though it allows the agent to game effort exertion. We show that this insight continues to hold when the agent can borrow or postpone sales between periods. However, if the time periods are dependent (e.g., a fixed amount of inventory has to be sold across the two time periods), then a period-by-period contract can strictly dominate under certain conditions. We also derive implications for the agent’s multi-period effort profile and show that various effort profiles that may appear to be sub-optimal, e.g., delaying effort, may be induced optimally by the firm; i.e., the firm may not want to induce high effort in both periods under all conditions. Throughout the paper, we show that two often ignored factors, namely the agent’s outside option and level of limited liability, are important determinants of the optimal time horizon and contract form, and of the agent’s effort response.

Keywords: Salesforce compensation; dynamic incentives; time horizon; limited liability; effort dynamics; inventory.

INTRODUCTION

Salesforce expenditures account for 10%–40% of the revenues of US firms (Albers and Mantrala 2008); this is of the order of hundreds of billions of dollars annually and is about three times the amount spent on advertising (Zoltners et al. 2008). Consequently, how to best motivate salespeople is of prime importance to firms, and salesforce compensation design problems have drawn significant attention from economists and marketing researchers, prominent early papers in each area being Hölstrom (1979) and Basu et al. (1985), respectively. In most cases, demand outcomes are uncertain and sales effort is not fully observable by the firm; this makes the determination of compensation, which is to reimburse for effort exerted, a difficult problem. Typical compensation contracts used by firms are comprised of a fixed part (e.g., base salary) and a variable part (e.g., commissions on sales or discrete bonuses awarded on achieving a quota of sales in a specified time period). According to Joseph and Kalwani (1998), who
conducted a survey of Fortune 500 firms, 58% of the firms use commissions and over 90% use quota-based contracts in their compensation plans. The advantage of quota-based plans is that they provide stronger incentives to salespeople to reach a high level of sales, as variable compensation is rewarded only in that case. However, if the quota is too difficult or too easy to achieve, the effort exertion by the salesperson will be sub-optimal. This issue does not exist in a commission-based contract because every additional sales unit brings in the same additional reward. Determining whether quota-based bonuses or commissions should be used is an important problem that firms face.

Furthermore, firms employ salespeople for extended periods of time and they have to determine how frequently salespeople should be evaluated and rewarded, e.g., monthly, quarterly, semi-annually, annually, a combination of these, etc. Coughlan and Joseph (2012) refer to this as the time horizon over which rewards are offered to the salesperson, and state that essentially all firms face this problem as well. As they discuss, when quota-based incentives are used in such a multi-period setting, the issue of dynamic gaming arises. This is because in a multi-period scenario the agents may strategically choose their effort exertion over time, based on how uncertain outcomes are realized and how the contract will determine reward in current and future periods.

As a canonical example, consider a scenario in which the salesperson is paid a bonus if a particular sales quota is reached in six months. To reach his six-month quota with minimum effort, the agent may strategically shirk work in the first quarter hoping for a high demand outcome without much effort, and exert greater effort only in the second quarter in case of low demand realization in the first quarter. For similar strategic reasons, sales agents who have already achieved the quota in early sales cycles may not have the incentive to put in extra effort later. This phenomenon of agents dynamically adjusting their sales effort — both postponement of effort exertion and shirking after achieving the quota — has been widely documented (e.g., Oyer 1998, Chen 2000, Steenburgh 2008, Misra and Nair 2011, Jain 2012, Kishore et al. 2013). Specifically, postponement of effort exertion is considered an especially undesirable outcome from the firm’s point of view because it involves not exerting effort before the sales quota is reached; it is sometimes known as the “hockey stick” pattern (because effort exertion is flat in early periods and increases sharply in later periods, thus taking the shape of a hockey stick; Chen 2000). Such perverse gaming incentives are, however, not present in commission-based contracts where every additional unit of demand brings in the same additional compensation.

Clearly, the firm has multiple options for structuring the time horizon for the contract. The firm
could offer a short time horizon contract that evaluates and rewards the agent frequently (e.g., every quarter). Alternatively, the firm could offer a longer time horizon contract that evaluates and rewards the agent less frequently (e.g., six months); in this case, the firm would also have to determine the rewards for a larger number of possible realizations. In each of these cases, the firm would have to determine the structure of the contract, for instance, should it include only commissions or should it include rewards for reaching quotas. If the firm chooses a longer time horizon with a quota-based contract, it would be more exposed to gaming.

We therefore see that the two problems of determining the time horizon of compensation and determining the optimal compensation structure are inter-related. This is an issue that essentially every company that uses a salesforce must resolve — non-linear quota-based incentive contracts lead to stronger incentives but invite gaming, while linear commission-based incentive contracts reduce gaming but also weaken incentives. A number of empirical papers have studied this tradeoff but have not reached a clear answer regarding which factor — the incentive effect or the gaming effect — dominates in a multi-period dynamic incentives scenario under which conditions. Oyer (1998) analyzes aggregate data spanning scores of industries (from the Survey of Income and Program Participation (SIPP)) in which quota-based plans are used and detects dynamic gaming effects, and suggests that this gaming hurts more than the incentive effect helps. Steenburgh (2008) analyzes individual salesperson-level data from a Fortune 500 company that sells durable office products and uses quota-based plans, and determines that stronger incentives dominate the downside from gaming (and also states that analyzing these data in aggregate would produce results similar to those reported in Oyer (1998)). Misra and Nair (2011) uses a dynamic structural model to analyze data from a Fortune 500 contact lens manufacturer and shows that a plan that uses only commissions performed better than a quota-based plan (that was originally in use at the company); using only commissions makes the time horizon decision irrelevant. Kishore et al. (2013) studies this question using data from a large pharmaceutical firm in an emerging market and finds that commissions do better than quotas by preventing gaming, but this comes at the cost of neglecting non-incentivized tasks. Chung et al. (2014) uses a dynamic structural model to analyze data from a Fortune 500 office durable goods manufacturer and determines that quotas, through higher effort motivation, perform better than plans without quotas in spite of gaming effects being present; it also finds that both short term and long term quotas have roles to play. Chung and Narayandas (2017) studies the effect of time horizon under a quota plan and finds that a shorter horizon generally ensures more consistent
effort but may not be more profitable for the firm due to a change in the product mix being sold. Across these studies, choosing a better (even if not “optimal”) compensation plan can lead to very significant increases in revenues and profits, of the order of 5% to 20%. These papers also carefully document the effort exertion profiles of agents induced by different types of contracts in a multi-period scenario. They consistently report delaying effort and shirking after a good outcome as an issue of concern in long time horizon contracts. Overall, existing empirical research has found the problem of determining the optimal time horizon (and contract form) to be highly relevant across a wide variety of scenarios but has reached mixed conclusions regarding this.

In this paper, we conduct a theoretical investigation to shed light on this fundamental question that, arguably, any firm in any industry that employs a salesforce faces (and, in a recent review article, Coughlan and Joseph (2012) list as a very important yet under-researched issue in salesforce management): What time horizon should the firm use to evaluate and compensate the salesperson, and what should be the associated contract? Should the firm offer multiple sequential short time horizon contracts (which enables the firm to have more control over the effort exertion of the salesperson in every period) or should it offer a long time horizon contract (which allows the salesperson more freedom to adjust his effort profile to “game” the system but also allows the firm to make variable compensation contingent on an outcome that is more difficult to achieve)? What are some key factors that influence this decision? Furthermore, what effort profile(s) will be induced by the optimal incentive contract?

To answer these questions, we build a stylized principal-agent model in which a firm interacts with a salesperson for two time periods. In this context, using short time horizon for evaluation implies offering two period-by-period contracts where each contract is determined at the start of a period and pays at the end of the period based on the outcome of the period. On the other hand, using long time horizon for evaluation implies offering a two-period contract that is determined at the start of the first period and pays once at the end of the second period based on the outcomes of the two periods. (In the rest of the paper, we will use “long time horizon contracting” interchangably with “two-period contract,” and “short time horizon contracting” interchangably with “period-by-period contracts.”) We assume the demand outcome in each period to be stochastically dependent on the effort exerted in that period, and assume the demand outcomes in the two periods to be independent of each other. In the two-period contract, the agent can dynamically adjust his effort level in the later period based on the early period’s demand outcome which also influences his first-period effort exertion decision. We assume that the firm and the
salesperson are risk neutral, and that the agent has limited liability. Limited liability can be thought of as protection from downside risk for the salesperson, i.e., he will be guaranteed a minimum payment even in the case of an unfavorable market outcome (which is a robust feature of real-world compensation plans). We assume that the agent’s limited liability can be lower or higher than his outside option; the latter can happen, for instance, when the salespeople’s skills are most valuable in a sales context and they cannot expect comparable compensation in other professions (Kim 1997, Oyer 2000).

Our analysis shows that, for the firm, a fully flexible two-period contract weakly dominates a period-by-period contract, as expected. Interestingly, however, we find that the two-period contract, even though it allows gaming of effort by the agent, strongly dominates the period-by-period contract under certain conditions. In the optimal two-period contract it is sufficient to determine compensation based on the cumulative sales for the two periods and, under different conditions (discussed shortly), the optimal two-period contract is either an “extreme” contract that concentrates the reward only at the highest cumulative output level, or a “gradual” contract with rewards at all (but the lowest) cumulative output levels. In fact, similar to Hölstrom and Milgrom (1987), the optimal gradual two-period contract can be interpreted as identical to a commission contract. Furthermore, we obtain an interesting equivalence result that states that the optimal two-period gradual (commission) contract is identical in all ways (i.e., in terms of expected effort exertion, sales outcomes and total compensation) to a sequence of optimal period-by-period contracts; in other words, a long time horizon contract with commissions achieves the same outcomes as a sequence of optimal short time horizon contracts.

Whether the extreme long time horizon contract or the gradual long time horizon contract (equivalently, a sequence of short time horizon contracts) is optimal can be explained by understanding the two familiar countervailing effects at play. The first is the beneficial “incentive effect,” which is that, given the agent’s limited liability, an extreme plan provides a larger incentive to work compared to a gradual plan because any output lower than the highest possible does not provide any additional reward. However, the extreme plan also leads to a negative “gaming effect,” that is, dynamic gaming of effort in the second period based on the outcome of the first period hurts the principal. The extreme contract is optimal when the incentive effect is stronger than the gaming effect, and this is the case when the effectiveness of the agent’s effort is either low or high. This is because in the extreme contract the loss in demand due to the gaming effect is larger for higher effort effectiveness, but in the optimally designed contract the probability that this loss will happen is lower for higher effort effectiveness. Therefore,
the expected demand loss due to the gaming effect in the extreme contract is highest for intermediate effectiveness levels, and this loss is large enough to offset the incentive effect, so that in this region the extreme contract is not optimal. As limited liability decreases (fixing the agent’s outside option) the friction from moral hazard becomes smaller and the incentive effect becomes less important, so that the gradual contract becomes optimal in a larger parameter space.

In terms of the agent’s effort exertion, we find that multiple effort exertion profiles are possible under different conditions under the optimal contract — effort exertion in both periods; effort exertion in the first period and effort exertion in the second period conditional on high or low realized outcome in the first period; and no effort exertion in the first period and effort exertion in the second period conditional on high or low realized outcome in the first period. The last pattern is especially interesting as it implies that in the optimal contract the firm induces delaying of effort (or “hockey stick” effort profile). This effort postponement is typically interpreted negatively (Chen 2000), and as something to avoid; our analysis shows that it indeed can be generated under an optimal contract even with independent periods, and this happens when limited liability is intermediate. In other words, observing less than high effort exertion either early or late may not necessarily imply that the contract is “not effective” — that may, in fact, be optimal for the firm and, indeed, induced by it through the contract. This implies that one has to carefully understand and consider the setting and environmental factors when making inferences about contract efficiency from dynamic effort profiles of agents.

Next, we extend our basic model such that the two time periods are not completely independent. Specifically, we introduce the idea of an exogenous and limited amount of product inventory that has to be sold in the two periods, such that the contract design decisions for the principal in the two time periods become dependent. (Note that demand outcomes in the two periods are still independent.) We find that in this scenario the principal may find it optimal to use a period-by-period contract in which the second-period contract is decided based on the outcome of the first period. Such a period-by-period contract can strongly dominate the two-period contract because it gives the principal more flexibility in adjusting the contract. This cannot be reproduced by a two-period contract under the (reasonable) assumption that total compensation cannot be decreasing in sales. Furthermore, with limited inventory, the principal’s incentive to induce effort in the first period is lesser, i.e., the principal may optimally desire effort postponement by the agent in a larger parameter space.

A number of papers, including Oyer (1998), Steenburgh (2008), Misra and Nair (2011), Jain (2012),
Chung et al. (2014) document another kind of gaming (in addition to effort gaming) in a dynamic incentives setting — they show that in a multi-period setting with non-linear contracts, sales agents pull in orders from future periods if they would otherwise fall short of a sales quota in one cycle, whereas they push out orders to the future if quotas are either unattainable or have already been achieved. We extend our basic model to study such strategic sales pull in and push out behavior, which also introduces dependence between the periods. Allowing this affects short time horizon contracts because it gives the agent more freedom to game the system, but it does not affect long time horizon contracts. In accordance with this insight, we find that if sales pull in and push out is possible then a long time horizon contract becomes more attractive to the principal, because it evaluates the agent only for the output at the end of the two periods.

Our research is related to the body of work on dynamic incentives with repeated moral hazard. One stream of this work assumes the firm to be risk neutral but agents to be risk averse, which leads to contracting frictions. Under this paradigm, a seminal paper, Hölstrom and Milgrom (1987) shows that a linear contract is optimal for the principal when a number of other assumptions hold. We note that the gradual two-period contract that we derive as optimal for the firm under certain conditions can be interpreted as a linear contract as well, but it is only optimal for intermediate values of effort effectiveness (recall that we assume the agent to be risk neutral). A number of papers in the risk aversion paradigm revisit the assumptions of Hölstrom and Milgrom (1987) and show the optimality of non-linear contracts (Rogerson 1985, Spear and Srivastava 1987, Schättler and Sung 1993, Sung 1995, Hellwig and Schmidt 2002, Sannikov 2008, Rubel and Prasad 2015).

A second stream of the work on dynamic incentives assumes agents to be risk neutral with limited liability, which leads to a different kind of contracting friction (our paper falls under this paradigm). Bierbaum (2002) studies how to induce high effort from the agent in each of two periods (which may not be profit maximizing for the principal), while we allow different effort profiles to be induced by the optimal contracts under different conditions. Kräkel and Schöttner (2016) study the firm’s choice between commissions and bonuses and determines conditions under which one or the other (or a combination) is optimal when the reward must be paid at the end of multiple periods. Schöttner (2016) studies optimal contracting when the agent’s effort costs change over time. None of these papers, however, consider whether long time horizon or short time horizon contracts are optimal. Relatedly, they also do not allow the agent to strategically borrow or postpone sales between periods, neither do they consider the case of
limited inventory to be sold across two periods which creates a particular form of dependence between periods. In addition, these papers normalize the values of the outside option and the limited liability and are unable to study comparative statics with respect to these quantities on the optimal contract and the effort profile induced.

More broadly, our research adds to the extensive literature on salesforce incentives in marketing which, in addition to the papers already cited, includes Raju and Srinivasan (1996), Godes (2004), Simester and Zhang (2010) and Zhang (2016), among many others. Our extension with limited inventory is related to the work on salesforce compensation when operational considerations are important (Chen 2000, Plambeck and Zenios 2003, Dai and Jerath 2013, Saghaifian and Chao 2014, Dai and Jerath 2016, Dai and Jerath 2018).

The rest of the paper is organized as follows. In the following section, we describe the model. Next, we conduct the analysis and obtain our key insights regarding the different forces at play, and the comparison between short and long time horizon contracts. Following this, we allow for periods to be dependent by assuming that the principal has limited inventory to be sold in the two periods. In the final section, we conclude with a discussion. The proofs are provided in an Appendix and an Online Appendix (indicated appropriately).

**MODEL**

We develop a simple agency theoretic model in which a firm (the principal) hires a salesperson (the agent) to exert demand-enhancing effort. There are two time periods denoted by $t \in \{1, 2\}$. Demand in both periods is uncertain and independent. Let $D_t$ be the demand realization in period $t$, which can be either $H$ or $L$ with $H > L > 0$. The agent’s effort increases the probability of realizing high demand levels. The effort level in period $t$, denoted by $e_t$, can be either 1 or 0, i.e., the agent either “works” or “shirks” in each period; however, the principal does not observe the effort level. We can think of effort level 0 as a salesperson making a client visit (which is observable and verifiable) and effort level 1 as the salesperson’s additional effort spent in talking to and convincing the client to make the purchase (which the firm cannot observe or verify). Without effort exertion ($e_t = 0$) demand is realized as $H$ with a probability of $q$, and with effort exertion ($e_t = 1$) this probability increases to $p$ ($0 < q < p < 1$). A larger $p$ implies greater effectiveness of the salesperson’s effort, while $q$ can be interpreted as the natural market outcome. We assume that all the demand created can be met and each unit sold gives a revenue
of 1 and has a marginal cost of zero. The cost of effort is given by $\phi > 0$ for $e_t = 1$ and is normalized to zero for $e_t = 0$.

We assume that both the firm and the salesperson are risk neutral. Unlike the firm, however, the salesperson has limited liability, implying that he must be protected from downside risk. Specifically, we assume that the salesperson has a limited liability of $K$ in each period, i.e., to employ the agent for one period, the principal must guarantee a compensation of at least $K$ under any demand outcome. Limited liability is a widely observed feature of salesforce contracts in the industry, and this assumption is a standard one in the literature (Sappington 1983, Park 1995, Kim 1997, Oyer 2000, Laffont and Martimort 2009, Simster and Zhang 2010, Dai and Jerath 2013). The limited liability assumption also implies the existence of a wage floor for the salesperson, which is aligned with industry practice. We assume that the salesperson’s reservation utility is $U$ for each period, and that the limited liability can be either lower or higher relative to the agent’s reservation utility. For instance, if the salesperson’s alternative employment opportunities are attractive, then limited liability can be relatively low compared with reservation utility, but if salespeople’s skills are most valuable in a sales context and they cannot expect comparable compensation in other professions, then limited liability can be relatively high compared with reservation utility (as also discussed in Kim 1997, Oyer 2000).

The agent is reimbursed for effort using an incentive contract. Effort is unobservable to the firm and demand is random but can be influenced by effort, so the firm and the agent sign an outcome-based contract. The firm can contract using a short time horizon, i.e., two period-by-period contracts, where each contract is determined at the start of each period and pays at the end of the period based on the outcome of the period. Alternatively, the firm can propose a long time horizon, single two-period contract that is determined at the beginning of the first period and pays once at the end of the second period based on the outcomes of the two periods.\footnote{The discrete demand distribution that we have assumed ensures that effort will not change the support of the demand distribution; otherwise, the principal may be able to infer the agent’s effort from the demand outcome and would induce the agent to work by imposing a large penalty for demand outcomes that cannot be obtained under equilibrium effort but can be obtained under off-equilibrium efforts, as argued in Mirrlees (1976).} The timings of the different games will be clarified in the relevant sections. We now proceed to the analysis.

**ANALYSIS**

**First-Best Scenario**

We start by presenting the first-best solution (for instance, if the agent’s effort is observable). In this
case, the two periods are independent and equivalent and it is sufficient to study just one period. The firm can implement any effort level $e_t$ in either period, by reimbursing the agent a fixed salary $s_t$ which must be at least $K$ while ensuring the agent’s participation. The principal’s problem in each period is the following.

$$\max_{s_t} E[D_t|e_t] - E[s_t|e_t]$$

s.t. $U_A(e_t) \geq U$ \hspace{1cm} (PC_t)

$s_t \geq K$ \hspace{1cm} (LL_t)

Here, (PC_t) is the agent’s participation constraint, where $U_A(e_t)$ stands for the salesperson’s expected net utility on exerting effort $e_t$, which is equal to $s_t - \phi$ if the agent exerts effort and is equal to $s_t$ if the agent does not exert effort. It states that to employ the sale agent, the principal needs to provide a fixed salary that makes the agent’s expected net utility from exerting effort $e_t$ no less than his outside option, which simplifies as $s_t \geq U + \phi$ if effort is exerted, and as $s_t \geq U$ if effort is not exerted. (LL_t) stands for the agent’s limited liability constraint, which ensures that the agent receives a fixed salary $s_t$ no less than his limited liability $K$.

If the contract specifies effort exertion in period $t \in \{1, 2\}$, i.e., $e_t = 1$, the principal’s expected profit is equal to the expected market demand subject to the agent’s effort exertion, $pH + (1 - p)L$, minus the minimal salary to ensure effort exertion, $\max\{U + \phi, K\}$, i.e., $pH + (1 - p)L - \max\{U + \phi, K\}$. If $e_t = 0$, the principal gets the natural market outcome and pays the minimal salary to employ the salesperson, i.e. $qH + (1 - q)L - \max\{U, K\}$. This leads to the following first-best solution (the proof is in Section A1 in the Appendix).

**Proposition 1 (Optimal First-Best Solution)** The first-best contract and outcomes are as per the following table.

<table>
<thead>
<tr>
<th>$U - K$</th>
<th>$H - L$</th>
<th>$e^*_{FB}$</th>
<th>$s^*_{FB}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U - K \geq 0$</td>
<td>$H - L \geq \frac{\phi}{p-q}$</td>
<td>$\null$</td>
<td>$U + \phi$</td>
</tr>
<tr>
<td>$-\phi \leq U - K &lt; 0$</td>
<td>$H - L \geq \frac{\phi}{p-q} + \frac{U-K}{p-q}$</td>
<td>1</td>
<td>$U + \phi$</td>
</tr>
<tr>
<td>$U - K &lt; -\phi$</td>
<td>$H - L \geq 0$</td>
<td>$K$</td>
<td>$\null$</td>
</tr>
<tr>
<td>$U - K \geq 0$</td>
<td>$H - L &lt; \frac{\phi}{p-q}$</td>
<td>0</td>
<td>$U$</td>
</tr>
<tr>
<td>$-\phi \leq U - K &lt; 0$</td>
<td>$H - L &lt; \frac{\phi}{p-q} + \frac{U-K}{p-q}$</td>
<td>$\null$</td>
<td>$K$</td>
</tr>
</tbody>
</table>

In the table in Proposition 1, the first column gives the condition on $U - K$, the second column gives
the condition on $H - L$, the third column gives the effort exertion under the optimal salary, and the fourth column gives the optimal salary. Figure 1 depicts the first-best solution with respect to the range of the demand distribution $(H - L)$, the agent’s effectiveness parameter $(p - q)$, and the agent’s outside option relative to his limited liability $(U - K)$. From Figure 1, we can infer that the principal would like the agent to exert effort when the upside market potential is large, or when the effectiveness of the agent’s effort is high, or when the agent’s limited liability is large relative to his outside option.

Intuitively, the firm would like to direct the salesperson to work hard if and only if the increase in the expected demand subject to the agent’s effort exertion (given by $(p - q)(H - L)$) outweighs the marginal cost for soliciting effort (given by $\max\{U + \phi, K\} - \max\{U, K\}$). When limited liability is low relative to the agent’s outside option (given by $K \leq U$), the principal only needs to compensate the agent for his outside option plus cost of effort. Therefore the additional cost for soliciting effort is $\phi$, and the principal solicits effort exertion if and only if $H - L \geq \frac{\phi}{p - q}$. When limited liability is intermediate (i.e., $U < K \leq U + \phi$), even if the principal does not solicit effort, he still has to pay the agent his limited liability, so the additional cost for soliciting effort becomes $\phi + U - K$. In this case as $K$ increases, the additional cost for soliciting effort decreases, thus the principal solicits effort in a larger parameter space. When limited liability increases beyond $U + \phi$, the principal pays the agent his limited liability regardless of effort levels and there is no additional cost for soliciting effort, therefore, the principal instructs the agent to exert effort given any $H \geq L$. The above arguments give the following counterintuitive result.

**Corollary 1** In the first-best scenario, as limited liability increases the principal solicits effort in a weakly larger parameter space.

**Period-by-Period (Short Time Horizon) Contract**

In this scenario, the principal contracts over a short time horizon and specifies a one-period contract at the beginning of the first period, and then specifies another one-period contract at the beginning of the second period. The effort for each period is rewarded separately, and therefore we call this a disaggregate contract. As the two periods are identical and independent, it is sufficient to study just one period.

Consider the problem for period $t \in \{1, 2\}$. Since demand follows a binomial distribution, the principal offers quota-bonus contracts with quota levels $\chi_t \in \{H, L\}$ and bonuses $b_{\chi_t, t} \geq 0$, where the bonus $b_{\chi_t, t}$ is paid to the salesperson if and only if the sales reach the quota $\chi_t$, together with a fixed salary of $s_t$. Indeed, it suffices for the principal to consider only two of the decision variables. Without
loss of generality, we normalize \( b_{L,t} \) to 0 and simplify the notation of \( b_{H,t} \) as \( b_t \), i.e., the principal does not issue bonus when the demand outcome is \( L \) and issues bonus \( b_t \) when the demand outcome is \( H \). The principal’s problem in each period is the following.

\[
\begin{align*}
\max_{s_t,b_t} & \quad E[D_t|e_t] - E[s_t + b_t|e_t] \\
\text{s.t.} & \quad U_A(e_t) > U_A(\bar{e}_t) \quad (IC_t) \\
& \quad U_A(e_t) \geq U \quad (PC_t) \\
& \quad s_t, s_t + b_t \geq K \quad (LL_t)
\end{align*}
\]

The participation constraint \( (PC_t) \) and the limited liability constraint \( (LL_t) \) can be interpreted in a similar way as in the first-best scenario. In addition, the contract needs to satisfy an incentive compatibility constraint \( (IC_t) \), which states that to induce effort \( e_t \), the principal needs to ensure that the agent gains a higher net utility by exerting effort \( e_t \) compared with a different effort level \( \bar{e}_t \).

Before solving the optimal contract for the principal, we first derive the best contract for the principal to induce any given effort level. To implement \( e_t = 1 \), from the incentive compatibility constraint \( (IC_t) \), the principal needs to set \( b_{H,t} \) satisfying \( s_t + pb_t - \phi \geq s_t + qb_t \), which simplifies into \( b_t \geq \frac{\phi}{p-q} \). The participation constraint \( (PC_t) \) requires that the agent’s expected utility from exerting effort no lower than his reservation utility, that is, \( s_t + \frac{p\phi}{p-q} - \phi \geq U \). To meet the limited liability constraint \( (LL_t) \) we need the guaranteed salary no less than the agent’s limited liability, i.e., \( s_t \geq K \). The solution is that to implement \( e_t = 1 \), the principal offers a fixed salary \( s_t = \max\{K, U - q\frac{\phi}{p-q}\} \), and a bonus \( b_t = \frac{\phi}{p-q} \) if the demand outcome is high. To implement \( e_t = 0 \), it is enough for the principal to only offer the agent a fixed salary \( s_t = \max\{K, U\} \). The overall solution to the optimal period-by-period contract is specified in the following proposition (the proof is in Section A2 in the Appendix).

**Proposition 2 (Optimal Period-by-Period Contract)** The optimal period-by-period contract and outcomes are as per the following table.
Figure 2 depicts the optimal period-by-period contract with respect to the range of the demand distribution \((H - L)\), the agent’s effectiveness parameter \((p - q)\), and the agent’s reservation utility relative to his limited liability \((U - K)\). In Region I, the principal does not want to induce effort even in the first-best scenario. In Region II, the principal wants to induce effort in the first-best scenario but not in the period-by-period contracting scenario. In Region III, the principal wants to induce effort in the period-by-period scenario. Note that when limited liability is relatively small \((K \geq U - \frac{q}{p-q} \phi)\), even if effort is unobservable, the principal can still achieve the first-best solution by penalizing the agent for low demand realization and rewarding the agent for high demand realization. As limited liability increases beyond \(U - \frac{q}{p-q} \phi\), the principal cannot pay the agent less than his limited liability when demand realization is \(L\), therefore the first-best solution is no longer achievable. This leads the principal to induce effort in a smaller parameter space as limited liability increases. When limited liability exceeds \(U\), the agent needs to be guaranteed his limited liability, with or without a bonus to induce effort. Therefore, the principal induces effort if and only if the extra cost for inducing effort \(\frac{p}{(p-q)^2} \phi\) is offset by the increase in expected demand from exerting effort \((H - L)(p - q)\). From Figure 2, we can see that as limited liability increases, while in the first-best scenario the principal solicits effort in a weakly larger parameter space (as per Corollary 1; represented by the dashed line), with unobservable effort he will induce effort in a weakly smaller parameter space (represented by the solid line).

Two-Period (Long Time Horizon) Contract

In this scenario, the firm proposes a long time horizon two-period contract at the beginning of the first period and pays once at the end of the second period based on the outcomes of the two periods. The timeline of the game is as follows. At the beginning of period 1, i.e., \(T = 1\), the principal proposes the contract and the agent decides whether or not to accept the offer. If accepted, the agent then decides
on his effort in the first period, $e_1$. At the end of $T = 1$, the agent and the principal observe the demand outcome for the first period, $D_1$. The agent then chooses his second period effort $e_2$. At the end of $T = 2$, the agent and the principal observe the second period demand outcome $D_2$. The agent then gets paid according to the contract.

A key feature of this scenario introduced due to unobservability of effort and the contract paying at the end of two periods is that the agent can “game” the system — the agent can choose effort in period 2 based on the outcome of period 1 (and, realizing this, can also choose the effort in period 1 strategically). We denote the two-period effort profile by $(e_1, e^H_2, e^L_2)$, where the second period’s effort $e^D_2$ is contingent on the first period’s demand realization, $D_1$.

In full generality, this contract involves a guaranteed salary for employing the agent for two periods, plus a bonus issued at the end of the two periods that is contingent on the whole history of outputs. We denote the fixed salary as $S$, and denote the bonus paid at the end of $T = 2$ by $b_2(D_1, D_2)$. Such a contract thus stipulates four possible bonuses, $b_2(L, L), b_2(L, H), b_2(H, L)$ and $b_2(H, H)$. To prevent the agent from restricting sales to $L$ when demand is $H$, we impose a constraint on the two-period contract given by $b(H, H) \geq \max\{b(H, L), b(L, H)\}$, i.e., the bonus paid when demand in both periods is realized as $H$ should be no lower than that paid when demand in only one of the periods is realized as $H$. (Note that we do not need this assumption to derive our result as this constraint does not bind in the optimal contract, but this is a natural assumption and simplifies the analysis.) Under this constraint, we obtain the following lemma (the detailed proof is in Section A3.1 in the Appendix).

**Lemma 1** When the two periods are independent of each other, in the weakly dominant two-period contract, $b_2(H, L) = b_2(L, H)$.

Lemma 1 implies that it is sufficient for the principal to pay the agent at the end of two periods a bonus according to cumulative sales (which can be $2L, H + L$ or $2H$) and independent of the sales history.\(^2\)\(^,3\) We denote the fixed salary by $S$, normalize the bonus payment when the total sales are $2L$ as $0$, denote the bonus payment when the total sales across two periods are $H + L$ by $B_1$, and denote

\(^2\)Only the contract to induce $(0, 1 - q)$ is history-dependent, but we find it sub-optimal for the principal when the two periods are independent. However, later analysis, we will show that such a history-dependent contract can be optimal when the two periods become dependent.

\(^3\)The lemma holds without discounting and with risk-neutral agents. As shown by Spear and Srivastava (1987) and Sannikov (2008), if agents discount their future utility, or if the agent is risk averse, a path-dependent contract can be optimal.
the bonus payment when the total sale are \(2H\) by \(B_2\). We formulate the principal’s problem as follows.

\[
\max_{B_1, B_2} \ E[D|e_1, e_2^H, e_2^L] - E[S + B_1 + B_2|e_1, e_2^H, e_2^L]
\]

s.t. \(U_A(e_2^H) > U_A(\bar{e}_2^H)\) \(\text{(IC}_2^H\)\)

\(U_A(e_2^L) > U_A(\bar{e}_2^L)\) \(\text{(IC}_2^L\)\)

\(U_A(e_1|e_2^H, e_2^L) > U_A(\bar{e}_1|e_2^H, e_2^L)\) \(\text{(IC}_1\)\)

\(U_A(e_1, e_2^H, e_2^L) \geq 2U\) \(\text{(PC)}\)

\(S, S + B_1, S + B_2 \geq 2K\) \(\text{(LL)}\)

\(\text{(IC}_2^H\) stands for the agent’s incentive compatible constraint in the second period following \(D_1 = H\), where \(U_A(e_2^H)\) represents the agent’s net payoff in Period 2 upon exerting effort \(e_2^H\). If the agent exerts effort, he will get \(S + B_2 - \phi\) with probability \(p\) and \(S + B_1 - \phi\) otherwise; without exerting effort, he will get \(S + B_2\) with probability \(q\) and \(S + B_1\) otherwise. To induce \(e_2^H\), the principal needs to ensure that the agent gets a higher payoff upon exerting effort \(e_2^H\), compared with a different effort level \(\bar{e}_2^H\).

Similarly, \(\text{(IC}_2^L\) stands for the incentive compatible constraint for inducing effort level \(e_2^L\) in the second period following \(D_1 = L\). Then, \(\text{(IC}_1\) represents the incentive compatible constraint in the first period.

\(U_A(e_1|e_2^H, e_2^L)\) denotes the agent’s net payoff across two periods upon exerting \(e_1\) in the first period, given that the agent is induced to exert effort \((e_2^H, e_2^L)\) in the second period. If \(e_1 = 1\), his total net payoff will be \(U_A(e_2^H) - \phi\) with probability \(p\) and \(U_A(e_2^L) - \phi\) otherwise; if \(e_1 = 0\), his total net payoff will be \(U_A(e_2^L)\) with probability \(q\) and \(U_A(e_2^L)\) otherwise. To induce \(e_1\), the principal needs to ensure that the agent gets a higher total net payoff on exerting \(e_1\), compared with a different effort level \(\bar{e}_1\). The participation constraint \((PC)\) and the limited liability constraint \((LL)\) are similar to that in the period-by-period case, except for that we multiply the right-hand sides by two under a two-period contracting.

To arrive at an optimal contract for the principal, it is crucial to understand how the agent’s effort profile in the two periods changes with the bonuses \(B_1\) (provided for \(H + L\)) and \(B_2\) (provided for \(2H\)) in the two-period contract. The following lemma describes this effort profile (the proof is immediate from the proof of Lemma 1).\(^4\) Note that since \(e_2\) depends on \(D_1\), which is random, we write \(e_2\) in terms of its expectation value. For instance, if the agent exerts effort in period 1 and will exert effort in period 2 only if the outcome in period 1 is \(H\), then \(e_2 = 1\) with probability \(p\), so we write this effort profile as

\(^4\)We make the assumption that when the agent is indifferent between exerting effort or not, he will choose to exert effort.
Lemma 2 (Agent’s Response to Two-period Contract) Given $B_1$ and $B_2$, the agent’s expected effort profile $(e_1, E[e_2])$ is as per the following table.

<table>
<thead>
<tr>
<th>$(B_1, B_2)$</th>
<th>$(e_1, E[e_2])$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_1 \geq \frac{\phi}{p-q}$, $B_2 - B_1 \geq \frac{\phi}{p-q}$</td>
<td>$(1, 1)$</td>
</tr>
<tr>
<td>$0 \leq B_1 &lt; \frac{\phi}{p-q}$, $pB_2 + (1-p-q)B_1 \geq \frac{\phi}{p-q} + \phi$</td>
<td>$(1, p)$</td>
</tr>
<tr>
<td>$B_2 - B_1 &lt; \frac{\phi}{p-q}$, $qB_2 + (1-p-q)B_1 \geq \frac{\phi}{p-q} - \phi$</td>
<td>$(1, 1-p)$</td>
</tr>
<tr>
<td>$B_2 - B_1 \geq \frac{\phi}{p-q}$, $pB_2 + (1-p-q)B_1 &lt; \frac{\phi}{p-q} + \phi$</td>
<td>$(0, q)$</td>
</tr>
<tr>
<td>$B_1 \geq \frac{\phi}{p-q}$, $qB_2 + (1-p-q)B_1 &lt; \frac{\phi}{p-q} - \phi$</td>
<td>$(0, 1-q)$</td>
</tr>
<tr>
<td>$0 \leq B_1 &lt; \frac{\phi}{p-q}$, $0 \leq B_2 - B_1 &lt; \frac{\phi}{p-q}$</td>
<td>$(0, 0)$</td>
</tr>
</tbody>
</table>

Figure 3 illustrates Lemma 2 graphically. The $x$-axis, $B_1$, is the incremental reward when total sales increase from $2L$ to $H + L$; the $y$-axis, $B_2 - B_1$, is the incremental reward when total sales increase from $H + L$ to $2H$. If both rewards are small, there is no effort exertion in either period, denoted by $e = (0, 0)$, which is Region I. If both rewards are large, the agent will put in effort in both periods, i.e., $e = (1, 1)$, which is Region IV. For other regions, the effort exertion decisions are more involved. If the agent does not secure the bonus $B_1$ after period 1 with the demand outcome $L$, he will not expend additional effort if $B_1 \leq \frac{\phi}{p-q}$. If the agent secures the bonus $B_1$ after period 1 with the demand outcome $H$, he will not expend additional effort if $B_2 - B_1 \leq \frac{\phi}{p-q}$. In other words, $B_1$ and $B_2 - B_1$ motivate the agent to exert effort in the second period if demand in the first period turns out to be $L$ and $H$, respectively. Furthermore, the agent’s effort exertion at $T = 1$ depends on the values of both $B_1$ and $B_2 - B_1$.

In Regions II and VI, the agent does not work in period 1 and chooses to “ride his luck” in period 1. However, in Region II, he works in period 2 if the demand outcome is unfavorable, i.e., $L$, in period 1, and in Region VI, he works in period 2 if the demand outcome is favorable, i.e., $H$, in period 1. In Regions III and V, the agent works in period 1. However, in Region III, he works in period 2 if the demand outcome is unfavorable, i.e., $L$, in period 1, and in Region V, he works in period 2 if the demand outcome is favorable, i.e., $H$, in period 1.

We now determine the optimal compensation plan for the firm. We find the optimal contract by balancing the expected revenue $E[D]$ less the expected compensation cost $E[S + B_1 + B_2]$. Proposition 3 characterizes the optimal two-period contract for the principal (the detailed proof is in Section A3.2 in the Appendix).
Proposition 3 (Optimal Two-period Contract) The optimal two-period contract and outcomes are as per the following table.

| Region | $U - K$ | $H - L$ | $(e_1,E|e_2|)$ | $S^*$ | $B^*_1$ | $B^*_2$ |
|--------|---------|---------|----------------|-------|---------|---------|
| I      | $\frac{p^2}{2(p - q)}\phi < U - K < \frac{q^2}{2(p - q)}\phi$ | $H - L \leq \frac{q}{p - q}$ & $2U$ | $(0,0)$ | 2U | 0 0 |
|        | $U - K < 0$ | $H - L \leq \frac{q}{p - q}$ & $2U$ | $2U$ | 0 0 |
| II     | $\frac{\phi}{p - q} \leq U - K < \frac{q}{2(p - q)}\phi$ | $H - L \leq \frac{q}{p - q}$ & $2U$ | $2U - \frac{\phi}{p - q}$ | 0 0 |
|        | $\frac{p^2}{2(p - q)}\phi \leq U - K < \frac{q^2}{2(p - q)}\phi$ | $H - L \leq \frac{p^2}{2(p - q)}\phi$ & $2K$ | $2K$ | 0 0 |
|        | $0 \leq U - K < \frac{q}{2(p - q)}\phi$ | $H - L \leq \frac{q}{p - q}$ & $2U$ | $2U - \frac{\phi}{p - q}$ | 0 0 |
|        | $U - K < 0$ | $H - L \leq \frac{q}{p - q}$ & $2U$ | $2U - \frac{\phi}{p - q}$ | 0 0 |
| III    | $\frac{\phi}{p - q} \leq U - K < \frac{q}{2(p - q)}\phi$ | $H - L \leq \frac{p^2}{2(p - q)}\phi$ & $2U$ | $2U - \frac{\phi}{p - q}$ | 0 0 |
| IV     | $\frac{p^2}{2(p - q)}\phi < U - K < \frac{q}{2(p - q)}\phi$ | $H - L \geq \frac{q}{p - q}$ & $2K$ | $2K$ | $\frac{\phi}{p - q}$ | $\frac{\phi}{p - q}$ |
|        | $0 \leq U - K < \frac{q}{2(p - q)}\phi$ | $H - L \geq \frac{p^2}{2(p - q)}\phi$ & $2K$ | $2K$ | $\frac{\phi}{p - q}$ | $\frac{\phi}{p - q}$ |

We illustrate the result with the aid of Figure 4. The optimal contract is either a “gradual contract” (in which $B_1 > 0$, i.e., it rewards bonuses at both $H + L$ and $2H$) or an “extreme contract” (in which $B_1 = 0$, i.e., it rewards bonuses only at $2H$). In Region I, the principal does not want to motivate effort. In Region II, the principal finds it optimal to use the extreme contract to motivate the effort profile $(0,q)$ by giving a bonus $B_2 = \frac{\phi}{p - q}$. In Region III, the principal finds it optimal to use the extreme contract to motivate the effort profile $(1,p)$ by giving a bonus $B_2 = \frac{1 + p - q}{p(p - q)}\phi$ (which is larger than $\frac{\phi}{p - q}$). In Region IV, the principal finds it optimal to use the optimal gradual two-period contract to motivate the effort profile $(1,1)$.

To develop the intuition behind these results, we first focus on the case when limited liability is sufficiently high. Specifically, assume $K = U$, in which case the principal pays a fixed salary of $S = 2K$ for inducing any effort profile. From Figure 4, we can see that in this case the optimal contract is either the extreme two-period contract with $B_2 = \frac{1 + p - q}{p(p - q)}\phi$ to implement $e = (1,p)$, or the gradual two-period contract with bonuses $B_1 = \frac{\phi}{p - q}, B_2 = 2\frac{\phi}{p - q}$ to implement $e = (1,1)$. To understand why, we discuss two effects that are operative, namely the “incentive effect” and the “gaming effect.”

First, we discuss the incentive effect. In Figure 5, we vary $p - q$, the effectiveness of the agent’s effort, keeping $H - L$ fixed. Generally speaking, more effective agents require lower incentives to work because the outcome is a better signal of effort exerted. In line with this, the expected bonus payments under the extreme contract, $(p^2 + p - pq)\frac{\phi}{p - q}$, and under the gradual contract, $2p\frac{\phi}{p - q}$, both decrease with $p$. 

17
However, the difference between them, $E[B]_{\text{gradual}} - E[B]_{\text{extreme}} = p\left(\frac{1}{p-q} - 1\right)\phi$, is always positive, as shown by the solid line in Figure 5. This means that the principal always pays a smaller expected bonus under the extreme contract than under the gradual contract. Therefore, on the positive side, the extreme contract benefits from the incentive effect: it provides more effective incentives for an agent with limited liability, thus saving on the bonus payment for the principal. The reason behind this is that under limited liability, the principal concentrates compensation at a high level of sales. In a period-by-period contract the highest level of sales at which reward can be given is $H$ while in a two-period contract this level is $2H$; this can lead to higher incentive provision in a two-period contract (even though the reward is given only once). Another interesting observation from Figure 5 is that the incentive effect, as measured by the solid line, shrinks as $p$ increases. This is because as moral hazard frictions decrease with more effective agents, so will the comparative advantage of the extreme contract on saving incentive costs.

However, in a dynamic setting, such a non-linear reward structure will suffer from the agent’s gaming. As a consequence, on the negative side, the principal obtains less demand under the extreme contract, as the dashed line in Figure 5 illustrates. Mathematically, $E[D]_{\text{gradual}} - E[D]_{\text{extreme}} = (1-p)(p-q)(H-L)$ is always positive. As we have mentioned, due to the non-linear structure of the extreme contract, an agent will game the system by varying his effort in a dynamic setting. Specifically, the agent exerts effort in the first period, but if the first period outcome turns out to be $L$, the agent will give up on effort exertion in the second period, leading to a demand loss for the principal. Interestingly, as $p$ gets larger, agents under both contracts generate higher sales, but the difference between the sales they generate, caused by the gaming effect, takes an inverse-U shape. This is because when $p$ increases, the demand loss, if it happens, $(p-q)(H-L)$, gets larger, but the probability of its happening, $(1-p)$, decreases.

Combining the incentive effect and the gaming effect, we can see from Figure 5 that if $p-q$ is small, the incentive effect dominates and the extreme plan outperforms the gradual plan — the gaming loss under the extreme contract is relatively small compared with its advantage in providing incentives. Above a threshold of $p-q$, the gaming loss becomes dominant and the gradual contract is preferred by the principal. However, as $p-q$ continues to increase, the gaming loss begins to decline, rendering the extreme contract better again. Overall, when $p-q$ is either very small or very large, the incentive effect will be more significant than the gaming effect and the extreme plan outperforms the gradual plan.

The above analysis is based on the premise that limited liability is sufficiently high. Now we discuss the optimal contract as limited liability decreases. We fix $H-L$ and $p-q$ at a low level so that when
limited liability is sufficiently high the principal does not want to induce effort. As limited liability decreases, the friction due to moral hazard becomes smaller, and the principal starts to motive effort using the extreme two-period contract, which provides more effective incentives than the period-by-period contract. Since limited liability is still relatively high in this scenario, the principal only induces $e_2^H = 1$ through a low ultimate bonus and the full effort profile is $e = (0, q)$—that is, there is no early effort exertion in the first period, and there is effort exertion in the second period if the early period realizes as high. As limited liability continues to decreases further, the principal implements $e_1 = 1$ through a high ultimate bonus and the full effort profile is $e = (1, p)$—that is, the agent exerts effort in the first period, and he will continue exerting effort in the second period if the early period realizes as high. When limited liability becomes small enough, the principal will implement effort $e = (1, 1)$ using the gradual two-period contract.

Put together, the preceding discussion explains the patterns in Figure 4. When limited liability is not too small (relative to the agent’s outside option), in a market with small upside demand potential, and with either very inefficient or very efficient salespeople, the extreme contract performs best for the principal. In other circumstances, it is profitable for firms to propose a gradual contract to motivate hard work in both periods. Next, we state an interesting corollary.

**Corollary 2** Under a two-period contract, both $e = (1, p)$ and $e = (0, q)$ can be optimally induced expected effort profiles by the firm.

The corollary has interesting and important implications for observed effort profiles of agents. It states that a pattern of “giving up” in the second period (expected effort profile $e = (1, p)$, i.e., effort is exerted in the first period but then not exerted in the second period if the first period demand is realized as low) is actually optimally induced by the firm under certain conditions. It also states that a pattern of “delaying effort” or a “hockey stick” effort profile (expected effort profile $e = (0, q)$, i.e., effort is not exerted in the first period but then exerted in the second period only if the first period demand is realized as high) is also optimally induced by the firm under certain conditions. To see why, note that effort postponement is optimal for the firm when the limited liability, the demand upside potential, and agent’s effort effectiveness are all at intermediate levels. In this case, the principal would like to induce effort using an extreme two-period contract with a low ultimate bonus, which provides the most effective incentives. In other parameter spaces, early effort exertion is preferred by the principal under
a two-period contract, i.e., $e_1^* \geq E[e_2^*]$, as happens when the optimal two-period contract induces either $e = (1, 1)$ or induces $e = (1, p)$.

**Comparison Between Two-Period and Period-by-Period Contracts**

We now compare the outcomes in the period-by-period contract scenario and the two-period contract scenario from the point of view of the principal. We find, not surprisingly, that the principal weakly prefers the two-period contract to the period-by-period contract. However, more interestingly, our analysis shows that, under certain conditions, the principal strongly prefers a two-period contract over a period-by-period contract (even though the latter gives the principal more control over the agent’s action while the former allows the agent the freedom to exert effort to game the contract). Furthermore, with independent periods, a two-period contract that rewards bonus on the basis of the total sales in the two periods suffices, i.e., achieves the same outcome as a contract that rewards for the full sequence of outcomes. We obtain the following proposition.

**Proposition 4** In Regions II and III as defined in Proposition 3, the principal strongly prefers a two-period contract over a period-by-period contract.

The reason is that the gradual contract with $B_1 = \frac{\phi}{p-q}, B_2 = 2\frac{\phi}{p-q}$ is essentially a replicate of the period-by-period contract. Therefore, whenever the principal prefers the extreme contract over the gradual contract in the two-period contract, it strongly prefers the two-period contract over the period-by-period contract. This happens when the effectiveness of effort of the salesperson is either very high or very low (but high enough that it is worthwhile to have effort exertion). Also, as the limited liability decreases, the strong preference for the two-period contract reduces. We also note that the preferred extreme two-period contract may be the one that pays a small bonus for high sales in both periods, which induces effort only in the second period if the outcome in the first period is high, or it may be the one that pays a large bonus for high sales in both periods, which induces effort in the first period and in the second period only if the outcome in the first period is high.

**Extension: Sales Push-out and Pull-in Between Periods**

Salespeople working under quota-based plans may resort to modifying demand in particular periods to meet quotas in those periods. Oyer (1998) empirically demonstrates the existence of demand pull-in and push-out between fiscal cycles when salespeople face non-linear contracts. In particular, Oyer (1998) reveals that sales agents will pull in orders from future periods if they would otherwise fall short of a
sales quota in one cycle, whereas they push out orders to the future if quotas are either unattainable or have already been achieved. We ignore such sales push-out and pull-in phenomena in previous sections, by assuming that agents cannot shift sales between two periods. In this section, we relax this assumption and allow the agents to push extra sales to (or borrow sales from) the later period. While the two-period optimal contract, which pays at the end, is not affected, the period-by-period contract, which pays in the interim, is subject to sales push-out and pull-in effects, and thus has to be reanalyzed. We provide a sketch of the analysis below, with details provided in Section OA1 in the Online Appendix.

In the period-by-period contract, at the end of the first period the agent observes the actual sales $D_1$ ahead of the principal. He can then strategically push out sales to (or pull in sales from) the second period, if necessary. The principal only observes the sales level after the agent’s manipulation, which we denote by $D_1'$, and pays the agent according to $D_1'$. For instance, the principal will observe $D_1' = H$ if $D_1 = H$ or if $D_1 = L$, but the agent pulls in $H - L$ from the second period. Likewise, observing $D_1' = L$ may imply that $D_1 = L$ or that $D_1 = H$ and the agent pushed out the demand $H - L$ to the second period.

To derive the optimal contract, consider first the principal’s problem at $T = 2$. In this case, the problems of the two periods are dependent. At the beginning of the second period, a new contract is initiated. To induce a specific effort level in the second period, the principal will set the second period’s quota level and bonus value based on the observed earlier outcome $D_1'$, which we denote by $\chi_2(D_1')$ and $b_2(D_1')$, respectively. While we relegate the details to Appendix OA1, the result is that to induce $(e_H^2, e_L^2) = (1, 0)$ the principal sets $\chi_2(D_1') = 2H - D_1'$, and to induce $(e_H^2, e_L^2) = (0, 1)$, $\chi_2(D_1')$ is set to be $H + L - D_1'$. In both cases, $b_2(D_1') = \frac{p}{p-q}$. This implies that the principal readjusts the quota level but not the bonus amount to achieve a desired effort profile in the second period. Anticipating this, if the first period’s quota level is not high enough, agents prefer to pull in sales to achieve the first-period quota and obtain the first period bonus rather than exerting effort. To induce $e_1 = 1$, in turn, the principal then sets $\chi_1$ high enough (for instance $H + L$), such that it becomes impossible for the agent to simply secure early bonus by pulling in sales if $D_1 = L$, but it is still achievable in case $D_1 = H$. The fixed salary at each period is chosen such that each of them is no lower than the agent’s limited liability, and the two combined together ensures the agent’s participation. In Table 1, we summarize the optimal contract to induce different effort profiles.

---

5We assume that the agent can pull in at most $L$ from the second period to the first, and we focus on the case when $H < 2L$. This ensures that even if $D_1 = L$, the agent can manage to report $D_1' = H$ by pulling in $H - L < L$ from $T = 2$.\[21]
The period-by-period contract in the presence of sales push-out and pull-in still performs weakly worse for the principal than the two-period contract. Namely, it performs the same as the two-period contract for inducing $e = (1, p)$, $e = (1, 1 - p)$, $e = (0, q)$ and $e = (0, 0)$, but it performs worse then the two-period contract for inducing $e = (0, 1 - q)$ and it fails to induce $e = (1, 1)$. In other words, when the principal anticipates that agents may push-out and pull-in sales, and agents also realize that the principal will respond optimally to it, the principal is unable to induce consistently high efforts. As a result, the period-by-period contract is still dominated by the two-period contract (which is unaffected by sales push-out and pull-in).

INTER-DEPENDENT PERIODS: LIMITED INVENTORY

Until now, we have assumed that the two time periods are independent of each other (except for the extension with demand transfer). In this section, we allow the two periods to be dependent on each other. There may be many ways due to which the periods can be interdependent. We consider one such way, in which we assume that the principal has a limited amount of product to sell, such that the demand outcome in the first period can change incentive provision for inducing demand in the second period. In the model until now, the agent had the incentive to dynamically adjust his effort; in the model with limited inventory, the principal also has the incentive to dynamically adjust the contract in the two time periods.

We extend the model by assuming that the principal has limited inventory, denoted by $\Omega$, to be sold across two periods. The inventory cannot be replenished before period 2 starts and any demand more than $\Omega$ is lost, i.e., actual sales $\tilde{D} = \min\{D_1 + D_2, \Omega\}$. Therefore, the two periods become dependent through $\Omega$. We assume zero inventory costs for simplicity. We keep everything else in the model the same as before. To focus on the interesting cases, we only consider the case when $0 \leq U - K \leq \frac{q}{p - q} \phi$ so that the optimal contract varies with the agent’s limited liability, and $H - L \geq \frac{p}{(p - q)^2} \phi$ so that the market upside potential is large enough to justify effort induction in both periods given unlimited inventory.

Period-by-Period (Short Time Horizon) Contract with Limited Inventory

Recall that in the previous analysis with a period-by-period contract, independence across the two periods implied that the optimal contract stays the same for $t = 1$ and $t = 2$. However, in the presence of limited inventory, the principal’s decision at $T = 2$, after observing $D_1$, is affected by the remaining inventory level $\Omega - D_1$. In other words, with limited inventory, the principal’s decision variables become
Proposition 5 (Optimal Period-by-Period Contract with Limited Inventory) With limited inventory, the optimal period-by-period contract and the outcomes are as per the following table.

<table>
<thead>
<tr>
<th>Region</th>
<th>$\Omega$</th>
<th>$e_1, E[e_2]$</th>
<th>$(s_1, s_2^{H}, s_2^{L})$</th>
<th>$(b_1, b_2^{H}, b_2^{L})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$[2L, \omega_3 - \frac{U-K}{p-q}]$</td>
<td>(0, 0)</td>
<td>(0, 0, 0)</td>
<td></td>
</tr>
<tr>
<td>II</td>
<td>$[\omega_3 - \frac{U-K}{p-q}, \max{\omega_2 - \frac{1+q-p}{q(p-q)}(U-K), \omega_2' - \frac{1+q-p}{(1-p)(p-q)}(U-K)}]$</td>
<td>(0, 1 - $q$)</td>
<td>(0, 0, $\frac{\phi}{p-q}$)</td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>$[\max{\omega_2 - \frac{1+q-p}{q(p-q)}(U-K), \omega_2' - \frac{1+q-p}{(1-p)(p-q)}(U-K)}, \omega_1 - \frac{U-K}{p-q}]$</td>
<td>(1, 1 - $p$)</td>
<td>(K, K, K)</td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>$[\omega_1 - \frac{U-K}{p-q}, 2H]$</td>
<td>(1, 1)</td>
<td>(K, K, K)</td>
<td></td>
</tr>
</tbody>
</table>

When the inventory level is high enough (Region IV) and does not lead to a bottleneck, the principal induces $e_1 = (1, 1)$, consistent with the case without inventory concerns. For a smaller $\Omega$ (Region III), although $e_1 = 1$ remains, $e_2$ becomes contingent on $D_1$; a successful first period will cause an inventory shortage later and no extra effort is needed. The expected effort in the second period thus is the probability of realizing $D_1$ as $L$, which is $1 - p$, and the resulting effort profile is $e = (1, 1 - p)$. If $\Omega$ is further below a threshold (Region II), the principal abandons early effort induction. This leads to an equilibrium effort profile $e = (0, 1 - q)$. For a yet smaller $\Omega$ (Region I), inventory levels are too low to justify any effort induction, i.e. $e = (0, 0)$. It is noteworthy that the set of optimal effort profiles excludes $e = (0, 1)$ and $e = (1, 0)$. We state the following corollary.

Corollary 3 Under a period-by-period contract with limited inventory, both $e = (1, 1 - p)$ and $e = (0, 1 - q)$ can be optimally induced expected effort profiles by the firm.

The expected effort profile $e = (1, 1 - p)$ indicates that the agent exerts effort in the first period and exerts effort in the second period only if the demand outcome of the first period is low; this can be interpreted as “getting lazy” in the second period as no effort is exerted if the first period is a success.

\[ \omega_1 \equiv H + L + \frac{p}{(p-q)^2} \phi, \omega_2 \equiv H + L - \frac{1-p}{q} (H-L) + \frac{p-q+p}{q(p-q)} \phi, \omega_2' \equiv 2L + \frac{p-p^2+p}{q(p-q)} \phi, \omega_3 \equiv 2L + \frac{p}{(p-q)^2} \phi, \omega_4 \equiv \frac{1+q-p}{(1-p)(p-q)} \phi \]
The expected effort profile \( e = (0, 1 - q) \) indicates that the agent does not exert effort in the first period and exerts effort in the second period only if the demand outcome of the first period is low; this can be interpreted as “delaying and rushing” in the second period as no effort is exerted if the first period is a success. On the face of it, these behaviors of the agent may appear sub-optimal for the firm. However, our analysis shows that they can be optimally induced by it. A key insight is that when the working environment is easy enough for the agent, i.e., the total amount of product to be sold, \( \Omega \), is small, or \( H \) or \( p \) is large, the principal may have the incentive to induce agents to work hard only in the late period, i.e., postpone effort.

**Two-Period (Long Time Horizon) Contract with Limited Inventory**

In this case, we solve the firm’s contracting problem by replacing the total demand \( D \) by its truncated value \( \tilde{D} = \min\{D_1 + D_2, \Omega\} \). Note that we assume that compensation cannot be decreasing in total sales. We obtain the following proposition (the proof is in Section OA3 in the Online Appendix).

**Proposition 6 (Optimal Two-Period Contract with Limited Inventory)** With limited inventory, the optimal two-period contract and the outcomes are as per the following table.\textsuperscript{7}

<table>
<thead>
<tr>
<th>Region</th>
<th>( \Omega )</th>
<th>( U - K )</th>
<th>( (e_1, E[e_2]) )</th>
<th>Contract</th>
</tr>
</thead>
<tbody>
<tr>
<td>( I )</td>
<td>( [\mu_2, \frac{q}{(p-q)\phi}] )</td>
<td>( [2L, \omega_7 - \frac{2(U-K)}{(p-q)(p-q)}] )</td>
<td>( (0, 0) )</td>
<td>( S = 2U, B_1 = 0, B_2 = 0 )</td>
</tr>
<tr>
<td>( III )</td>
<td>( [\frac{q}{(p-q)\phi}, \mu_3] )</td>
<td>( [H + L + \frac{\phi}{p-q}, 2H] )</td>
<td>( (1, p) )</td>
<td>( S = 2U - \frac{q}{p-q} \phi, B_1 = 0, B_2 = \frac{p+q}{2}\phi )</td>
</tr>
<tr>
<td>( IV )</td>
<td>( [\mu_3, \frac{q}{(p-q)\phi}] )</td>
<td>( [H + L + \frac{\phi}{p-q}, 2H] )</td>
<td>( (1, 1) )</td>
<td>( S = 2K, B_1 = \frac{p}{p+q}, B_2 = \frac{p}{p+q} )</td>
</tr>
<tr>
<td>( V )</td>
<td>( [\mu_3, \frac{q}{(p-q)\phi}] )</td>
<td>( \max \left{ \omega_8 - \frac{2(U-K)}{(p-q)(p-q)}, \omega_6 - \frac{2(U-K)}{(p-q)(p-q)} \right}, H + L + \frac{\phi}{p-q} )</td>
<td>( (1, 1 - p) )</td>
<td>( S = 2K, B_1 = \frac{p}{p+q}, B_2 = 2 \frac{p}{p+q} )</td>
</tr>
<tr>
<td>( VI )</td>
<td>( [\mu_2, \frac{q}{(p-q)\phi}, \mu_3] )</td>
<td>( [2L + \frac{\phi}{p-q}, \max \left{ \omega_6 - \frac{2(U-K)}{(p-q)(p-q)}, H + L + \frac{\phi}{p-q} \right}] )</td>
<td>( (0, 1 - q) )</td>
<td>( S = 2U - \frac{q}{p-q} \phi, b(L, H) = \frac{p}{p+q}, b(H, L) = 0, b(H, H) = \frac{p}{p+q} )</td>
</tr>
<tr>
<td>( \mu_3, \frac{q}{(p-q)\phi} )</td>
<td>( [2L + \frac{\phi}{p-q}, H + L + \frac{\phi}{p-q}] )</td>
<td>( (0, 1 - q) )</td>
<td>( S = 2U - \frac{q}{p-q} \phi, b(L, H) = \frac{p}{p+q}, b(H, L) = 0, b(H, H) = \frac{p}{p+q} )</td>
<td></td>
</tr>
<tr>
<td>( \mu_2, \frac{q}{(p-q)\phi} )</td>
<td>( [\omega_7 - \frac{2(U-K)}{(p-q)(p-q)}, H + L + \frac{\phi}{p-q} \phi] )</td>
<td>( (0, 1 - q) )</td>
<td>( S = 2K, b(L, H) = \frac{p}{p+q}, b(H, L) = 0, b(H, H) = \frac{p}{p+q} )</td>
<td></td>
</tr>
</tbody>
</table>

Figures 6 illustrates the parametric regions with the different effort profiles under the optimal contract. In our previous analysis, we showed that without limited inventory, the optimal contract is either a gradual contract inducing \( e = (1, 1) \) or an extreme contract inducing \( e = (1, p) \) or \( e = (0, q) \). In this scenario, if \( \Omega \) is relatively high, we are in Region IV in which a gradual contract induces \( e = (1, 1) \) or in Region III in which an extreme contract induces \( e = (1, p) \). For a small \( \Omega \), we are in Region V in which a gradual contract induces effort \( e = (1, 1 - p) \). In this case, the agent still exerts early effort,

\[
\omega_7 = H + L - \frac{1-q}{p-q} \left( H - L \right) + \frac{p^2 + p(q-p)q_2}{(p-q)(p-q)} \phi, \quad \omega_6 = H + L - \frac{1-q}{p-q} \left( H - L \right) + \frac{p+q-(p-q)^2}{(p-q)(p-q)} \phi, \quad \omega_8 = 2L + \frac{p+q-(p-q)^2}{(p-q)(p-q)} \phi, \quad \omega_5 = 2L + \frac{p+q-(p-q)^2}{(p-q)(p-q)} \phi, \quad \mu_3 = \frac{1}{2} \left[ \frac{p+q}{p-q} \phi - (1-q)(p-q)(H - L) \right], \quad \mu_2 = \frac{1}{2} \left[ \frac{p+q}{p-q} \phi - (1-q)(p-q)(H - L) \right].
\]
but will exert effort in the second period only when the first period’s outcome is \( L \). For a yet smaller \( \Omega \), we are in Region VI in which a history-dependent contract inducing effort \( e = (0, 1 - q) \) is optimal for the principal. Under this contract, the principal offers \( b_2(H, L) = 0 \) and \( b_2(L, H) = b_2(H, H) = \frac{\phi}{p-q} \). In this case, the bonus payment is not affected by the first period’s demand outcome, and will be issued if the second period realizes as \( H \).\(^8\) Under such a contract, the agent exerts no effort in the first period, and will exert effort in the second period only if the outcome of the first period is \( L \). This is again the interesting case of the “hockey stick” effort profile with effort postponement.\(^9\)

**Comparison between Period-by-Period and Two-period Contracts with Limited Inventory**

As a result of limited inventory, the conclusion from the basic model that firms weakly prefer two-period contracting over period-by-period contracting does not always hold true. Overall, the period-by-period contract outperforms the two-period contract when limited liability is very small or very large, since it gives the principal more flexibility in adjusting the contracts (note that we maintain the assumption that compensation is non-decreasing in sales). We state the following proposition (the proof is in Section OA4 in the Online Appendix).

**Proposition 7** In the presence of limited inventory, the period-by-period contract and the two-period contract compare as per the following table; specifically, the principal prefers the period-by-period contract to the two-period contract in Regions III and IV.\(^10\)

<table>
<thead>
<tr>
<th>Region</th>
<th>( U - K )</th>
<th>( \Omega )</th>
<th>(( \zeta_1, \zeta_2(\zeta_1) ))</th>
<th>Contract</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>2L &lt; ( \Omega ) ≤ ( \omega_1 - \frac{L}{p-q} )</td>
<td>( 0 \leq U - K &lt; \frac{L}{p-q} )</td>
<td>(1) ( p )</td>
<td>Period-by-period / Two-period</td>
</tr>
<tr>
<td></td>
<td>2L &lt; ( \Omega ) ≤ ( \omega_2 - \frac{2p(U - K)}{q} )</td>
<td>( \frac{L}{p-q} \leq U - K &lt; \frac{L}{p-q} )</td>
<td>(0, 1 - ( q ))</td>
<td>Period-by-period</td>
</tr>
<tr>
<td></td>
<td>2L &lt; ( \Omega ) ≤ 2L + ( \frac{L}{p-q} )</td>
<td>( \frac{L}{p-q} \leq U - K &lt; \frac{L}{p-q} )</td>
<td>(0, 1 - ( q ))</td>
<td>Period-by-period</td>
</tr>
<tr>
<td>II</td>
<td>( \omega_1 - \frac{L}{p-q}(U - K) ) ≤ ( \Omega ) ≤ ( 2L )</td>
<td>( \mu_1 \leq U - K &lt; \frac{L}{p-q} )</td>
<td>(1)</td>
<td>Period-by-period / Two-period</td>
</tr>
<tr>
<td>III</td>
<td>( \omega_2 - \frac{2p(U - K)}{q} ) ≤ ( \Omega ) ≤ ( \omega_1 - \frac{L}{p-q}(U - K) )</td>
<td>( \mu_1 \leq U - K &lt; \frac{L}{p-q} )</td>
<td>(0, 1 - ( q ))</td>
<td>Period-by-period</td>
</tr>
<tr>
<td>IV</td>
<td>( \omega_1 - \frac{L}{p-q}(U - K) ) ≤ ( \Omega ) ≤ ( 2L )</td>
<td>( \mu_1 \leq U - K &lt; \frac{L}{p-q} )</td>
<td>(1) ( p )</td>
<td>Period-by-period / Two-period</td>
</tr>
<tr>
<td>V</td>
<td>( \omega_1 - \frac{L}{p-q}(U - K) ) ≤ ( \Omega ) ≤ ( 2L )</td>
<td>( \mu_1 \leq U - K &lt; \frac{L}{p-q} )</td>
<td>(1) ( p )</td>
<td>Period-by-period / Two-period</td>
</tr>
<tr>
<td>VI</td>
<td>( \omega_1 - \frac{L}{p-q}(U - K) ) ≤ ( \Omega ) ≤ ( 2L )</td>
<td>( \mu_1 \leq U - K &lt; \frac{L}{p-q} )</td>
<td>(1) ( p )</td>
<td>Period-by-period / Two-period</td>
</tr>
</tbody>
</table>

\(^8\)Note that this is the only case where the non-decreasing constraint (that compensation should not be decreasing in sales) binds in the optimal contract. In particular, to induce \( \zeta_2(\zeta_1) = 1 \), we need \( b_2(L, H) = \) at least \( \frac{\phi}{p-q} \). Given \( b_2(L, H) = \frac{\phi}{p-q} \), \( b_2(H, H) \) cannot be less than \( \frac{\phi}{p-q} \) due to the non-decreasing constraint \( b_2(H, H) \geq b_2(L, H) \).

\(^9\)Note that with limited inventory the effort profile \( (0, q) \) is not induced under the optimal contract, while without limited the effort profile \( (0, 1 - q) \) is not induced under the optimal contract.

\(^10\)\( \omega_8 \equiv H + L + \frac{p^2}{(p+q)(p-q)^2} \phi, \omega_9 \equiv 2L + \frac{q^2 + pq - p^2 - q^2}{(1-q)(p-q)^2} \phi, \omega_9' \equiv H + L - \frac{1}{q} (H - L) + \frac{p+q-pq-p^2-q^2}{2-p} \phi, \omega_{10} \equiv H + L + \frac{1}{p} (H - L) - \frac{p+q-pq-p^2}{(p-q)^2} \phi, \mu_4 = \frac{p+q-pq-p^2}{2-p} \phi \) (H - L).
Figure 7 illustrates the results of the proposition. We now discuss the above results in greater detail with respect to the agent’s limited liability, keeping $\Omega$ fixed at $\Omega = H - L$. In Region III, where the agent’s limited liability is large, the period-by-period contract implements $e = (0, 1 - q)$ and performs the best for the principal. This is because, under the constraint that compensation cannot be decreasing in sales (to prevent the agent from restricting sales) the two-period contract rewards the agent more than the period-by-period contract when demand outcomes at both periods are $H$. In Region IV, where the agent’s limited liability is small, the period-by-period contract implements $e = (1, 1 - p)$ and performs the best for the principal. The two-period contract cannot replicate the period-by-period contract for inducing $e = (1, 1 - p)$, because it suffers from the agent’s dynamic gaming. To ensure early effort exertion, the principal pays higher bonus under the two-period contract when demand outcomes at both periods are $H$, compared with under the period-by-period contract. In those scenarios, a period-by-period contract that gives the principal flexibility to adjust quota levels performs better than a two-period contract. In Region V where the limited liability is intermediate, the two-period contract induces $e = (0, 1 - q)$ performs better than the period-by-period contract, since it can provide more effective incentives by paying the agent for the aggregate at the end of the second period. We also state the following corollary.

**Corollary 4** With limited inventory, $e = (1, p)$, $e = (1, 1 - p)$ and $e = (0, 1 - q)$ can all be optimally induced expected effort profiles by the firm.

We note that, as per our discussions earlier, different realizations under these expected effort profiles can be variourly interpreted as “giving up,” “getting lazy” and “delaying effort,” which makes it appear that the contract is not efficient. However, our analysis shows that these behaviors may be optimal for the firm and it is not necessary that the firm wants to induce high effort in both periods in all cases.

**CONCLUSIONS AND DISCUSSION**

Firms employ and reward salespeople over multiple time periods. We address a fundamental question that arises in this context: What should be the time horizon of rewarding salespeople, and what should be the structure of the contract used? We employ a two-period repeated moral hazard framework with stochastic demand and unobservable effort, and assume the agent to be risk neutral with limited liability. We allow the agent’s limited liability to be greater than or smaller than his outside option.

We find that, with independent periods, a long time horizon contract weakly dominates a short time horizon contract and, interestingly, can also do strictly better than the latter. The reason is that using
a quota-based contract over a long time horizon allows the firm to reward the salesperson only when a relatively high quota level is reached, which it cannot do in a short time horizon contract. Setting reward at a large quota allows the firm to incentivize the salesperson more strongly (the “incentive effect”). Even though the salesperson has the ability to game the contract by adjusting effort levels strategically across periods (the “gaming effect”), the incentive effect dominates when the effectiveness of the salesperson’s effort is either low or high, but not when it is intermediate. When the effectiveness is intermediate, the firm uses a contract that rewards the salesperson for all output levels, essentially on a commission basis; we show that this is equivalent to using a sequence of short time horizon optimal contracts.

We extend our analysis to a case in which a fixed amount of inventory has to be sold across multiple periods — this introduces dependence between periods as the principal’s preferred effort exertion in the later period depends on the outcome of the early period. In this case, we find that, under the assumption that compensation cannot be decreasing in sales, a sequence of short time horizon contracts can strongly dominate a long time horizon contract.

We also study the effort exertion profile of the agent. We show that the optimal effort profile may not always be high effort exertion in both periods; under different conditions, the principal may optimally induce lower effort exertion in either or both periods. Figure 8 shows three canonical expected effort profiles that the firm may optimally induce — the leftmost plot indicates high effort in both periods, the middle plot indicates high effort in the first period but low expected effort in the second period and the rightmost plot indicates no effort in the first period and low expected effort exertion in the second period (the low value can vary under different conditions, we keep it at one level for simplicity). The latter two expected effort profiles, and especially the second-period optimal effort actions that are possible under each, conditional on the first-period demand realization after the first-period optimal effort actions, can be interpreted variously as “giving up” or “getting lazy” when late effort is not exerted and “delaying effort” or “delaying and rushing” when early effort is not exerted. These sound like sub-optimal behaviors by the agent, even though the firm may want exactly these effort profiles under optimality. In essence, we show that a number of different effort profiles are possible under the optimal contract, and high effort exertion in every period is actually not always desired by the principal. Therefore, one has to be careful in making inferences about contract efficiency from realized multi-period effort exertion profiles of agents.

We obtain an interesting and useful interpretation of our model if we impose the restriction that in
a long time horizon contract only the total sales at the end of the multiple periods can be observed. Similar to Kräkel and Schöttner (2016), this models a situation in which a salesperson, within a time period, has the opportunity to sell to multiple potential consumers one at a time; however, the outcome of each interaction is not observable and only the total sales at the end are observable. We show that a gradual contract (that rewards for intermediate levels of sales) or an extreme contract (that rewards only for reaching a large enough level of sales) may be optimal for the principal in this setting, and that effectiveness of the salesperson’s effort, degree of limited liability, inventory constraints, etc., will all influence this. Furthermore, the multiple effort profile shapes discussed above will be observed here too under the optimal contract.

We conclude with a brief discussion of some of our assumptions and limitations. We have restricted our analysis to the case in which the principal chooses between a two-period contract and a period-by-period contract, but not allowed a mixed contract. We note that the principal cannot do better by mixing these two types of contracts, as using the better of the period-by-period and the two-period contracts is a weakly dominant strategy for the principal. (A formal proof is available on request.) This is because the tradeoff facing the principal remains the same — providing larger incentives or avoiding gaming losses. The mixed contract cannot provide higher incentives compared with the two-period extreme contract, and it cannot further avoid gaming losses compared with the gradual two-period contract (or the period-by-period contract). We have assumed inventory to be exogenous and have assumed away inventory costs for simplicity. However, it is straightforward to make the inventory decision endogenous by incorporating inventory costs such as marginal cost of goods, inventory holding cost for holding leftover inventory across periods, salvage costs for leftover inventory, etc. These costs will determine the choice of Ω, and given Ω the results and insights that we have derived will hold. We have assumed binomial demand and binary effort levels. However, since our main insights are driven by the tradeoff between providing incentives at the cost of gaming losses, we expect them to hold in a continuous setting as well. By the same token, if we allow periods to be dependent in other ways (e.g., a high demand outcome in the early period makes a high demand outcome in the second period more or less likely), our key insights will hold. We have also not considered phenomena such as “racheting” in which future quota targets for a salesperson are determined based on past and current performance, and we leave these considerations for future research.
References


Figure 1: First-Best Contract Effort Outcomes. In both plots, in Region I, no effort is induced even under the first-best case, and in Region II high effort is induced in both periods.

Figure 2: Optimal Period-by-Period Contract Effort Outcomes. In both plots, in Region I, no effort is induced even under the first-best case, in Region II no effort is induced in the period-by-period contracting case (but high effort would be induced in the first-best case, and in Region III high effort is induced in both periods.
Figure 3: Agent’s Effort Responses to Two-Period Contracts

Figure 4: Optimal Two-Period Contract and Effort Outcomes

Figure 5: Incentive and Gaming Effects in the Two-Period Contract. The x-axis in the figure varies $p - q$ while keeping $H - L$ fixed.
Figure 6: Optimal Two-Period Contract and Effort Profile with Limited Inventory

Figure 7: Optimal Contract Comparison and Effort Profile Under Limited Inventory

Figure 8: Different Optimally Induced Effort Profiles in a Long Time Horizon Contract
<table>
<thead>
<tr>
<th>$(e_1, E[e_2])$</th>
<th>$s_1 + s_2$</th>
<th>$\chi_1$</th>
<th>$b_1$</th>
<th>$\chi_2(D'_1)$</th>
<th>$b_2(D'_1)$</th>
<th>$E[b]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>$\max{2K, 2U}$</td>
<td>0</td>
<td>0</td>
<td>2L</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(0, $1 - q$)</td>
<td>$\max{2K + \frac{p + q - p^2}{p - q} \phi, 2U + (1 - q)\phi}$</td>
<td>0</td>
<td>0</td>
<td>$H + L - D'_1$</td>
<td>$\frac{\phi}{p-q}$</td>
<td>$(p + q - pq) \frac{\phi}{p-q}$</td>
</tr>
<tr>
<td>(0, q)</td>
<td>$\max{2K + \frac{p^2}{p - q} \phi, 2U + q\phi}$</td>
<td>0</td>
<td>0</td>
<td>$2H - D'_1$</td>
<td>$\frac{\phi}{p-q}$</td>
<td>$pq \frac{\phi}{p-q}$</td>
</tr>
<tr>
<td>(1, $1 - p$)</td>
<td>$\max{2K + \frac{2p - p^2 + pq}{p - q} \phi, 2U + (2 - p)\phi}$</td>
<td>$H + L$</td>
<td>$q \frac{\phi}{p-q}$</td>
<td>$H + L - D'_1$</td>
<td>$\frac{\phi}{p-q}$</td>
<td>$(2p - p^2 + pq) \frac{\phi}{p-q}$</td>
</tr>
<tr>
<td>(1, p)</td>
<td>$\max{2K + \frac{p^2 + p - pq}{p - q} \phi, 2U + (1 + p)\phi}$</td>
<td>$H + L$</td>
<td>$(1 - q) \frac{\phi}{p-q}$</td>
<td>$2H - D'_1$</td>
<td>$\frac{\phi}{p-q}$</td>
<td>$(p^2 + p - pq) \frac{\phi}{p-q}$</td>
</tr>
</tbody>
</table>

Table 1: Optimal Period-by-Period Contract with Sales Push-Out and Pull-In
Appendix

A1 First-Best Solution

We first list the expected payment to the agent in different regions of the parameter space.

- When \( U - K \geq 0 \), if the principal instructs the agent to exert effort, he pays the agent a fixed salary of \( U + \phi \), and if there is no effort exertion, the principal pays the agent a fixed salary of \( U \).

- When \( -\phi \leq U - K < 0 \), if the principal instructs the agent to exert effort, he pays the agent a fixed salary of \( U + \phi \), and if there is no effort exertion, the principal pays the agent a fixed salary of \( K \).

- When \( U - K < -\phi \), regardless of whether the principal instructs the agent to exert effort or not, he pays the agent a fixed salary of \( K \).

Now we compare the principal’s profits when instructing the agent to exert effort or not. The incremental payment to the agent when instructing the agent to exert effort, compared with not exerting effort, is given by

\[
E[s_t|e_t = 1] - E[s_t|e_t = 0] = \begin{cases} 
\phi, & \text{if } U - K \geq 0, \\
\phi + (U - K), & \text{if } -\phi \leq U - K < 0, \\
0, & \text{if } U - K < -\phi.
\end{cases}
\]

Compared with not instructing the agent to exert effort, instructing him to exert effort can increase the expected demand by

\[
E[D_t|e_t = 1] - E[D_t|e_t = 0] = (p - q)(H - L).
\]

The principal would like to instruct the agent to exert effort if the increase in expected demand offsets the increase in payment to the agent, i.e., when \( E[D_t|e_t = 1] - E[D_t|e_t = 0] \geq E[s_t|e_t = 1] - E[s_t|e_t = 0] \).
This is equivalent to the following condition:

\[
H - L \geq \begin{cases} 
\frac{\phi}{p-q}, & \text{if } U - K \leq 0, \\
\frac{\phi}{p-q} + \frac{U - K}{p-q}, & \text{if } -\phi \leq U - K < 0, \\
0, & \text{if } U - K < -\phi.
\end{cases}
\]

### A2 Period-by-Period (Short Time Horizon) Contract

To induce effort \( e = 1 \), the expected payment to the agent is \( \max\{K, U - \frac{q}{p-q} \phi + \frac{p}{p-q} \phi\} \). To induce effort \( e = 0 \), the payment to the agent is \( \max\{K, U\} \). Comparing the two cases, the increase in expected payment to the agent when inducing effort (compared with not inducing effort) can be simplified as

\[
E[s_t + b_t|e_t = 1] - E[s_t + b_t|e_t = 0] = \begin{cases} 
\phi, & \text{if } U - K > \frac{q}{p-q} \phi, \\
\frac{p}{p-q} \phi - (U - K), & \text{if } 0 \leq U - K < \frac{q}{p-q} \phi, \\
\frac{p}{p-q} \phi, & \text{if } U - K \leq 0.
\end{cases}
\]

Furthermore, inducing effort from the agent, compared with not inducing effort, increases the expected demand by

\[
E[D_t|e_t = 1] - E[D_t|e_t = 0] = (p - q)(H - L).
\]

The principal would like to induce effort exertion from the agent if the increase in expected demand offsets the increase in expected payment, i.e., \( E[D_t|e_t = 1] - E[D_t|e_t = 0] \geq E[s_t + b_t|e_t = 1] - E[s_t + b_t|e_t = 0] \). This is equivalent to the following condition:

\[
H - L \geq \begin{cases} 
\frac{1}{p-q} \phi, & \text{if } U - K \geq \frac{q}{p-q} \phi, \\
\frac{p}{(p-q)^2} \phi - \frac{U - K}{p-q}, & \text{if } 0 \leq U - K < \frac{q}{p-q} \phi, \\
\frac{p}{(p-q)^2} \phi, & \text{if } U - K < 0.
\end{cases}
\]
A3 Two-Period (Long Time Horizon) Contract

A3.1 General Two-Period Contract

By enumerating the optimal contract to incentivize any possible effort profile, we show that under the weakly-dominant long-term contract, $b_2(L, H) = b_2(H, L)$. Therefore, it is sufficient for the principal to focus on the long-term contract that pays at the end according to cumulative sales.

In the following, we use the labels $(IC^H_2 - ge)$, $(IC^H_2 - l)$, $(IC^L_2 - ge)$, $(IC^L_2 - l)$, $(IC_1 - ge)$, $(IC_1 - l)$ and $(LL)$ to denote the following constraints:

$(IC^H_2 - ge)$ denotes $b_2(H, H) - b_2(H, L) \geq \frac{\phi}{p - q}$ and $(IC^H_2 - l)$ denotes $b_2(H, H) - b_2(H, L) < \frac{\phi}{p - q}$;

$(IC^L_2 - ge)$ denotes $b_2(L, H) \geq \frac{\phi}{p - q}$ and $(IC^L_2 - l)$ denotes $b_2(L, H) < \frac{\phi}{p - q}$;

$(IC_1 - ge)$ denotes $U_H - U_L \geq \frac{\phi}{p - q}$ and $(IC_1 - l)$ denotes $U_H - U_L < \frac{\phi}{p - q}$;

$(LL)$ denotes $S + b_2(H, H), S + b_2(H, L), S + b_2(L, H) \geq 2K$.

To induce $e = (1, 1)$, the principal’s problem is:

$$\min_{S, b_2(H, H), b_2(H, L), b_2(L, H)} S + p^2 b_2(H, H) + p(1 - p)(b_2(H, L) + b_2(L, H))$$

s.t.

$$S + p^2 b_2(H, H) + p(1 - p)(b_2(H, L) + b_2(L, H)) - 2\phi \geq 2U \quad (PC)$$

and

$(IC^H_2 - ge)$, $(IC^L_2 - ge)$, $(IC_1 - ge)$, $(LL)$

In $(IC_1 - ge)$, $U_H = S + pb_2(H, H) + (1 - p)b_2(H, L) - \phi$ is the agent’s expected utility in the second period given $D_1 = H$, and $U_L = S + pb_2(L, H) - \phi$ is the agent’s expected utility in the second period given $D_1 = L$.

The optimal contract to induce this effort profile is given by: $S = 2\max\{K, U - \frac{q}{p - q}\phi\}$, $b_2(L, H) = \frac{\phi}{p - q}$, $b_2(H, L) \leq \frac{\phi}{p - q}$, $pb_2(H, H) + (1 - p)b_2(H, L) = (1 + p)\frac{\phi}{p - q}$. The following history-independent contract lies within the optimal contract set: $S = 2\max\{K, U - \frac{q}{p - q}\phi\}$, $b_2(L, H) = \frac{\phi}{p - q}$, $b_2(H, L) = \frac{\phi}{p - q}$, $b_2(H, H) = 2\frac{\phi}{p - q}$. The expected payment to the agent is $\max\{2\frac{p}{p - q}\phi + 2K, 2\phi + 2U\}$. 

38
To induce $e = (1, p)$, the principal’s problem is:

$$\min_{S, b_2(H, H), b_2(L, H), b_2(L, H)} S + p^2 b_2(H, H) + p(1 - p)b_2(H, L) + p(1 - q)b_2(L, H)$$

s.t. 

$$S + p^2 b_2(H, H) + p(1 - p)b_2(H, L) + p(1 - q)b_2(L, H) - (1 + p)\phi \geq 2U$$

and 

$$(IC_2^H\text{-ge}), (IC_2^L\text{-ge}), (IC_1\text{-ge}), (LL)$$

In $(IC_1\text{-ge})$, $U_H = pb_2(H, H) + (1 - p)b_2(H, L) - \phi$, $U_L = qb_2(L, H)$. The optimal contract to induce this effort profile is given by: $S = 2 \max\{K, U - \frac{q}{2 - q}\phi\}$, $b_2(L, H) = 0$, $0 \leq b_2(H, L) \leq \frac{1 + p - q}{1 - p - q}$, $pb_2(H, H) + (1 - p)b_2(H, L) = (1 + p - q)\frac{\phi}{p - q}$. The following history-independent contract lies within the optimal contract set: $S = 2 \max\{K, U - \frac{q}{2 - q}\phi\}$, $b_2(L, H) = 0$, $b_2(H, L) = 0$, $b_2(H, H) = (1 + \frac{1}{p - q})\frac{\phi}{p - q}$.

To induce $e = (1, 1 - p)$, the principal’s problem is:

$$\min_{S, b_2(H, H), b_2(H, L), b_2(L, H)} S + pq b_2(H, H) + p(1 - q)b_2(H, L) + (1 - p) pb_2(L, H)$$

s.t. 

$$S + pq b_2(H, H) + p(1 - q)b_2(H, L) + (1 - p) pb_2(L, H) - (2 - p)\phi \geq 2U$$

and 

$$(IC_2^H\text{-ge}), (IC_2^L\text{-ge}), (IC_1\text{-ge}), (LL)$$

In $(IC_1\text{-ge})$, $U_H = qb_2(H, H) + (1 - q)b_2(H, L)$, $U_L = pb_2(L, H) - \phi$.

The optimal contract to induce this effort profile is given by: $S = 2 \max\{K, U - \frac{q}{2 - q}\phi\}$, $b_2(L, H) = \frac{\phi}{p - q}$, $0 \leq b_2(H, L) \leq \frac{1 + q}{1 - q}\frac{\phi}{p - q}$, $qb_2(H, H) + (1 - q)b_2(H, L) = (1 + q)\frac{\phi}{p - q}$. The following history-independent contract lies within the optimal contract set: $S = 2 \max\{K, U - \frac{q}{2 - q}\phi\}$, $b_2(L, H) = \frac{\phi}{p - q}$, $b_2(H, L) = \frac{\phi}{p - q}$, $b_2(H, H) = 2\frac{\phi}{p - q}$.

To induce $e = (0, q)$, the principal’s problem is:

$$\min_{S, b_2(H, H), b_2(H, L), b_2(L, H)} S + qpb_2(H, H) + q(1 - p)b_2(H, L) + (1 - q)qb_2(L, H)$$

s.t. 

$$S + qpb_2(H, H) + q(1 - p)b_2(H, L) + (1 - q)qb_2(L, H) - q\phi \geq 2U$$

and 

$$(IC_2^H\text{-ge}), (IC_2^L\text{-ge}), (IC_1\text{-ge}), (LL)$$

In $(IC_1\text{-l})$, $U_H = pb_2(H, H) + (1 - p)b_2(H, L) - \phi$, $U_L = qb_2(L, H)$.

The unique optimal contract to induce this effort profile is given by: $S = 2 \max\{K, U - \frac{q}{2 - q}\phi\}$, $b_2(L, H) = 0$, $b_2(H, L) = 0$, $b_2(H, H) = \frac{\phi}{p - q}$. The above contract is clearly history-independent.
To induce $e = (0, 1 - q)$, the principal’s problem is:

$$\min_{S, b_2(H, H), b_2(H, L), b_2(L, H)} S + q^2 b_2(H, H) + q(1 - q)b_2(H, L) + (1 - q)pb_2(L, H)$$

subject to:

$$S + q^2 b_2(H, H) + q(1 - q)b_2(H, L) + (1 - q)pb_2(L, H) - (1 - q)\phi \geq 2U \quad (PC)$$

and

$$(IC_2^H - l), (IC_2^L - ge), (IC_1 - l), (LL)$$

In $(IC_1 - l)$, $U_H = qb_2(H, H) + (1 - q)b_2(H, L)$, $U_L = pb_2(L, H) - \phi$.

The unique optimal contract to induce this effort profile is given by: $S = 2 \max\{K, U - \frac{1}{2} \frac{q}{p - q} \phi\}$, $b_2(L, H) = \frac{\phi}{p - q}$, $b_2(H, L) = \frac{\phi}{p - q}$, $b_2(H, H) = \frac{\phi}{p - q}$ due to the non-decreasing constraint. $b_2(L, H)$ needs to be at least $\frac{\phi}{p - q}$ to induce $e_2^L = 1$. Because $b_2(H, H)$ cannot be lower than $b_2(L, H)$ according to the non-decreasing constraint, we will also have $b_2(H, H) = \frac{\phi}{p - q}$. Indeed, the non-decreasing constraint matters only when inducing $e = (0, 1 - q)$.

We will reach at the same contract for inducing any other profile regardless of imposing the non-decreasing constraint or not.

Note that the optimal contract for inducing $e = (0, 1 - q)$ turns out to be history-dependent. However, we can prove that inducing $e = (0, 1 - q)$ is sub-optimal for the principal when the two periods are independent (see Appendix A3.2 for details), thus is out of consideration.

The cases when the principal would like to induce effort $e = (1, 0)$ or $e = (1, 0)$ are trivially dominated by the case when he would like to induce $e = (1, 1)$ so there is no need for consideration.

We observe that, although the contract to induce $e = (0, 1 - q)$ is history dependent, it is sub-optimal for the principal; in all other cases, the optimal contract is characterized by $b_2(L, H) = b_2(H, L)$. Therefore, when the two periods are independent, it suffices for the principal to pay the agent the same at the end of the second period based on the cumulative sales across two periods.

We summarize in Table A1 the expected sales and payments to the agent for different effort profiles that the principal induces.
\begin{tabular}{|c|c|c|}
\hline
\((e_1, E[e_2])\) & \(E[D]\) & \(S + E[B]\) \\
\hline
\((0, 0)\) & \(2qH + (2 - 2q)L\) & \(\max\{2K, 2U\}\) \\
\((0, 1 - q)\) & \((p + q^2 + q - pq)H + (1 - q)(2 - p + q)L\) & \(\max\{2K + \frac{p + q^2 - pq}{p - q} \phi, 2U + (1 - q)\phi\}\) \\
\((0, q)\) & \((pq + 2q - q^2)H + (2 - pq - 2q + q^2)L\) & \(\max\{2K + \frac{pq}{p - q} \phi, 2U + q\phi\}\) \\
\((1, 1 - p)\) & \((2p - p^2 + pq)H + (2 - 2p + p^2 - pq)L\) & \(\max\{2K + \frac{2p^2 + pq}{p - q} \phi, 2U + (2 - p)\phi\}\) \\
\((1, p)\) & \((p + p^2 + q - pq)H + (2 - p - p^2 - q + pq)L\) & \(\max\{2K + \frac{2p^2 + pq}{p - q} \phi, 2U + (1 + p)\phi\}\) \\
\((1, 1)\) & \(2pH + (2 - 2p)L\) & \(\max\{\frac{2K + 2p \phi}{p - q}, 2U + 2\phi\}\) \\
\hline
\end{tabular}

Table A1: Two-period Contract

### A3.2 Optimal Two-Period Contract

We first rule out the optimality of inducing \(e = (0, 1 - q)\) and \(e = (1, 1 - p)\) for the principal. Inducing \(e = (0, 1 - q)\) is sub-optimal for the principal due to the following reasons. The saving in expected payment in inducing \(e = (0, 1 - q)\) compared with inducing \(e = (1, p)\) is given by \((p + q)\phi\), regardless of the value of \(U - K\). (This is because when \(K < -\frac{q}{2(p - q)} \phi\), the expected payment to the agent for inducing \(e = (0, 1 - q)\) is \((1 - q)\phi + 2U\), and the expected payment to the agent for inducing \(e = (1, p)\) is \((1 + p)\phi + 2U\). When \(K > -\frac{q}{2(p - q)} \phi\), the expected payment to the agent for inducing \(e = (0, 1 - q)\) is \(\frac{p + q^2 - pq}{p - q} \phi + 2K\), and the expected payment to the agent for inducing \(e = (1, p)\) is \(\frac{p + q^2 - pq}{p - q} \phi + 2K\).) In addition, the loss in expected demand in inducing \(e = (0, 1 - q)\) compared with inducing \(e = (1, p)\) is given by \((p^2 - q^2)(H - L)\). Therefore, inducing \(e = (0, 1 - q)\) is dominated by inducing \(e = (1, p)\) if \(H - L \geq \frac{\phi}{p - q}\).

This is the parameter space we consider when \(U - K \geq 0\). For \(U - K < 0\), inducing \(e = (0, 1 - q)\) generates a lower profit for the principal compared with inducing \(e = (0, 0)\) when \(H - L < \frac{p + q^2 - pq}{(1 - q)(p - q)} \phi\). Since \(\frac{1}{p - q} \phi < \frac{p + q^2 - pq}{(1 - q)(p - q)^2} \phi\), inducing \(e = (0, 1 - q)\) will be either dominated by inducing \(e = (1, p)\) or inducing \(e = (0, 0)\) under the parameter space we are considering.

We can rule out the optimality of inducing \(e = (1, 1 - p)\) using a similar rationale. In particular, when \(U - K \geq 0\), inducing \(e = (1, 1 - p)\) is dominated by inducing \(e = (1, 1)\) given \(H - L \geq \frac{\phi}{p - q}\), the parameter space we focus on. When \(U - K < 0\), inducing \(e = (1, 1 - p)\) is dominated by inducing either \(e = (1, 1)\) or \(e = (0, 0)\) for the principal and is also sub-optimal. In particular, when \(H - L < \frac{2p^2 + pq}{(2 - p)(p - q)} \phi\), we can prove that the incremental expected demand when inducing \(e = (1, 1 - p)\) compared with inducing \(e = (0, 0)\), given by \((2 - p)(p - q)(H - L)\), is no more than the incremental expected payment, given
by \( \frac{2p-p^2+pq}{p-q} \phi \). Therefore inducing \( e = (0, 0) \) dominates inducing \( e = (1, 1 - p) \) for the principal when 
\[
H - L < \frac{2p-p^2+pq}{(2-p)(p-q)} \phi - \frac{\phi}{p-q}.
\]
Since \( \frac{2p-p^2+pq}{(2-p)(p-q)} \phi > \frac{\phi}{p-q} \), inducing \( e = (1, 1 - p) \) will be dominated either by 
inducing \( e = (1, 1) \) or inducing \( e = (0, 0) \) for the principal and thus is sub-optimal.

Next, we compare the principal’s profits for inducing the remaining effort profiles, i.e., \( e = (0, 0) \), 
\( e = (0, q) \), \( e = (1, p) \), and \( e = (1, 1) \). We now solve for the optimal two-period contract.

- **Case 1:** \( U - K \leq 0 \).

  In this case, under the first-best scenario it is in the principal’s interest to induce effort if and only
  if \( H - L > \frac{\phi + (U - K)}{p-q} \).

  \[
  E[S + B] = \begin{cases} 
  2K, & \text{if } e = (0, 0) \\
  2K + \frac{pq}{p-q} \phi, & \text{if } e = (0, q) \\
  2K + \frac{p^2+p-pq}{p-q} \phi, & \text{if } e = (1, p) \\
  2K + \frac{2p}{p-q} \phi, & \text{if } e = (1, 1)
  \end{cases}
  \]

  Under this scenario, we find that inducing \( e = (0, q) \) is sub-optimal, since the principal gets a lower
  profit by inducing \( e = (0, q) \) compared with inducing \( e = (1, p) \) when \( H - L \geq \frac{p(1+p-2q)}{(1+p-q)(p-q)\phi} \),
  and the principal gets a lower profit inducing \( e = (0, q) \) compared with inducing \( e = (0, 0) \) when
  \( H - L < \frac{p}{(p-q)^2} \phi \). Since \( \frac{p(1+p-2q)}{(1+p-q)(p-q)^2} \phi < \frac{p}{(p-q)^2} \phi \), it is sub-optimal for the principal to induce 
  \( e = (0, q) \) under this scenario. Comparing the expected demands and payments for inducing 
  \( e = (0, 0) \), \( e = (1, p) \) and \( e = (1, 1) \), we get the optimal contract for the principal as follows:

  \[
  (e_1, E[e_2])^* = \begin{cases} 
  (1, 1), & \text{if } H - L \geq \frac{p(1+p-q)}{(1-p)(p-q)\phi}, \\
  (1, p), & \text{if } \frac{p(1+p-q)}{(1+p-q)(p-q)\phi} \leq H - L < \frac{p(1+p+q)}{(1-p)(p-q)^2} \phi, \\
  (0, 0), & \text{o.w.}
  \end{cases}
  \]

- **Case 2:** \( 0 < U - K \leq \frac{q^2}{2(p-q)} \phi \).

  Case 2 is the same with Case 1, except that the expected payment to the agent for inducing 
  \( e = (0, 0) \) becomes \( 2U \), rather than \( 2K \). The other payments are the same as in Case 1. Comparing 
  the principal’s profit for inducing the other effort profiles, we get the solution under this case as
follows:

\[(e_1, E[e_2])^* = \begin{cases} 
(1, 1), & \text{if } H - L \geq \frac{p(1+p+q)}{(1-p)(p-q)^2} \phi, \\
(1, p), & \text{if } 0 \leq U - K < \frac{pq}{2(1+p-q)(p-q)} \phi, H - L > \frac{p^2+p-q}{(1+p)(p-q)^2} \phi - \frac{2(U-K)}{(1-p)(p-q)^2} \phi, \\
(0, q), & \text{if } \frac{pq}{2(1+p-q)(p-q)} \phi \leq U - K < \frac{q^2}{2(p-q)} \phi, \frac{p(1+p-2q)}{(1+p-q)(p-q)} \phi \leq H - L < \frac{p(1+p+q)}{(1-p)(p-q)^2} \phi, \\
(0, 0), & \text{if } \frac{pq}{2(1+p-q)(p-q)} \phi \leq U - K < \frac{q^2}{2(p-q)} \phi, H - L \leq \frac{p}{(p-q)^2} \phi - \frac{q}{q(p-q)} L. 
\end{cases} \]

- **Case 3:** \(\frac{q^2}{2(p-q)} \phi < U - K \leq \frac{q}{p-q} \phi\).

The expected payment for inducing \(e = (0, q)\) becomes \(q \phi + 2U\) in this case. The other payments are the same as in Case 2. The solution for this case is as follows:

\[(e_1, E[e_2])^* = \begin{cases} 
(1, 1), & \text{if } H - L \geq \frac{p(1+p+q)}{(1-p)(p-q)^2} \phi, \\
(1, p), & \text{if } \frac{p+(p-q)}{(1+p)(p-q)^2} \phi - \frac{2(U-K)}{(1-p)(p-q)^2} \phi \leq H - L < \frac{p(1+p+q)}{(1-p)(p-q)^2} \phi, \\
(0, q), & \text{if } \frac{p}{p-q} \phi \leq H - L < \frac{p(p-q)^2}{(1+p)(p-q)} \phi - \frac{2(U-K)}{(1+p-q)(p-q)^2} \phi, \\
(0, 0), & \text{if } H - L \leq \frac{p}{p-q}. 
\end{cases} \]

- **Case 4:** \(\frac{q}{2(p-q)} \phi < U - K \leq \frac{q}{p-q} \phi\).

The expected payment for inducing \(e = (1, p)\) becomes \((1 + p) \phi + 2U\) in this case. The other payments are the same as in Case 3. The solution for this case is as follows:

\[(e_1, E[e_2])^* = \begin{cases} 
(1, 1), & \text{if } H - L \geq \frac{p(1+p+q)}{(1-p)(p-q)^2} \phi, \\
(1, p), & \text{if } \frac{p}{p-q} \phi \leq H - L < \frac{p(1+p+q)}{(1-p)(p-q)^2} \phi - \frac{2(U-K)}{(1-p)(p-q)^2} \phi, \\
(0, 0), & \text{if } H - L \leq \frac{p}{p-q}. 
\end{cases} \]

- **Case 5:** \(U - K > \frac{q}{p-q} \phi\).

The expected payment for inducing \(e = (1, 1)\) becomes \(2 \phi + 2U\) in this case. The other payments are the same as in Case 4. The solution for this case is as follows:

\[(e_1, E[e_2])^* = \begin{cases} 
(1, 1), & \text{if } H - L \geq \frac{1}{p-q}, \\
(0, 0), & \text{if } H - L < \frac{1}{p-q}. 
\end{cases} \]
Online Appendix for “How Often Should You Reward Your Salesforce? Multi-Period Incentives and Effort Dynamics”

OA1 Period-by-Period Contract with Sales Push-Out and Pull-In

We derive the optimal period-by-period contract using backward induction. We start from the principal’s problem of inducing a specified effort profile in the second period.

• **Case 1:** Consider the case when the principal induces \( e^H_2 = 1 \), i.e., effort is exerted by the agent when the first period’s sales realization is \( D_1 = H \).

Given the reported sales level \( D'_1 \), even though the principal cannot observe the real realization of \( D_2 \) but can only observe the reported sales level \( D'_2 \), he can still infer \( D_2 \) from \( D'_2 \) by readjusting the quota level at the second period and setting bonus value high enough to ensure that if \( D_2 = H \) the agent will not restrict sales. In particular, given \( D_1 = H \), no matter what sales level the agent reports for the early period \( D'_1 \), if \( D_2 \) also realizes as \( H \), the principal expects to observe the second period’s sales level as \( D'_2 = 2H - D'_1 \) (conditional on the bonus being high enough).

Consequently, to induce \( e^H_2 = 1 \), it suffices to set the quota level \( \chi^H_2 \) equal to \( 2H - D'_1 \), and to provide bonus \( b_2 \) equal to \( \frac{\phi}{p-q} \). Additionally, since \( \chi_2 = 2H - D'_1 \geq H + L - D'_1 \), in case \( D_1 = L \), no matter how much the agent reports, the later quota level will never be met. This implies that inducing \( e^H_2 = 1 \) will lead to \( e^L_2 = 0 \).

Combined together, by setting \( \chi_2 = 2H - D'_1 \geq H \) and \( b_2 = \frac{\phi}{p-q} \), the principal induces \((e^H_2, e^L_2) = (1,0)\). We will discuss the level of the fixed salary later, as the agent decides whether to accept the principal’s contract at the beginning by weighing his utilities across two periods, anticipating that the principal may readjust quota levels later.

• **Case 2:** Inducing \( e^L_2 = 1 \), i.e., motivating effort exertion when the first period’s sales realization is \( D_1 = L \).

In a similar way as the case above, we have that the principal needs to set the quota level at \( \chi^L_2 = H + L - D'_1 \) and the bonus level at \( b_2 = \frac{\phi}{p-q} \) to induce \( e^L_2 = 1 \). Also, since \( \chi_2 = H + L - D'_1 \), the quota level in the second period will always be met in case \( D_1 = H \) so that there will be no effort exerted. This implies \( e^L_2 = 1 \) will lead to \( e^H_2 = 0 \). Combined together, by setting \( \chi_2 = H + L - D'_1 \geq H \) and \( b_2 = \frac{\phi}{p-q} \), the principal induces \((e^H_2, e^L_2) = (0,1)\).

To summarize, to induce a specific effort level, the principal can adjust the quota level in the second period \( \chi_2 \) based on reported sales \( D'_1 \) in the first period. To induce \((e^H_2, e^L_2) = (1,0)\), the principal sets \( \chi_2(D'_1) = 2H - D'_1 \). To induce \((e^H_2, e^L_2) = (0,1)\), the principal sets \( \chi_2(D'_1) = H + L - D'_1 \). In both cases, the principal offers a bonus of \( b_2 = \frac{\phi}{p-q} \) once the sales meet the quota level. Furthermore, \((e^H_2, e^L_2) = (1,1)\) is not incentive compatible when a salesperson can push out or pull in sales since the range of quota levels required to motivate \( e^H_2 = 1 \) has no overlap with that to induce \( e^L_2 = 1 \).
Now we move to the principal’s problem in the first period. To induce \( e_1 = 1 \), the principal needs to set the corresponding quota level \( \chi_1 \) at such a level that, after accounting for sales push out and pull in, when \( D_1 = L \) the agent cannot meet the quota, and when \( D_1 = H \) he can meet the quota; this is derived as \( 2L < \chi_1 \leq H + L \). To see this, given \( \chi_1 > 2L \), when \( D_1 = L \), the agents cannot make \( \chi_1 \) even by pulling in all available sales \( L \) from the second period. Given \( \chi_1 \leq H + L \), when \( D_1 = H \), the agent can meet the quota by pulling in \( \chi - H < L \). Without loss of generality, it is enough to consider \( \chi_1 = H + L \). The early bonus level \( b_1 \) as well as the two fixed wage levels \( s_1 \) and \( s_2 \) are chosen by accounting for the second period’s effort profile. Now we discuss all the possible scenarios.

(1) To induce \( e = (1, p) \), the principal needs to provide sufficient \( b_1 \) for agents to exert effort, which is given by \( (1-q)\frac{\phi}{p-q} \). This is because, if \( D_1 = H \), the agent earns \( s_1 + b_1 + s_2 + p\frac{\phi}{p-q} - \phi \); if \( D_1 = L \), the agent earns \( s_1 + s_2 \). To induce \( e_1 = 1 \), the principal needs to make sure \( b_1 + p\frac{\phi}{p-q} - \phi \geq \frac{\phi}{p-q} \), which simplifies into \( b_1 \geq (1-q)\frac{\phi}{p-q} \). Following this, the principal pays the agent \( s_1 + s_2 + p(b_1 + p\frac{\phi}{p-q}) = s_1 + s_2 + p(1+p-q)\frac{\phi}{p-q} \) in expectation. Fixed wages are chosen such that the fixed wage in each period is no lower than the limited liability, and the two fixed wages combined can ensure the agent’s participation, namely, \( s_1 \geq K, s_2 \geq K, s_1 + s_2 \geq 2U + p(1+p-q)\frac{\phi}{p-q} - (1-p)\phi = 2U + \frac{q}{p-q}\phi \).

To summarize, to induce \( e = (1, p) \), at \( T = 1 \), the principal sets \( \chi_1 = H + L \) and \( b_1 = (1-q)\frac{\phi}{p-q} \). Under this contract, the agent exerts effort at \( T = 1 \). If \( D_1 = H \), the agent pulls in \( L \) from \( T = 2 \) to meet the early quota. Then at \( T = 2 \), the principal readjusts the quota level to \( \chi_2 = H - L \) and sets \( b_2 = \frac{\phi}{p-q} \) to encourage effort exertion. If \( D_1 = L \), the agent reports \( D_1 = L \) as it is. The principal then readjusts the quota level at the second period to \( \chi_2 = 2H - L \) which is higher than \( H \) so the quota level is not achievable and the agent gives up.

(2) Similarly, to induce \( e = (1, 1-p) \), the principal sets \( \chi_1 = H + L \) and offers \( b_1 = q\frac{\phi}{p-q} \). To see this, if \( D_1 = H \), the agent gets paid \( s_1 + b_1 + s_2 + \frac{\phi}{p-q} \); otherwise, the agent gets paid \( s_1 + s_2 + p\frac{\phi}{p-q} - \phi \). It requires \( b_1 + \frac{\phi}{p-q} - (p\frac{\phi}{p-q} - \phi) \geq \frac{\phi}{p-q} \) to motivate \( e_1 = 1 \), which is equivalent to \( b_1 \geq q\frac{\phi}{p-q} \). The principal pays the agent \( s_1 + s_2 + (2p - p^2 + pq)\frac{\phi}{p-q} \) on expectation. Finally, the fixed wages are set such that each period’s wage is no lower than the limited liability and the two wages in combination will ensure the agent’s participation, i.e., \( s_1 \geq K, s_2 \geq K, s_1 + s_2 \geq 2U + \frac{2q}{p-q}\phi \).

In this scenario, if \( D_1 = H \), agents again pull in \( L \) from the second period to earn early bonus; the principal later sets \( \chi_2 = 0 \) inducing no effort. If \( D_1 = L \) instead, agents cannot make early bonus and simply report \( D_1 = L \); the principal will set \( \chi_2 = H \) to induce effort.

(3) In other circumstances when the principal does not want to induce early effort, the contract is straightforward to derive and is shown in Table 1.

**OA2 Optimal Period-by-period Contract with Limited Inventory**

With limited inventory, we first update the expected demands generated under different effort profiles. Here, we write the effort level in the second period in terms of its expectation. For instance, effort profile \( e = (0, 1-q) \) implies that the principal does not induce effort in the first period, and has a probability of \( 1-q \) to induce effort in the second period (i.e., when the first period demand realizes as \( L \)). The result is given by Table OA1.
\[
\begin{array}{|c|c|c|}
\hline
(e_1, E[e_2]) & 2L < \Omega \leq H + L & H + L < \Omega \leq 2H \\
\hline
(0, 0) & (2q - q^2)\Omega + (1 - q)^22L & q^2\Omega + (2q - q^2)(H + L) + (1 - q)^22L \\
(0, 1 - q) & (p + q - pq)\Omega + (1 - p)(1 - q)2L - (q^2 + p - pq)\frac{\Omega}{1-q} & q^2\Omega + (p - q^2 + q - pq)(H + L) + (1 - p)(1 - q)2L - (q^2 + p - pq)\frac{\Omega}{1-q} \\
(0, q) & (2q - q^2)\Omega + (1 - q)^22L - pq\frac{\Omega}{p-q} & pq\Omega + (1 + q^2 - q - pq)(H + L) + (1 - q)^22L - pq\frac{\Omega}{p-q} \\
(1, 1 - p) & (2p - p^2)\Omega + (1 - p)^22L - (2p - p^2 + pq)\frac{\Omega}{p-q} & pq\Omega + (2p - pq - p^2)(H + L) + (1 - p)^22L - (2p - p^2 + pq)\frac{\Omega}{p-q} \\
(1, p) & (p + q - pq)\Omega + (1 - p)(1 - q)2L - (p^2 + p - pq)\frac{\Omega}{p-q} & p^2\Omega + (p - p^2 + q - pq)(H + L) + (1 - p)(1 - q)2L - (p^2 + p - pq)\frac{\Omega}{p-q} \\
(1, 1) & (2p - p^2)\Omega + (1 - p)^22L - 2p\frac{\Omega}{p-q} & p^2\Omega + (2p - 2p^2)(H + L) + (1 - p)^22L - 2p\frac{\Omega}{p-q} \\
\hline
\end{array}
\]

Table OA1: Expected Demand under Period-by-period Contract with Limited Inventory

![Figure OA1: Optimal Period-by-period Contract with Limited Inventory](image)

Figure OA1: Optimal Period-by-period Contract with Limited Inventory

We then derive the expected payments to the agent when different effort profiles are induced. Based on the solution for the period-by-period contract, if the induced effort level is \( e_t = 1 \), then the principal pays \( \max\{K + \frac{p}{p-q}\phi, U + \phi\} \) to the agent in expectation. If the induced effort level is \( e_t = 0 \), then the principal pays \( \max\{K, U\} \) to the agent. We thus get the expected payments to the agent when the two periods are dependent, by adding up the expected payments to the agent in each period. Throughout this online appendix, we use \( E[S + B] \) as the total expected payment to the agent during the two periods, including both fixed salaries and bonuses. The result is given by the following table.

\[
\begin{array}{|c|c|}
\hline
(e_1, E[e_2]) & E[S + B] \\
\hline
(0, 0) & 2\max\{K, U\} \\
(0, 1 - q) & (1 - q)\max\{K + \frac{p}{p-q}\phi, U + \phi\} + (1 + q)\max\{W, U\} \\
(0, q) & q\max\{K + \frac{p}{p-q}\phi, U + \phi\} + (2 - q)\max\{K, U\} \\
(1, 1 - p) & (2 - p)\max\{K + \frac{p}{p-q}\phi, U + \phi\} + p\max\{K, U\} \\
(1, p) & (1 + p)\max\{K + \frac{p}{p-q}\phi, U + \phi\} + (1 - p)\max\{K, U\} \\
(1, 1) & 2\max\{K + \frac{p}{p-q}\phi, U + \phi\} \\
\hline
\end{array}
\]

Table OA2: Expected Payment under Period-by-period Contract with Limited Inventory
We solve the optimal period-by-period contract for the principal when the two periods are dependent, by considering $U - K$ in different ranges. Before solving the problem, we first rule out the optimality of inducing $e = (1, p)$ and inducing $e = (0, q)$ for the principal.

- Below, we first prove that it is sub-optimal for the principal to induce $e = (1, p)$.
  - When $\Omega < H + L$, we can observe that inducing $e = (1, p)$ is dominated by inducing $e = (0, 1 - q)$ for the principal. This is because, given $\Omega < H + L$, inducing $e = (1, p)$ generates the same demand in expectation as inducing $e = (0, 1 - q)$, while leading to a higher payment in expectation.
  - When $\Omega > H + L$ and $H - L \geq \frac{p}{(p - q)^2} \phi$, inducing $e = (1, p)$ is dominated by inducing $e = (1, 1)$ for the principal. Indeed, given $\Omega > H + L$, the difference in expected demands between inducing $e = (1, 1)$ and inducing $e = (1, p)$ is given by,
    \[ E_{\Omega > H + L}[D | e = (1, 1)] - E_{\Omega > H + L}[D | e = (1, p)] = \left(2p(1 - p) - (1 - p)(p + q)\right)(H + L) + \left((1 - p)^2 - (1 - p)(p - q)\right)(H + L)2L \]
    \[ = (1 - p)(p - q)(H - L) \]
    Meanwhile, the difference in their expected payments is,
    \[ E[S + B | e = (1, 1)] - E[S + B | e = (1, p)] = \begin{cases} p(1 - p)\phi, & \text{if } U - K \leq 0, \\ \frac{p(1 - p)}{p - q}\phi + (1 - p)(U - K), & \text{if } 0 < U - K \leq \frac{q}{p - q}\phi, \\ (1 - p)\phi, & \text{if } U - K > \frac{q}{p - q}\phi. \end{cases} \]

The above expression is no larger than $\frac{p(1 - p)}{p - q}\phi$, where the maximal value takes place when $U - K \leq 0$. We can see the incremental demand in inducing $e = (1, 1)$ relative to inducing $e = (1, p)$ is no less than the maximal incremental payment in this scenario. That is, given $H - L \geq \frac{p}{(p - q)^2} \phi$,

    \[ (1 - p)(p - q)(H - L) \geq \frac{p(1 - p)}{p - q}\phi. \]

This implies that $e = (1, p)$ is dominated by inducing $e = (1, 1)$ when $\Omega > H + L$ and $H - L \geq \frac{p}{(p - q)^2} \phi$.

- When $\Omega > H + L$ and $H - L < \frac{p}{(p - q)^2} \phi$, we will show that inducing $e = (1, p)$ is dominated by either inducing $e = (1, 1)$ or inducing $e = (0, 0)$. Consider the two dimensional parameter space $(U - K, \Omega)$.
  * Below, we first show that at $\Omega = 2H$ and $U - K = (p - q)\left(\frac{p}{(p - q)^2} \phi - (H - L)\right)$ (referred as “focal point”), the principal is indifferent between inducing $e = (1, 1)$, $e = (1, p)$ and $e = (0, 0)$. Since $\frac{p}{p - q}\phi - (p - q)(H - L) \in \left(0, \frac{q}{p - q}\phi\right)$, we can write down the expected payments at the focal point without ambiguity. Let’s compare the cases for inducing...
When $\Omega = \{e = (1, p), e = (1, 1)\}$. The difference in their expected demands at $\Omega = 2H$ is $(1 - p)(p - q)(H - L)$. At $U - K = \frac{p}{p - q} \phi - (p - q)(H - L)$, the difference in their expected payments is, $\frac{p(1 - p)}{p - q} \phi - (1 - p)(U - K) = (1 - p)(p - q)(H - L)$, which is equal to the difference in their expected demands. We then compare the cases for inducing $e = (1, p)$ and inducing $e = (0, 0)$ at $\Omega = 2H, U - K = \frac{p}{p - q} \phi - (p - q)(H - L)$. Given $\Omega = 2H$, the difference in their expected demands is $(1 - p)(p - q)(H - L)$. The difference in their expected payments is $\frac{p(1 + p)}{p - q} \phi - (1 + p)(U - K)$, which is equal to the incremental payment (i.e., $(1 - p)(p - q)(H - L)$) when $U - K = \frac{p}{p - q} \phi - (p - q)(H - L)$.

* We have shown that without inventory concern, the principal is indifferent between inducing $e = (1, 1), e = (1, p), e = (0, 0)$ at $\Omega = 2H$ and $U - K = (p\frac{p}{p - q}) (\frac{p}{(p - q)^2} \phi - (H - L))$. Now consider $\forall \Omega \in (H + L, 2H)$.

* $\forall \Omega \in (H + L, 2H)$, when $U - K > \frac{p}{p - q} \phi - (p - q)(H - L)$, compared to the focal point, the incremental demand in inducing $e = (1, 1)$ relative to inducing $e = (1, p)$ remains the same, but the incremental payment becomes smaller. This implies that $\forall \Omega \in (H + L, 2H)$, inducing $e = (1, p)$ performs worse for the principal relative to inducing $e = (1, 1)$ when $U - K > \frac{p}{p - q} \phi - (p - q)(H - L)$.

* $\forall \Omega \in (H + L, 2H)$ when $U - K < \frac{p}{p - q} \phi - (p - q)(H - L)$, compared to the focal point, the incremental demand in inducing $e = (1, p)$ relative to inducing $e = (0, 0)$ gets smaller, but the incremental payment becomes larger. This implies that inducing $e = (1, p)$ performs worse for the principal relative to inducing $e = (0, 0)$ when $U - K < \frac{p}{p - q} \phi - (p - q)(H - L)$.

* Combined together, $\forall \Omega \in (H + L, 2H)$, we have that inducing $e = (1, p)$ is sub-optimal for the principal given $H - L < \frac{p}{(p - q)^2} \phi$.

- Put the three cases above together, we can reach the conclusion that inducing $e = (1, p)$ is sub-optimal for the principal under the period-by-period contract with limited inventory.

- In a similar way, we can prove that inducing $e = (0, q)$ is sub-optimal for the principal.

  - When $\Omega < H + L$, we can observe that inducing $e = (0, q)$ is dominated by inducing $e = (0, 0)$ for the principal.

  **Proof.** This is because, given $\Omega < H + L$, inducing $e = (0, q)$ generates the same demand in expectation as inducing $e = (0, 0)$, while leading to a higher payment in expectation.

  - When $\Omega > H + L$ and $H - L \geq \frac{p}{(p - q)^2} \phi$, inducing $e = (0, q)$ is dominated by inducing $e = (1, 1 - p)$ for the principal. Indeed, given $\Omega > H + L$,

    $E_{\Omega > H + L}[D|e = (1, 1 - p)] - E_{\Omega > H + L}[D|e = (0, q)]$

    $= (p(2 - p - q) - q(2 - p - q))(H + L) + (1 - p)^2 - (1 - q)^2)2L$

    $= (2 - p - q)(p - q)(H - L)$. 

- 5
\[ E[S+B|e = (1,1-p)] - E[S+B|e = (0,q)] = \begin{cases} 
  p^{(2-p-q)\phi}, & \text{if } U-K \leq 0, \\
  p^{(2-p-q)\phi} + (2-p-q)(U-K), & \text{if } 0 < U-K \leq \frac{q}{p-q} \phi, \\
  (2-p-q)\phi, & \text{if } U-K > \frac{q}{p-q} \phi.
\end{cases} \]

The above expression is no larger than \( \frac{p^{(2-p-q)\phi}}{p-q} \), where the maximal value takes place when \( U-K \leq 0 \). Indeed, we can show that the incremental demand in inducing \( e = (1,1-p) \) relative to inducing \( e = (0,q) \), is no less than the maximal incremental payment in this scenario. That is,

\[ (2-p-q)(p-q)(H-L) \geq \frac{p^{(2-p-q)\phi}}{p-q}, \text{ given } H-L \geq \frac{p}{(p-q)^2}\phi. \]

This implies that \( e = (0,q) \) is dominated by inducing \( e = (1,1-p) \) when \( \Omega > H+L \) and \( H-L \geq \frac{p}{(p-q)^2}\phi. \)

* When \( \Omega > H+L \) and \( H-L < \frac{p}{(p-q)^2}\phi \), we will show that inducing \( e = (0,q) \) is dominated by either inducing \( e = (1,1-p) \) or inducing \( e = (0,0) \).

* Below, we first show that without inventory concern at \( \Omega = 2H \), \( U-K = \frac{p}{p-q}\phi - (p-q)(H-L) \), the principal is indifferent between inducing \( e = (0,q) \), \( e = (1,1-p) \) and \( e = (0,0) \). Notice that \( \frac{p}{p-q}\phi - (p-q)(H-L) \) lies between 0 and \( \frac{q}{p-q}\phi \), therefore we can write down the expected payments at the focal point without ambiguity. Let’s compare the case for inducing \( e = (0,q) \) and \( e = (1,1-p) \) at the focal point of \( \Omega = 2H \), \( U-K = \frac{p}{p-q}\phi - (p-q)(H-L) \). The difference in their expected demands at \( \Omega = 2H \) is \( (2-p-q)(p-q)(H-L) \). In addition, at \( U-K = \frac{p}{p-q}\phi - (p-q)(H-L) \), we get that the difference in their expected payments, \( \frac{p^{(2-p-q)\phi}}{p-q} - (2-p-q)(U-K) = (2-p-q)(p-q)(H-L) \), which is equal to the difference in their expected demands. Now compare the cases for inducing \( e = (0,q) \) and inducing \( e = (0,0) \). Given \( \Omega = 2H \), the difference in their expected demands is \( q(p-q)(H-L) \). The difference in their expected payments is \( \frac{p}{p-q}\phi - q(U-K) \), which is equal to the incremental payment (i.e., \( q(p-q)(H-L) \)) when \( U-K = \frac{p}{p-q}\phi - (p-q)(H-L) \). Therefore, we have shown that without inventory concern, the principal is indifferent between inducing \( e = (1,1-p) \), \( e = (0,q) \), \( e = (0,0) \) at the focal point of \( \Omega = 2H \), \( U-K = \frac{p}{p-q}\phi - (p-q)(H-L) \). Now let’s consider the case with \( \forall \Omega \in (H+L, 2H) \).

* \( \forall \Omega \in (H+L, 2H) \), when \( U-K > \frac{p}{p-q}\phi - (p-q)(H-L) \), compared to the focal point, the incremental demand in inducing \( e = (1,1-p) \) relative to inducing \( e = (0,q) \) remains the same, but the incremental payment becomes smaller. This implies that inducing \( e = (1,1-p) \) performs worse for the principal relative to inducing \( e = (0,q) \) in this case.

* \( \forall \Omega \in (H+L, 2H) \) when \( U-K < \frac{p}{p-q}\phi - (p-q)(H-L) \), compared to the focal point, the incremental demand in inducing \( e = (0,q) \) relative to inducing \( e = (0,0) \) gets smaller, but the incremental payment becomes larger. This implies that inducing \( e = (0,q) \) is dominated by inducing \( e = (0,0) \) when in this case.

* To summarize, it is sub-optimal for the principal to induce \( e = (0,q) \) when \( H+L < \Omega < 2H \).
\[2H \text{ and } H-L < \frac{p}{(p-q)^2}\phi.\]

- Put the three cases above together, we reach the conclusion that inducing \(e = (0, q)\) is suboptimal for the principal under the period-by-period contract with limited inventory.

Now we have ruled out the optimality of inducing \(e = (1, p)\) and \(e = (0, q)\) for the principal. Below, we focus on the remaining possible effort profiles, i.e., \(e = (1, 1-p), (0, 1-q), (0, 0)\) and compare the principal’s profits under these effort profiles. To solve the problem, we start our analysis by considering \(U-K\) in different ranges. In the following analysis, we focus on the parameter space where the upside demand potential is large enough, i.e., \(H-L \geq \frac{p}{(p-q)^2}\phi\). The other case when \(H-L < \frac{p}{(p-q)^2}\phi\) is less interesting, since in that case, when \(U-K \leq 0\), the principal will not induce effort in equilibrium regardless of the inventory level.

- Case 1: \(U-K \geq \frac{q}{p-q}\phi\):

- First, we compare the principal’s profits when inducing \(e = (1, 1)\) and when inducing \(e = (1, 1-p)\), by considering \(\Omega \in [2L, H+L)\) and \(\Omega \in [H+L, 2H)\) separately.

  * For \(\Omega \in [2L, H+L)\), inducing \(e = (1, 1)\) generates the same profit as inducing \(e = (1, 1-p)\), while paying more to the agent in expectation, thus is dominated by inducing \(e = (1, 1-p)\).

    For \(\Omega \in [H+L, 2H)\),
    \[
    E_{\Omega>H+L}[D|e = (1, 1)] - E_{\Omega>H+L}[D|e = (1, 1-p)] = (p^2 - pq)\Omega + (2p(1-p) - p(2-p-q))(H+L)
    = p\left(p - q\right)(\Omega - (H+L))
    \]

    \[
    E[S+B|e = (1, 1)] - E[S+B|e = (1, 1-p)] = (2U + 2\phi) - (2U + (2-p)\phi)
    = p\phi.
    \]

    Therefore, the principal prefers inducing \(e = (1, 1)\) to inducing \(e = (1, 1-p)\), when the incremental demand is larger then the incremental payment, which requires,

    \[
    p\left(p - q\right)(\Omega - (H+L)) > p\phi, \quad \Omega > H+L + \frac{1}{p-q}\phi
    \]

    As a result, for \(\Omega \in [H+L + \frac{1}{p-q}\phi, 2H)\), the principal prefers \(e = (1, 1)\) to \(e = (1, 1-p)\).

    For \(\Omega \in [H+L, H+L + \frac{1}{p-q}\phi)\), the principal prefers \(e = (1, 1-p)\) to \(e = (1, 1)\).

  * Combined together, we have that when \(\Omega > H+L + \frac{1}{p-q}\phi\), inducing \(e = (1, 1)\) generates a higher profit for the principal compared with inducing \(e = (1, 1-p)\).
Next, we compare the principal’s profits when inducing \( e = (1, 1 - p) \) and when inducing \( e = (0, 1 - q) \).

* For \( \Omega < H + L \),
\[
E_{2L<\Omega<H+L}[D|e = (1, 1 - p)] - E_{2L<\Omega<H+L}[D|e = (0, 1 - q)] \\
= (1 - p)(p - q)(\Omega - 2L),
\]
\[
E[S + B|e = (1, 1 - p)] - E[S + B|e = (0, 1 - q)] = (1 + q - p)\phi.
\]

Therefore the principal prefers inducing \( e = (1, 1 - p) \) to inducing \( e = (0, 1 - q) \), when the incremental demand is larger then the incremental payment, which requires,
\[
(1 - p)(p - q)(\Omega - 2L) > (1 + q - p)\phi,
\]
\[
\Omega < 2L + \frac{1 + q - p}{(1 - p)(p - q)} \phi = \omega_4.
\]

Notice that the above value is smaller than \( H + L \) (since \( H - L \geq \frac{p}{(p - q)^2} \phi \) implies that \( H - L \geq \frac{1 + q - p}{(1 - p)(p - q)} \phi \)). Therefore, the principal prefers inducing \( e = (1, 1 - p) \) to inducing \( e = (0, q) \) when \( \omega_4 < \Omega < H + L \).

* For \( \Omega > H + L \), inducing \( e = (1, 1 - p) \) dominates inducing \( e = (0, q) \). The reason is as follows. The difference in the expected demands between inducing the above two effort profiles is given by,
\[
E_{\Omega>H+L}[D|e = (1, 1 - p)] - E_{\Omega>H+L}[D|e = (0, 1 - q)] \\
= \left( p - q\right) \left( q(\Omega - (H + L)) + (1 - p)(H - L) \right).
\]

The difference in the expected payments to the agent is the same as what we just derived. Thus the principal prefers inducing \( e = (1, 1 - p) \) to inducing \( e = (0, 1 - q) \) when,
\[
\left( p - q\right) \left( q(\Omega - (H + L)) + (1 - p)(H - L) \right) > (1 + q - p)\phi,
\]
\[
i.e., \, \Omega > H + L - \frac{1 - p}{q} (H - L) + \frac{1 + q - p}{q(1-p)} \phi.
\]

However, \( H + L - \frac{1 - p}{q} (H - L) + \frac{1 + q - p}{q(1-p)} \phi \) is smaller than \( H + L \), since \( H - L \geq \frac{p}{(p - q)^2} \phi \geq \frac{1 + q - p}{(1 - p)(p - q)} \phi \).

* Combined, if \( H + L \geq \frac{p}{(p - q)^2} \phi \), the principal prefers inducing \( e = (1, 1 - p) \) to inducing \( e = (0, 1 - q) \) when \( \Omega > \omega_4 \) (which is smaller than \( H + L \)).

Finally, we compare the principal’s profits when inducing \( e = (0, 1 - q) \) and when inducing \( e = (0, 0) \).
For $\Omega < H + L$,
\[
E_{\Omega<H+L}[D|e = (0, 1 - q)] - E_{\Omega<H+L}[D|e = (0, 0)] \\
= (1 - q)(p + q - 2q(1 - q))\Omega + (1 - p)(1 - q) - (1 - q)^2)2L \\
= (1 - q)(p - q)(\Omega - 2L),
\]
\[
E[S + B|e = (0, 1 - q)] - E[S + B|e = (0, 0)] = (1 - q)\phi.
\]
Therefore the principal prefers inducing $e = (0, 1 - q)$ to inducing $e = (0, 0)$, when the incremental demand is larger than the incremental payment, which requires,
\[
(1 - q)(p - q)(\Omega - 2L) > (1 - q)\phi, \\
\Omega > 2L + \frac{1}{p - q}\phi.
\]
We can see that $2L + \frac{1}{p - q}\phi$ is indeed smaller than $H + L$ since $H - L > \frac{1}{p - q}\phi$.

* For $\Omega > H + L$, we can easily see that inducing $e = (0, q)$ is sub-optimal compared with inducing $e = (0, 0)$. The reason is that when $\Omega > H + L$, inducing $e = (0, q)$ will generate the same profit for the principal as inducing $e = (0, 0)$ while paying a higher expected payment.

* The conclusion is that under this scenario, if $\Omega > 2L + \frac{1}{p - q}\phi$, it generates a higher profit for the principal by inducing $e = (1, 1)$ compared with inducing $e = (0, 0)$.

To summarize, given $U - K > \frac{q}{p - q}\phi$, the optimal contract for the principal with limited inventory is,
\[
(e_1, E[e_2])^* = \begin{cases} 
(1, 1), & \text{if } \Omega \in [H + L + \frac{1}{p - q}\phi, 2H), \\
(1, 1 - p), & \text{if } \Omega \in [\omega_4, H + L + \frac{1}{p - q}\phi), \\
(0, 1 - q), & \text{if } \Omega \in [2L + \frac{1}{p - q}\phi, \omega_4), \\
(0, 0), & \text{if } \Omega \in [2L, 2L + \frac{1}{p - q}\phi).
\end{cases}
\]

• Case 2: $0 < U - K \leq \frac{q}{p - q}\phi$:

  We first compare the principal’s profits when inducing $e = (1, 1)$ and when inducing $e = (1, 1 - p)$.

  * Since compared with Case 1, inducing $e = (1, 1)$ will only become less attractive relative to inducing $e = (1, 1 - p)$, the cutoff value of $\Omega$ for the principal to be indifferent between the two contracts should be larger than $H + L$. The difference in the expected demands is the same as in Case 1. But the difference in their expected payments to the agent
Next, we compare the principal’s profits when inducing $e = (1, 1)$ to inducing $e = (1, 1 - p)$, when

$$p(p - q)(\Omega - (H + L)) > \frac{p^2}{p - q} \phi - p(U - K),$$

$$\Omega > \omega_1 - \frac{U - K}{p - q},$$

where $\omega_1 \equiv H + L + \frac{p}{(p - q)^2} \phi$.

* For $\Omega < H + L$, again, inducing $e = (1, 1)$ is dominated by inducing $e = (1, 1 - p)$ since the difference in their expected payments only becomes larger compared with Case 1.

* Overall, we have that compared with inducing $e = (1, 1 - p)$, inducing $e = (1, 1)$ performs better for the principal when $\Omega > \omega_1 - \frac{U - K}{p - q}$.

Next, we compare the principal’s profits when inducing $e = (1, 1 - p)$ and when inducing $e = (0, 1 - q)$. The difference in the expected demands between inducing the above two effort profiles the same as in Case 1, given by $(p - q)(q(\Omega - (H + L)) + (1 - p)(H - L))$.

* For $\Omega \in [H + L, 2H)$,

$$E[S + B|e = (1, 1 - p)] - E[S + B|e = (0, 1 - q)]
= ((2 - p)K + pU + \frac{2p - p^2}{p - q} \phi) - ((1 - q)K + (1 + q)U + \frac{p - pq}{p - q} \phi) - (1 + q - p)(U - K)
= \frac{p - p^2 + pq}{p - q} \phi - (1 + q - p)(U - K).$$

Then the principal prefers inducing $e = (1, 1 - p)$ to inducing $e = (0, 1 - q)$, when the incremental demand is larger than the incremental payment, which requires,

$$(p - q)(q(\Omega - (H + L)) + (1 - p)(H - L)) > \frac{p^2 - pq}{p - q} \phi - (1 + q - p)(U - K),
\Omega > \omega_2 - \frac{1 + q - p}{q(p - q)}(U - K)$$
and $\Omega > H + L,$

where $\omega_2 \equiv H + L - \frac{1}{q(p - q)}(H - L) + \frac{p^2 - pq}{q(p - q)^2} \phi$. Here, if $H - L < \frac{q(1 - p + q)}{(1 - p)(p - q)^2} \phi$, the above cut off value, $\omega_2 - \frac{1 + q - p}{q(p - q)}(U - K)$ is smaller than $H + L$ for $\forall U - K \in (0, \frac{q}{p - q} \phi)$. For $H - L > \frac{q(1 - p + q)}{(1 - p)(p - q)^2} \phi$, it exceeds $H + L$ only when $U - K < \mu_1 \equiv \frac{p - pq}{p - q} \phi - \frac{1}{1 + q - p}(H - L)$, where $\mu_1 \in (0, \frac{q}{p - q} \phi)$. Thus the condition for $e = (1, 1 - p)$ to dominate $e = (0, 1 - q)$ can be further simplified as: if $H - L < \frac{q(1 - p + q)}{(1 - p)(p - q)^2} \phi$, we need $\Omega > H + L$, and if $H - L > \frac{q(1 - p + q)}{(1 - p)(p - q)^2} \phi$, we need $\Omega > \omega_2 - \frac{1 + q - p}{q(p - q)}(U - K)$ and $0 < U - K < \mu_1$. 

10
Finally, we compare the principal’s profits when inducing $e = (0, 1 - q)$ when the following conditions are met:

(1) if $H - L < \frac{q(1-p+q)}{(1-p)(p-q)}\phi$, 

$$\Omega > \omega_2 - \frac{1 + q - p}{q(p-q)}(U - K)$$

(2) if $H - L > \frac{q(1-p+q)}{(1-p)(p-q)}\phi$, 

$$\Omega > \omega'_2 - \frac{1 + q - p}{(1-p)(p-q)}(U - K)$$

These can be written together as

$$\Omega > \max \left\{ \omega_2 - \frac{1 + q - p}{q(p-q)}(U - K), \omega'_2 - \frac{1 + q - p}{(1-p)(p-q)}(U - K) \right\}$$

and $0 < \mu < \frac{q}{p-q}\phi$.

Finally, we compare the principal’s profits when inducing $e = (0, 1 - q)$ and when inducing $e = (0, 0)$.

* For $\Omega$ smaller than $H + L$, the difference in the expected demands between inducing the above two effort profiles is given by the same value as in Case 1, $\Omega = H + L$.

The difference in the expected payments to the agent between inducing the above two effort profiles is

$$E_0[S + B|e = (0, 1 - q)] - E_0[S + B|e = (0, 0)]$$

$$= \left( (1 - q)K + (1 + q)U + \frac{p - pq}{p - q} \phi \right) - 2U$$

$$= \frac{p - pq}{p - q} \phi - (1 - q)(U - K).$$

Therefore the principal prefers inducing $e = (0, 1 - q)$ to inducing $e = (0, 0)$, when the
incremental demand is larger than the incremental payment, which requires,

\[(1 - q)(p - q)\left(\Omega - 2L\right) > \frac{p - p_0}{p - q} \phi - (1 - q)(U - K),\]

\[\omega_3 = \frac{U - K}{p - q} < \Omega < H + L,\]

where \(\omega_3 = 2L + \frac{p}{p - q} \phi\). Indeed, we can prove that given \(H - L > \frac{1}{p - q} \phi\), we have \(\omega_3 - \frac{U - K}{p - q} < H + L\).

* For \(\Omega > H + L\), inducing \(e = (0, 1 - q)\) generates the same profit as inducing \(e = (0, 0)\) but paying more thus is dominated by inducing \(e = (0, 0)\).

– Combined together, given \(0 < U - K \leq \frac{q}{p - q} \phi\), the optimal contract for the principal with limited inventory is as follows:

\[
(e_1, E[e_2])^* = \begin{cases} 
(1, 1), & \text{if } \Omega \in \left[\omega_1 - \frac{U - K}{p - q}, 2H\right), \\
(1, 1 - p), & \text{if } \Omega \in \left[\max\{\omega_2 - \frac{1 + q - p}{q(p - q)}(U - K), \omega_1 - \frac{U - K}{p - q}\}, \omega_1 - \frac{U - K}{p - q}\right), \\
(0, 1 - q), & \text{if } \Omega \in \left[\omega_3 - \frac{U - K}{p - q}, \max\{\omega_2 - \frac{1 + q - p}{q(p - q)}(U - K), \omega_2 - \frac{1 + q - p}{(1 - p)(p - q)}(U - K)\}\right), \\
(0, 0), & \text{if } \Omega \in \left[2L, \omega_3 - \frac{U - K}{p - q}\right).
\end{cases}
\]

**Case 3: \(U - K \leq 0\):**

– First, we compare the principal’s profits when inducing \(e = (1, 1)\) and when inducing \(e = (1, 1 - p)\).

* For \(\Omega > H + L\),

\[
E_{\Omega > H + L} [D|e = (1, 1)] - E_{\Omega > H + L} [D|e = (1, 1 - p)]
= \left(p^2 - pq\right) \Omega + \left(2p(1 - p) - p(2 - p - q)\right)(H + L)
= p \left(p - q\right) \left(\Omega - (H + L)\right),
\]

\[
E [S + B|e = (1, 1)] - E [S + B|e = (1, p)]
= \left(2K + 2 \frac{p}{p - q} \phi\right) - \left(2K + \frac{2p - p_0^2}{p - q} \phi\right)
= \frac{p^2}{p - q} \phi.
\]

Therefore the principal prefers inducing \(e = (1, 1)\) to inducing \(e = (1, 1 - p)\), when the incremental demand is larger than the incremental payment, which requires,

\[p \left(p - q\right) \left(\Omega - (H + L)\right) > \frac{p^2}{p - q} \phi,\]

\[\Omega > H + L + \frac{p}{(p - q)} \phi = \omega_1.\]
* For $\Omega < H + L$, it is easy to see that it generates a higher profit for the principal by inducing $e = (1, 1)$ compared with inducing $e = (1, 1 - p)$. Thus, inducing $e = (1, 1)$ dominates inducing $e = (1, 1 - p)$ when $\Omega > \omega_1$.

- Next, we compare the principal’s profits when inducing $e = (1, 1 - p)$ and when inducing $e = (0, 1 - q)$.

* For $\Omega$ greater than $H + L$,

\[
E_{\Omega > H+L}[D|e = (1, 1 - p)] - E_{\Omega > H+L}[D|e = (0, 1 - q)] \\
= \left(\frac{pq - q^2}{p - q}\right)\Omega + \left(p(2 - p - q) - (1 - q)(p + q)\right)(H + L) \\
= \left(p - q\right)\left(q(\Omega - (H + L)) + (1 - p)(H - L)\right), \\
E[S + B|e = (1, 1 - p)] - E[S + B|e = (0, 1 - q)] \\
= \left(2K + \frac{2p - p^2}{p - q}\phi\right) - \left(2K + \frac{p - pq}{p - q}\phi\right) \\
= \frac{p - p^2 + pq}{p - q}\phi.
\]

Therefore the principal prefers inducing $e = (1, 1)$ to inducing $e = (1, 1 - p)$, when the incremental demand is larger then the incremental payment, which requires,

\[
\left(p - q\right)\left(q(\Omega - (H + L)) + (1 - p)(H - L)\right) > \frac{p - p^2 + pq}{p - q}\phi, \\
\Omega > H + L - \frac{1}{q}\left(1 - p\right)(H - L) + \frac{p - p^2 + pq}{q(1 - p)(p - q)^2}\phi = \omega_2
\]

* For $\Omega < H + L$, it generates a higher profit for the principal by inducing $e = (1, 1 - p)$ compared with inducing $e = (0, 1 - q)$ if

\[
(p - q)(1 - p)(\Omega - 2L) > \frac{p - p^2 + pq}{p - q}\phi, \\
\Omega > 2L + \frac{p - p^2 + pq}{(1 - p)(p - q)^2}\phi = \omega_2'
\]

* Indeed $\omega_2$ and $\omega_2'$ intersect at $H - L = \frac{q(1 - p + q)}{(1 - p)(p - q)^2}\phi$. Thus inducing $e = (1, 1 - p)$ dominates inducing $e = (0, 1 - q)$ if $\Omega > \max\{\omega_2, \omega_2'\}$.

- Finally, we compare the principal’s profits when inducing $e = (0, 1 - q)$ and when inducing $e = (0, 0)$. 

13
* For $\Omega$ smaller than $H + L$,

$$E_{\Omega < H+L}[D|e = (0, 1 - q)] - E_{\Omega < H+L}[D|e = (0, 0)]$$

$$= (1 - q)(p + q) - 2q(1 - q)\Omega + \left((1 - p)(1 - q) - (1 - q)^2\right)2L$$

$$= (1 - q)(p - q)\left(\Omega - 2L\right),$$

$$E[S + B|e = (0, 1 - q)] - E[S + B|e = (0, 0)]$$

$$= (2K + \frac{p - pq}{p - q}\phi) - 2K$$

$$= \frac{p - pq}{p - q}\phi.$$

Therefore the principal prefers inducing $e = (0, 1 - q)$ to inducing $e = (0, 0)$, when the incremental demand is larger then the incremental payment, which requires,

$$\left(1 - q\right)(p - q)\left(\Omega - 2L\right) > \frac{p - pq}{p - q}\phi,$$

$$\Omega > 2L + \frac{p - pq}{p - q}\phi = \omega_3,$$ and $\Omega < H + L$.

We can see that $\omega_3 < H + L$ when $H - L > \frac{p}{(p - q)^2}\phi$.

* For $\Omega > H + L$, it generates a higher profit for the principal by inducing $e = (0, 1 - q)$ compared with inducing $e = (0, 0)$, if

$$(1 - q)(p - q)(H - L) > \frac{p - pq}{p - q}\phi,$$

$$H - L > \frac{p}{(p - q)^2}\phi$$ and $\Omega > H + L$.

* Overall, it generates a higher profit for the principal by inducing $e = (0, 1 - q)$ compared with inducing $e = (0, 0)$ when $\Omega > \omega_3$, with $\omega_3 < H + L$.

As a summary, given $U - K \leq 0$, the optimal contract for the principal with limited inventory is as follows:

$$(e_1, E[e_2])^* = \begin{cases} 
(1, 1), & \text{if } \Omega \in [\omega_1, 2H), \\
(1, 1 - p), & \text{if } \Omega \in [\max\{\omega_2, \omega_2', \omega_1\}], \\
(0, 1 - q), & \text{if } \Omega \in [\omega_3, \max\{\omega_2, \omega_2'\}], \\
(0, 0), & \text{if } \Omega \in [2L, \omega_3].
\end{cases}$$

Combined, we have the optimal period-by-period contract for the principal with limited inventory as follows:
OA3 Optimal Two-period Contract with Limited Inventory

- Below, we first rule out the optimality for the principal to induce $e = (1, p)$ given $H - L > \frac{p}{(p-q)\phi}$.

  - When $\Omega < H + L$, we can observe that inducing $e = (0, q)$ is dominated by inducing $e = (0, 0)$ for the principal. This is because, given $\Omega < H + L$, inducing $e = (0, q)$ generates the same demand in expectation as inducing $e = (0, 0)$, while leading to a higher payment in expectation.

  - When $H + L < \Omega < H + L + \frac{\phi}{p-q}$, inducing $e = (0, q)$ is still dominated by inducing $e = (0, 0)$ for the principal. In this scenario, the difference between expected demands between inducing $e = (0, q)$ and inducing $e = (0, 0)$ is given by,

    \[ E_{H + L < \Omega \leq H + L + \frac{\phi}{p-q}}[D|e = (0, q)] - E_{H + L < \Omega \leq H + L + \frac{\phi}{p-q}}[D|e = (0, 0)] \]
    \[ = q(p - q)(\Omega - (H + L)), \]
    \[ \leq q(p - q)\left(\frac{\phi}{p-q}\right) = q\phi. \]

  Meanwhile the difference in their expected payments is,

    \[ E[S + B|e = (0, q)] - E[S + B|e = (0, 0)] = \begin{cases} \frac{pq\phi}{p-q}, & \text{if } U - K \leq 0, \\ \frac{pq\phi}{p-q} - 2(U - K), & \text{if } 0 < U - K \leq \frac{q^2}{2(p-q)}\phi, \\ q\phi, & \text{if } U - K > \frac{q^2}{2(p-q)}\phi. \end{cases} \]

  which is at least $q\phi$ (the minimal value is obtained when $U - K > \frac{q^2}{2(p-q)}\phi$).

  As we can see, compared with inducing $e = (0, 0)$, when the principal induces $e = (0, q)$, the minimal incremental payment is exactly equal to the maximal incremental demand. This
implies that the increase in expected payment cannot be justified by the increase in expected demand, therefore \( e = (0, q) \) is dominated by inducing \( e = (0, 0) \), given \( H + L < \Omega < H + L + \frac{\phi}{p-q} \).

- Finally, when \( \Omega > H + L + \frac{\phi}{p-q} \), inducing \( e = (0, q) \) is still dominated by inducing \( e = (1, p) \) for the principal. With \( \Omega > H + L + \frac{\phi}{p-q} \), the difference in expected demands between inducing \( e = (1, p) \) and inducing \( e = (0, q) \) is given by,

\[
E_{\Omega > H + L + \frac{\phi}{p-q}}[D|e = (1, p)] - E_{\Omega > H + L + \frac{\phi}{p-q}}[D|e = (0, q)]
= (p - q) \left( p(\Omega - (H + L)) + (1 - q)(H - L) \right),
> (p - q) \left( p \frac{\phi}{p-q} + (1 - q)(H - L) \right)
\]

Meanwhile the difference in their expected payments is no larger than \( \frac{p^2 + p - 2pq}{p-q} \phi \). Below, we show that compared with inducing \( e = (0, q) \), when the principal induces \( e = (1, p) \), even the maximal incremental payment is smaller than the minimal incremental demand in this scenario, thus the increase in demand will more than compensate the increase in payment. The condition for this to hold is,

\[
(p - q) \left( p \frac{\phi}{p-q} + (1 - q)(H - L) \right) > \frac{p^2 + p - 2pq}{p-q} \phi,
\]

i.e., \( H - L > \frac{p}{(p-q)^2} \phi \), which is the parameter space we focus on. This implies that \( e = (0, q) \) is dominated by inducing \( e = (0, 0) \) when \( \Omega > H + L + \frac{\phi}{p-q} \).

- Put the three cases above together, we reach the conclusion that inducing \( e = (0, q) \) is suboptimal for the principal (in the interesting parameter space where \( H - L > \frac{p}{(p-q)^2} \phi \)).

We now solve the optimal two-period contract with limited inventory by considering \( \Omega \) lying in the following intervals: \( H + L + \frac{1}{p-q} \phi \leq \Omega < 2H \), \( 2L + \frac{1}{p-q} \phi \leq \Omega < H + L + \frac{1}{p-q} \phi \), and when \( \Omega < 2L + \frac{1}{p-q} \phi \). The expected payment for inducing each effort profile remains the same as that in the two-period contracting case with independent periods (see Table A1 for details), and the expected demand for inducing each effort profile remains the same as that in the period-by-period contracting case with dependent periods (see Table OA1 for details).

- **Case 1:** \( H + L + \frac{1}{p-q} \phi \leq \Omega < 2H \).

  - In this case, we first rule out the optimality of inducing \( e = (1, 1-p) \) for the principal. Indeed, inducing \( e = (1, 1-p) \) is dominated by inducing \( e = (1, 1) \) in this region. The difference in the expected payment between inducing \( e = (1, 1-p) \) and inducing \( e = (0, 1-q) \) is \( p \phi \) regardless
of the value of $U - K$, i.e., $\forall U - K$,

$$E[S + B|e = (1, 1)] - E[S + B|e = (1, 1 - p)]$$

$$= \max \left\{ \frac{p^2 + p - pq}{p - q}(\phi + 2K, (1 + p)\phi + 2U) \right\} - \max \left\{ \frac{p + q^2 - pq}{p - q}(\phi + 2K, (1 - q)\phi + 2U) \right\}$$

$$= p\phi.$$

The difference in their expected demands is given by,

$$E_{\Omega \geq H + L + \frac{\phi}{p - q}}[D|e = (1, 1)] - E_{\Omega \geq H + L + \frac{\phi}{p - q}}[D|e = (1, 1 - p)]$$

$$= p(p - q)(\Omega - (H + L)),$$

$$\geq p(p - q)\frac{\phi}{p - q} = p\phi.$$

That means, the incremental demand in inducing $e = (1, 1)$ relative to inducing $e = (1, 1 - p)$ is no smaller than the incremental payment, therefore $e = (1, 1 - p)$ is sub-optimal given $\Omega \geq H + L + \frac{1}{p - q}\phi$.

- Next, we rule out the optimality of inducing $e = (0, 1 - q)$ for the principal under this scenario, by showing that inducing $e = (0, 1 - q)$ is dominated by inducing $e = (1, p)$ in this region. The difference in the expected demands they generate, given by $(p^2 - q^2)(\Omega - (H + L))$, is no smaller than than the difference in their expected payments to the agent, given by $(p + q)\phi$, when $\Omega \geq H + L + \frac{1}{p - q}\phi$.

From the analysis above, the possible optimal contracts for the principal are among inducing $e = (1, 1), e = (1, p), e = (0, 0)$.

- We compare the principal’s profits under $e = (1, 1)$ and $e = (1, p)$. We summarize the difference in their expected payments as,

$$E[S + B|e = (1, 1)] - E[S + B|e = (1, p)]$$

$$= \begin{cases} 
\frac{p^2 - pq}{p - q}\phi, & \text{if } U - K \leq \frac{q}{2(p - q)}\phi, \\
\frac{p + q^2 - pq}{p - q}\phi - 2(U - K), & \text{if } \frac{q}{2(p - q)}\phi < U - K \leq \frac{q}{p - q}\phi, \\
(1 - p)\phi, & \text{if } U - K > \frac{q}{p - q}\phi.
\end{cases}$$

The incremental demand is $(1 - p)(p - q)(H - L)$ in this scenario. Given $\Omega \geq H + L + \frac{\phi}{p - q}$, we can show that the principal is indifferent between inducing $e = (1, 1)$ and inducing $e = (1, p)$ when $U - K$ is at $\mu_3$ define below. That is,

$$(1 - p)(p - q)(H - L) = \frac{p + q^2 - pq}{p - q}\phi - 2(U - K),$$

i.e.,

$$U - K = \frac{1}{2}\left[\frac{p + q^2 - pq}{p - q}\phi - (1 - p)(p - q)(H - L)\right] \equiv \mu_3 \in \left(\frac{q}{2(p - q)}\phi, \frac{q}{p - q}\phi\right).$$

For $U - K \geq \mu_3$, the incremental payment becomes smaller thus inducing $e = (1, 1)$ dominates.
inducing \( e = (1, p) \). For \( U - K < \mu_3 \), the incremental payment becomes larger thus inducing \( e = (1, p) \) dominates inducing \( e = (1, 1) \).

- We now compare the principal’s profits under \( e = (1, p) \) and \( e = (0, 0) \). The difference in the expected demands between inducing the above two effort profiles is given by,

\[
E_{\Omega \geq H+L + \frac{\phi}{p-q}} [D | e = (1, p)] - E_{\Omega \geq H+L + \frac{\phi}{p-q}} [D | e = (0, 0)]
\]

\[
= \left(p^2 - q^2\right) \Omega + \left((1-p)(p+q) - 2q(1-q)\right)(H+L) - \left((1-p)(1-q) - (1-q)^2\right)2L
\]

\[
= \left(p^2 - q^2\right) \left(\Omega - (H+L)\right) + \left(p-q\right)\left(1-q\right)\left(H-L\right).
\]

The difference in the expected payments to the agent between inducing the above two effort profiles is,

\[
E [S + B | e = (1, p)] - E [S + B | e = (0, 0)] = \begin{cases} 
\frac{p^2 + p - pq}{p-q} \phi, & \text{if } U - K \leq 0, \\
\frac{p^2 + p - pq}{p-q} \phi - 2(U - K), & \text{if } 0 < U - K \leq \frac{q}{2(p-q)} \phi, \\
(1+p)\phi, & \text{if } U - K > \frac{q}{2(p-q)} \phi.
\end{cases}
\]

Therefore the principal prefers inducing \( e = (0, 1 - q) \) to inducing \( e = (0, 0) \), when the incremental demand is larger than the incremental payment, which requires,

1. if \( U - K > 0 \),

\[
\left(p^2 - q^2\right) \left(\Omega - (H+L)\right) + \left(p-q\right)\left(1-q\right)\left(H-L\right) > \frac{p^2 + p - pq}{p-q} \phi,
\]

i.e., \( \Omega > H + L - \frac{1-q}{p+q} (H-L) + \frac{p^2 + p - pq}{(p^2-q^2)(p-q)} \phi \equiv \omega_5; \)

2. if \( 0 < U - K < \frac{q}{2(p-q)} \phi \):

\[
\left(p^2 - q^2\right) \left(\Omega - (H+L)\right) + \left(p-q\right)\left(1-q\right)\left(H-L\right) > \frac{p^2 + p - pq}{p-q} \phi - 2(U - K),
\]

\[
\Omega > \omega_5 - \frac{2(U-K)}{p^2-q^2}.
\]

We can show that \( \omega_5 - \frac{2(U-K)}{p^2-q^2} > H + L + \frac{\phi}{p-q} \) only when \( U - K < \mu_2 \), with \( \mu_2 \in \left(0, \frac{q}{2(p-q)} \phi\right) \).

If \( U - K > \frac{q}{2(p-q)} \phi \), inducing \( e = (1, p) \) also dominates inducing \( e = (0, 0) \) since the difference in their expected payments will become smaller as \( U - K \) increases beyond \( \frac{q}{2(p-q)} \phi \).

To summarize, inducing \( e = (1, p) \) dominates inducing \( e = (0, 0) \) when \( \mu_2 < U - K < \mu_3 \), or when \( 0 < U - K < \mu_2 \) and \( \Omega > \omega_5 - \frac{2(U-K)}{p^2-q^2} \), or when \( U - K < 0 \) and \( \Omega > \omega_5 \).

- Therefore, given \( H + L + \frac{1}{p-q} \phi \leq \Omega < 2H \), the solution for the optimal contract for the principal is,
Case 2: $2L + \frac{1}{p-q}\phi \leq \Omega < H + L + \frac{1}{p-q}\phi$.

- First, we rule out the optimality of inducing $e = (1, 1)$ under this scenario. Compared with inducing $e = (1, 1-p)$, the incremental payment for inducing $e = (1, 1)$ is still $\phi$. However, the difference in their expected demands is given by,

$$E_{2L+\frac{1}{p-q}\phi}\Omega < H+L+\frac{1}{p-q}\phi [D|e = (1, 1)] - E_{2L+\frac{1}{p-q}\phi}\Omega < H+L+\frac{1}{p-q}\phi [D|e = (1, 1-p)] \leq p(p-q)(\Omega - (H + L)),$$

$$\leq p(p-q)\frac{\phi}{p-q} = p\phi.$$

That means, the incremental demand in inducing $e = (1, 1)$ relative to inducing $e = (1, 1-p)$ is no larger than the incremental payment, therefore $e = (1, 1)$ is sub-optimal.

- Next, we rule out the optimality of inducing $e = (1, p)$ for the principal by showing that inducing $e = (1, p)$ is dominated by inducing $e = (0, 1-q)$ in this region. The maximal difference in the expected demands they generate, given by $(p^2 - q^2)\frac{1}{p-q}\phi = (p+q)\phi$, is the same as the difference in their expected payments to the agent, given by $(p+q)\phi$. Therefore, inducing $e = (1, p)$ is dominated by inducing $e = (0, 1-q)$ for the principal.

Now we compare the principal’s profits under inducing $e = (1, 1-p)$, $e = (0, 1-q)$, $e = (0, 0)$.

- First consider the pair of $e = (1, 1-p)$ and $e = (0, 1-q)$.

$$E[S + B|e = (1, 1-p)] - E[S + B|e = (0, 1-q)] = \begin{cases} \frac{p-(p-q)}{p-q}\phi, & \text{if } U - K \leq \frac{q}{2(p-q)}\phi, \\ \frac{p+(q-p)^2}{p-q}\phi - 2(U - K), & \text{if } \frac{q}{2(p-q)}\phi < U - K \leq \frac{q}{p-q}\phi, \\ (1 - p + q)\phi, & \text{if } U - K > \frac{q}{p-q}\phi. \end{cases}$$

$$E[D|e = (1, 1-p)] - E[D|e = (0, 1-q)] = \begin{cases} (p - q)\left(q\left(\Omega - (H + L)\right) + (1 - p)(H - L)\right), & \text{if } \Omega > H + L, \\ (1 - p)\left(p - q\right)(\Omega - 2L), & \text{if } \Omega < H + L. \end{cases}$$

We find the boundary conditions where the principal is indifferent between inducing $e = (0, 1-q)$ and inducing $e = (1, 1-p)$. The result is that,
 Consider the case when \( U - K > \frac{q}{p-q}\phi \). At \( \Omega = 2L + \frac{1-p+q}{(1-p)(p-q)}\phi = \omega_4 \), inducing \( e = (0, 1-q) \) and inducing \( e = (1, 1-p) \) generate the same profits for the principal. Here, \( \omega_4 \) is the same as we defined in Appendix OA2, and it is smaller than \( H + L \). As such, the difference in their expected demands \((1-p)(p-q)(\Omega - 2L) = (1-p+q)\phi \) is equal to the difference in their expected payments which is also given by \((1-p+q)\phi \). Following this, given \( U - K > \frac{q}{p-q}\phi \), for \( \Omega > \omega_4 \), inducing \( e = (1, 1-p) \) dominates inducing \( e = (0, 1-q) \) and for \( \Omega < \omega_4 \), it is the opposite.

* Then consider the case when \( U - K \leq \mu_3 \). Indeed, at \( U - K = \mu_3 \) and \( \omega = H + L + \frac{\phi}{p-q} \), the principal is indifferent between inducing \( e = (0, 1-q) \) and inducing \( e = (1, 1-p) \). This is because at this point the principal is indifferent between inducing \( e = (1,1) \) and \( e = (1,1-p) \), and he is also indifferent between inducing \( e = (1, p) \) and \( (0, 1-q) \). Thus when \( U - K \) decreases from \( \mu_3 \) and the difference in expected payments becomes larger, inducing \( e = (0, 1-q) \) dominates inducing \( e = (1,1-p) \).

* When \( \mu_3 < U - K < \frac{q}{p-q}\phi \) and \( \Omega > H + L \), inducing \( e = (0, 1-q) \) and inducing \( e = (1, 1-p) \) generate the same profits for the principal if the following condition is met,

\[
\left(p - q\right)\left(q\left(\Omega - H + L\right) + \left(1-p\right)\left(H - L\right)\right) = \frac{p^3+q-p+q^2}{p-q}\phi - 2(U - K),
\]

i.e., \( \Omega = H + L - \frac{1-p}{q}\left(H - L\right) + \frac{p^3+q-p+q^2}{q(p-q)^2}\phi - \frac{2(U - K)}{q(p-q)} \).

Denote \( \omega_6 = H + L - \frac{1-p}{q}(H - L) + \frac{p^3+q-p+q^2}{q(p-q)}\phi \), then the boundary condition for this case is \( \Omega = \omega_6 - \frac{2(U - K)}{q(p-q)} \). Notice that \( \omega_6 - \frac{2(U - K)}{q(p-q)} \) intersects with \( \Omega = H + L + \frac{\phi}{p-q} \) at \( U - K = \mu_3 \), which aligns with our observation.

* For the case when \( \mu_3 < U - K < \frac{q}{p-q}\phi \) and \( \Omega < H + L \), inducing \( e = (0, 1-q) \) and inducing \( e = (1, 1-p) \) generate the same profits for the principal when,

\[
\left(1-p\right)\left(p - q\right)\left(\Omega - 2L\right) = \frac{p^3+q-p+q^2}{p-q}\phi - 2(U - K),
\]

i.e., \( \Omega = 2L + \frac{p^3+q-p+q^2}{(1-p)(p-q)^2}\phi - \frac{2(U - K)}{(1-p)(p-q)} \).

Denote \( \omega_6' = 2L + \frac{p^3+q-p+q^2}{(1-p)(p-q)^2}\phi \), then the boundary condition for this case is \( \Omega = \omega_6' - \frac{2(U - K)}{(1-p)(p-q)} \).

* Indeed, the above two cutoff values on \( \Omega \) intersect at \( \Omega = H + L \), therefore, when \( \mu_2 < U - K < \frac{q}{p-q}\phi \), the principal is indifferent between inducing the two effort profiles when \( \Omega = \max\{\omega_6 - \frac{2(U - K)}{(1-p)(p-q)}, \omega_6' - \frac{2(U - K)}{q(p-q)}\} \).

Combined together, under this scenario, we have inducing \( e = (1, 1-p) \) dominates inducing \( e = (0, 1-q) \) when \( \omega_4 \leq \Omega < H + L + \frac{1}{p-q}\phi, U - K > \frac{q}{p-q}\phi \), or when \( \max\{\omega_6 - \frac{2(U - K)}{(1-p)(p-q)}, \omega_6' - \frac{2(U - K)}{q(p-q)}\} \leq \Omega < H + L + \frac{1}{p-q}\phi, \mu_3 < U - K < \frac{q}{p-q}\phi \).

Now consider the pair of \( e = (0, 1-q) \) and \( e = (0, 0) \). The difference in the expected payments
to the agent between inducing the above two effort profiles is

\[
E[S + B|e = (0, 1 - q)] - E[S + B|e = (0, 0)] = \begin{cases} 
\frac{p+q^2-pq}{p-q}\phi, & \text{if } U - K \leq 0, \\
\frac{p+q^2-pq}{p-q}\phi - 2(U - K), & \text{if } 0 < U - K \leq \frac{q}{2(p-q)}\phi, \\
(1 - q)\phi, & \text{if } U - K > \frac{q}{2(p-q)}\phi.
\end{cases}
\]

The difference in the expected demands between inducing the above two effort profiles is given by,

\[
E[D|e = (0, 1 - q)] - E[D|e = (0, 0)] = \begin{cases} 
(1 - q)(p - q)(H - L), & \text{if } \Omega > H + L, \\
(1 - q)(p - q)(\Omega - 2L), & \text{if } \Omega < H + L.
\end{cases}
\]

We find boundary condition where the principal is indifferent between inducing \( e = (0, 1 - q) \) and inducing \( e = (0, 0) \). The result is as follows.

* Consider the case when \( U - K > \frac{q}{2(p-q)}\phi \). In this case, at \( \Omega = 2L + \frac{1}{(p-q)}\phi \) (which is smaller than \( H + L \)), inducing \( e = (0, 1 - q) \) and inducing \( e = (0, 0) \) generate the same profits for the principal. As such, the difference in their expected demands \((1 - q)(p - q)(\Omega - 2L) = (1 - q)\phi\) is equal to the difference in their expected payments which is also given by \((1 - q)\phi\). Following this, given \( U - K > \frac{q}{2(p-q)}\phi \), inducing \( e = (0, 1 - q) \) dominates inducing \( e = (0, 0) \) for \( \Omega > 2L + \frac{\phi}{p-q} \), and inducing \( e = (0, 1 - q) \) is dominated by inducing \( e = (0, 0) \) for \( \Omega < 2L + \frac{\phi}{p-q} \).

* When \( U - K < \mu_2 \), inducing \( e = (0, 0) \) dominates inducing \( e = (0, 1 - q) \). This is because at \( \Omega = H + L + \frac{\phi}{p-q} \) and \( U - K = \mu_2 \), the principal is indifferent between inducing \( e = (1, p) \) and inducing \( e = (0, 1 - q) \). According to Case 1, at this point, he is indifferent between inducing \( e = (1, p) \) and inducing \( e = (0, 0) \), therefore, he is also indifferent between inducing \( e = (0, 1 - q) \) and inducing \( e = (0, 0) \). To the southwestern corner of this point, as the inventory level becomes smaller the difference in their expected demands becomes smaller. In addition, as \( U - K \) becomes smaller, the difference in their expected demands becomes larger. Accordingly, inducing \( e = (0, 1 - q) \) will perform worse relative to inducing \( e = (0, 0) \). As a result, we have inducing \( e = (0, 1 - q) \) is dominated by inducing \( e = (0, 0) \) when \( U - K < \mu_2 \) and when \( \Omega < H + L + \frac{\phi}{p-q} \).

* When \( \mu_2 < U - K < \frac{q}{2(p-q)}\phi \), for the case when \( \Omega < H + L \), inducing \( e = (0, 1 - q) \) and inducing \( e = (0, 0) \) generate the same profits for the principal when,

\[
(1 - q)(p - q)(\Omega - 2L) = \frac{p+q^2-pq}{p-q}\phi - 2(U - K),
\]

i.e., \( \Omega = 2L + \frac{p+q^2-pq}{(1-q)(p-q)}\phi - \frac{2(U-K)}{(1-q)(p-q)} \).

Denote \( \omega_7 = 2L + \frac{p+q^2-pq}{(1-q)(p-q)}\phi \), then the boundary condition for this case is \( \Omega = \omega_7 - \)
Indeed, we can show that the above cutoff value on $\Omega$ is equal to $H + L$ at $U - K = \mu_2$, which makes sense since at $\Omega = H + L$ and $\mu_2$, the principal is indifferent between inducing the two effort profiles. Therefore, when $\mu_2 < U - K < \frac{q}{p - q} \phi$, the principal is indifferent between inducing the two effort profiles when $\Omega = \max\{\omega_5 - \frac{2(U - K)}{(1 - q)(p - q)} \omega_5, \omega_5' - \frac{2(U - K)}{q(p - q)}\}$. 

* To summarize, inducing $(0, 1 - q)$ dominates inducing $e = (0, 0)$ for the principal when 

$$ 2L + \frac{1}{p - q} \phi \leq \Omega < \max\{\omega_6 - \frac{2(U - K)}{(1 - p)(p - q)} \omega_6', \omega_6' - \frac{2(U - K)}{q(p - q)}\}, H + L + \frac{1}{p - q} \phi, U - K > \frac{q}{2(p - q)} \phi \text{ or} $$

when $\omega_7 - \frac{2(U - K)}{(1 - q)(p - q)} \leq \Omega < H + L + \frac{1}{p - q} \phi, \mu_2 < U - K < \frac{q}{2(p - q)} \phi$.

**Case 3:** $\Omega < 2L + \frac{1}{p - q} \phi$. Under this scenario, it is easy to see that the principal’s optimal strategy is not to induce effort, i.e., $e = (0, 0)$.

In summary, the overall solution to the optimal two-period contract under limited inventory for the principal is as follows:

$$(e_1, E[e_2])^* = \begin{cases} 
(1, 1), & \text{if } H + L + \frac{1}{p - q} \phi \leq \Omega < 2H,U - K > \mu_3, \\
(1, p), & \text{if } H + L + \frac{1}{p - q} \phi \leq \Omega < 2H, \mu_2 \leq U - K < \mu_3, \text{ or} \\
& \text{if } \omega_5 - \frac{2(U - K)}{(1 - q)(p - q)} \leq \Omega < 2H, 0 \leq U - K < \mu_2, \text{ or} \\
& \text{if } \omega_5 - \frac{2(U - K)}{p - q} \leq \Omega < 2H, \leq U - K < 0, \\
(1, 1 - p), & \text{if } \omega_4 \leq \Omega < H + L + \frac{1}{p - q} \phi, U - K > \frac{q}{p - q} \phi, \text{ or} \\
& \text{if } \max\{\omega_6 - \frac{2(U - K)}{(1 - p)(p - q)} \omega_6', \omega_6' - \frac{2(U - K)}{q(p - q)}\} \leq \Omega < H + L + \frac{1}{p - q} \phi, \mu_3 < U - K < \frac{q}{p - q} \phi, \\
(0, 1 - q), & \text{if } 2L + \frac{1}{p - q} \phi \leq \Omega < \max\{\omega_6 - \frac{2(U - K)}{(1 - p)(p - q)} \omega_6', \omega_6' - \frac{2(U - K)}{q(p - q)}\}, H + L + \frac{1}{p - q} \phi, U - K > \frac{q}{2(p - q)} \phi, \\
& \text{if } \omega_7 - \frac{2(U - K)}{(1 - q)(p - q)} \leq \Omega < H + L + \frac{1}{p - q} \phi, \mu_2 < U - K < \frac{q}{2(p - q)} \phi \text{ or}, \\
(0, 0), & \text{if } 2L < \Omega < \omega_5 - \frac{2(U - K)}{p - q}\mu_2 < U - K < \frac{q}{2(p - q)} \phi \text{ or}, \\
& \text{if } 2L \leq \Omega < \omega_5 - \frac{2(U - K)}{p - q}\mu_2 < U - K < \mu_2, \text{ or}, \\
& \text{if } 2L \leq \Omega < \omega_5, U - K < 0.
\end{cases}$$

**OA4** Comparison between Two-Period Contract and Period-by-Period contract with Limited Inventory

We prove the results with the aid of Figure 7, which provides a detailed comparison between the period-by-period and the two-period contract with dependent periods. We will show that (1) the period-by-period contract performs the same as the two-period contract in Region I and Region II, (2) the period-by-period contract performs better than the two-period contract in Region III and Region IV and, and (3) the period-by-period contract performs worse than the two-period contract in Region V and Region VI.

**First,** we prove that the period-by-period contract performs the same as the two-period contract in Region I and Region II.
In Region I, the principal induces \( e = (0, 0) \) under both the period-by-period contract and the two-period contract. The principal gets the profits regardless which contract he adopts.

In Region II, the principal induces \( e = (1, 1) \) under both the period-by-period contract and the two-period contract. The principal gets the same profits regardless which contract he adopts.

- **We then prove that the period-by-period contract performs better than the two-period contract in Region III.** We start our proof by establishing the boundary condition where the principal is indifferent between the two-period contract and the period-by-period contract in equilibrium.

\[
l_1 : U - K \leq 0, \quad \Omega = \omega_8 \equiv H + L + \frac{p^2}{(p+q)(p-q)^2} \phi.
\]

* Under this case, the principal induces \( e = (0, 1 - q) \) under the two-period contract, but he induces \( e = (1, 1) \) under the period-by-period contract. With \( \Omega = H + L + \frac{p^2}{(p+q)(p-q)^2} \phi > H + L \), the difference in their expected demands are

\[
E_{\text{period-by-period}}[D|e = (0, 1 - q)] - E_{\text{two-period}}[D|e = (1, p)] = - \left( p^2 - q^2 \right) \left( \Omega - (H + L) \right),
\]

\[
= - \frac{p^2}{p - q} \phi.
\]

The difference in their expected payments is given by,

\[
E_{\text{period-by-period}}[S + B|e = (0, 1 - q)] - E_{\text{two-period}}[S + B|e = (1, p)]
= \left( 2K + \frac{p - pq}{p - q} \phi \right) - \left( 2K + \frac{p^2 + p - pq}{p - q} \phi \right)
= - \frac{p^2}{p - q} \phi,
\]

which is equal to the difference in expected payments above.

* As \( \Omega \) decreases from \( \omega_8 \), the two-period contract inducing \((1, p)\) still generates higher demand relative to the period-by-period contract inducing \((0, q)\), however, the difference between the two demand levels decreases as \( \Omega \) decreases from \( \omega_8 \). As a result, in equilibrium, the period-by-period contract performs better than the two-period contract below \( l_1 \).

\[
l_2 : 0 < U - K < \frac{q^2}{(1+q)(p-q)} \phi \quad \text{and} \quad \Omega = \omega_8 - \frac{1+q}{p^2-q^2} (U - K).
\]

* Under this case, the principal induces \( e = (0, 1 - q) \) under the two-period contract, but induces \( e = (1, 1) \) under the period-by-period contract. With \( \Omega = H + L + \frac{p^2}{(p+q)(p-q)^2} \phi > \)
The difference in their expected demands is

\[
E_{\text{period-by-period}}[D|e = (0, 1 - q)] - E_{\text{two-period}}[D|e = (1, p)] = -\left(p^2 - q^2\right)\left(\Omega - (H + L)\right),
\]

\[
= (1 + q)(U - K) - \frac{p^2}{p - q} \phi.
\]

The difference in their expected payments is given by,

\[
E_{\text{period-by-period}}[S + B|e = (0, 1 - q)] - E_{\text{two-period}}[S + B|e = (1, p)] = \left((1 - q)K + (1 + q)U + \frac{p - pq}{p - q} \phi\right) - \left(2K + \frac{p^2 + p - pq}{p - q} \phi\right),
\]

\[
= (1 + q)(U - K) - \frac{p^2}{p - q} \phi,
\]

which is equal to the difference in expected payments above.

* To the lower-left of \(l_2\), as \(\Omega\) decreases, the two-period contract inducing \((1, p)\) still generates higher demand relative to the period-by-period contract inducing \((0, 1 - q)\), however, the difference between the two demand levels decreases. In addition, as \(U - K\) decreases, the two-period contract inducing \((1, p)\) pays the agent more relative to the period-by-period contract inducing \((0, 1 - q)\), and the difference between the two payments increases. As a result, in equilibrium, the period-by-period contract performs better than the two-period contract to the lower-left of \(l_2\).

\[l_3 : U - K = \frac{q^2}{(1+q)(p-q)} \phi, \forall \Omega \in \left[2L + \frac{1}{(p-q)^2}\left(p - \frac{q^2}{1+q}\right) \phi, H + L + \frac{\phi}{p-q}\right].\]

* In this case, both contracts, if optimally chosen, will induce \(e = (0, 1 - q)\). Under the period-by-period contract, the expected payment to the agent is given by,

\[
E_{\text{period-by-period}}[S + B|e = (0, 1 - q)] - E_{\text{two-period}}[S + B|e = (0, 1 - q)] = \left((1 - q)K + (1 + q)U + \frac{p(1-q)}{p - q} \phi\right) - \left(2K + \frac{p + q^2 - pq}{p - q} \phi\right),
\]

\[
= (1 + q)(U - K) - \frac{q^2}{p - q} \phi.
\]

Thus the expected payments to the agent are the same under the period-by-period contract and the two-period contract when \(U - K = \frac{q^2}{(1+q)(p-q)} < \frac{q}{2}\). In addition, both contracts induce \(e = (0, 1 - q)\) at optimal, thus generate the same expected demands. Together, we have that the principal’s equilibrium profits stay the same under the two types of contracts.

* To the left of \(l_3\), as \(U - K\) decreases, the two-period contract inducing \((0, 1 - q)\) pays the agent more relative to the period-by-period contract inducing \((0, 1 - q)\), and the difference between the two salaries increases. As a result, in equilibrium, the period-by-period contract performs better than the two-period contract to the left of \(l_3\).
– $U - K \leq 0$, $\Omega \in [2L, \omega_3]$; $0 < U - K \leq \frac{q^2}{(1+q)(p-q)} \phi$, $\Omega \in [2L, \omega_3 - \frac{U-K}{1-q}]$.

* This is the boundary condition where the principal is indifferent between inducing $e = (0,0)$ and inducing $e = (0, 1-q)$ under the period-by-period contract. Also, at this area, the principal induces $e = (0,0)$ under the two-period contract. Therefore, the two types of contract perform the same for the principal.

* Above of this line, as $\Omega$ increases, the principal gets a higher profit by inducing $e = (0, 1-q)$ under the period-by-period contract compared with inducing $e = (0,0)$, but he gets the same profit under the two-period contract by inducing $e = (0,0)$. As a result, the period-by-period contract performs better than the two-period contract in equilibrium.

• Similarly, we characterize the boundary condition of Region IV where the principal also prefers the period-by-period contract to the two-period contract.

– $l_4 : U - K = \frac{q}{p-q} \phi$, $\Omega \in [\omega_4, H + L + \frac{\phi}{p-q}]$, the principal is indifferent from the optimally chosen period-by-period contract and the optimally chosen two-period contract.

* Under both types of contracts, the principal induces $e = (1,1-p)$ and pays $2U + (2-p)\phi$.

* To the left of $l_2$, as $U - K$ decreases, the two-period contract still induces $e = (1,1-p)$, however the period-by-period contract pays the agent less relative to the two-period contract, therefore, is preferred by the principal.

– $l_5 : U - K \in [\omega_4, H + L]$ and $\Omega = \omega_9 - \frac{2-p}{(1-p)(p-q)}(U - K)$, where $\omega_9 \equiv 2L + \frac{p+q+pq-p^2-q^2}{(1-p)(p-q)}$.

* Under this case, the principal again induces $e = (0,1-q)$ under the two-period contract, but he induces $e = (1,1-p)$ under the period-by-period contract. With $\Omega = 2L + \frac{p+q+pq-p^2-q^2}{(1-p)(p-q)} - \frac{2-p}{(1-p)(p-q)}(U - K) < H + L$, the difference in their expected demands is

\[
E_{\text{period-by-period}}[D|e = (1,1-p)] - E_{\text{two-period}}[D|e = (0,1-q)]
\]

\[= (1-p)(p-q)(\Omega - 2L), \]

\[= - (2-p)(U - K) + \frac{p + q + pq - p^2 - q^2}{p-q}\phi. \]

The difference in their expected payments is again given by,

\[
E_{\text{period-by-period}}[S + B|e = (1,1-p)] - E_{\text{two-period}}[S + B|e = (0,1-q)]
\]

\[= (2-p)K + pU + \frac{2p-p^2}{p-q}\phi) - (2U + (1-q)\phi), \]

\[= - (2-p)(U - K) + \frac{p + q + pq - p^2 - q^2}{p-q}\phi, \]

which is equal to the difference in expected payments above.

* To the upper-right of $l_5$, as $\Omega$ increases, the two-period contract inducing $(0,1-q)$ still generates less demand relative to the period-by-period contract inducing $(1,1-p)$, and the difference between the two demand levels increases. In addition, as $U - K$ increases, the
two-period contract inducing \((0, 1 - q)\) pays the agent less relative to the period-by-period contract inducing \((1, 1 - p)\), but the difference between the two payments decreases. As a result, in equilibrium, the period-by-period contract performs better than the two-period contract to the upper-right of \(l_5\).

\[- l_6 : \Omega \in \left[ H + L, H + L + \frac{\phi}{p - q} \right] \text{ and } \Omega = \omega'_0 - \frac{(2-p)(U-K)}{q(p-q)} \text{, where } \omega'_0 \equiv H + L - \frac{1-p}{q}(H - L) + \frac{p+q+pq-p^2-q^2}{q(p-q)^2} \phi.\]

* Under this case, the principal induces \(e = (0, 1 - q)\) under the two-period contract, but induces \(e = (1, 1 - p)\) under the period-by-period contract. With \(\Omega = H + L - \frac{1-p}{q}(H - L) + \frac{p+q+pq-p^2-q^2}{q(p-q)^2} \phi - \frac{(2-p)(U-K)}{q(p-q)}\), the difference in their expected demands is

\[
E_{\text{period-by-period}}[D|e = (1, 1 - p)] - E_{\text{two-period}}[D|e = (0, 1 - q)]
= q(p - q)\left(\Omega - (H + L)\right) + (1 - p)(p - q)(H - L),
= -(2 - p)(U - K) + \frac{p + q + pq - p^2 - q^2}{p - q} \phi.
\]

The difference in their expected payments is given by,

\[
E_{\text{period-by-period}}[S + B|e = (1, 1 - p)] - E_{\text{two-period}}[S + B|e = (0, 1 - q)]
= (2 - p)K + pU + \frac{2p - p^2}{p - q} \phi - \left(2U + (1 - q)\phi\right),
= -(2 - p)(U - K) + \frac{p + q + pq - p^2 - q^2}{p - q} \phi,
\]

which is equal to the difference in expected payments above.

* To the upper-right of \(l_6\), the period-by-period contract performs better than the two-period contract due to the same reason as the case above.

\[- l_7 : \Omega > H + L + \frac{\phi}{p - q}, \mu < \mu_3, \Omega = \omega_{10} + \frac{2-p}{p(p-q)}(U - K), \text{ where } \omega_{10} = H + L + \frac{1-p}{p}(H - L) - \frac{p+q+pq-2p^2}{p(p-q)^2} \phi.\]

* Under this case, the principal again induces \(e = (1, p)\) under the two-period contract, and induces \(e = (1, 1 - p)\) under the period-by-period contract. With \(\Omega = H + L + \frac{1-p}{p}(H - L) - \frac{p+q+pq-2p^2}{p(p-q)^2} \phi + \frac{2-p}{p(p-q)}(U - K)\), the difference in their expected demands is

\[
E_{\text{period-by-period}}[D|e = (1, 1 - p)] - E_{\text{two-period}}[D|e = (1, p)]
= p\left(q - p\right)\left(\Omega - (H + L)\right) - (1 - p)(q - p)(H - L),
= -(2 - p)(U - K) + \frac{p + q + pq - 2p^2}{p - q} \phi.
\]
The difference in their expected payments is again given by,

$$E_{\text{period-by-period}}[S + B|e = (1, 1 - p)] - E_{\text{two-period}}[S + B|e = (1, p)]$$

$$= \left( (2 - p)K + pU + \frac{2p - p^2}{p - q} \phi \right) - \left( 2U + (1 + p)\phi \right),$$

$$= - (2 - p)(U - K) + \frac{p + q + pq - 2p^2}{p - q} \phi,$$

which is equal to the difference in expected payments above.

* To the lower-right of $l_7$, relative to the two-period contract inducing $e = (1, p)$, the period-by-period contract inducing $(1, 1 - p)$ induces more demand as $\Omega$ decreases. In addition, it pays lower as $U - K$ increases. As a result, in equilibrium, the period-by-period contract performs better than two-period contract to the lower-right of $l_7$.

$- l_8 : \Omega > H + L + \frac{\phi}{p - q}, \mu_3 < U - K < \frac{q}{p - q} \phi, \Omega = \omega_1 - \frac{1}{p - q}(U - K)$.

* Under this case, the principal again induces $e = (1, 1)$ under the two-period contract, and induces $e = (1, 1 - p)$ under the period-by-period contract. With $\Omega = H + L + \frac{p}{(p - q)^2} \phi - \frac{1}{p - q}(U - K)$, the difference in their expected demands is

$$E_{\text{period-by-period}}[D|e = (1, 1 - p)] - E_{\text{two-period}}[D|e = (1, 1)]$$

$$= - p \left( \frac{p - q}{p - q} \left( \Omega - (H + L) \right) \right),$$

$$= p(U - K) - \frac{p^2}{p - q} \phi.$$

The difference in their expected payments is again given by,

$$E_{\text{period-by-period}}[S + B|e = (1, 1 - p)] - E_{\text{two-period}}[S + B|e = (1, 1)]$$

$$= \left( (2 - p)K + pU + \frac{2p - p^2}{p - q} \phi \right) - \left( 2K + 2\frac{p}{p - q} \phi \right),$$

$$= p(U - K) - \frac{p^2}{p - q} \phi,$$

which is equal to the difference in expected payments above.

* To the lower-left of $l_8$, as $\Omega$ decreases, the two-period contract inducing $(1, 1)$ still generates higher demand relative to the period-by-period contract inducing $(1, 1 - p)$, but the difference between the two demand levels decreases. In addition, as $U - K$ decreases, the two-period contract pays the agent less relative to the period-by-period contract, and the difference between the two payments decreases. As a result, in equilibrium, the period-by-period contract performs better than two-period contract to the lower-left of $l_8$.

- Now we prove that the period-by-period contract performs worse than the two-period contract in Region V and Region VI.
In Region B, the principal induces \( e = (0, 1 - q) \) under the two-period contract, and induces either \( e = (0, 1 - q) \) or \( e = (0, 0) \). In the parameter space \( e = (0, 1 - q) \) is induced under the period-by-period contract, the two-period contract performs better the period-by-period contract. This is because Region V moves away from \( l_3 \) in the opposite direction as Region III does, thus we also get the opposite conclusion that the two-period contract performs better than the period-by-period contract to the right of \( l_3 \). Also, Region V and Region IV move away from \( l_5 \) and \( l_6 \) in the opposite directions, thus we get that the two-period contract performs better than the period-by-period contract to the lower-left of \( l_5 \) and \( l_6 \).

In the parameter space where \( e = (0, 0) \) is induced under the period-by-period contract, the two-period contract performs better since the principal’s profit will be higher when inducing \( e = (0, 1 - q) \) under the two-period contract.

– In region VI, the principal induces \( e = (1, p) \) under the two-period contract, and induces either \( e = (0, 1 - q) \) or \( e = (1, 1) \) under the period-by-period contract. Region VI and Region III move away from \( l_1 \) and \( l_2 \) in the opposite directions, therefore, we will get the result that the two-period contract performs better than the period-by-period contract above \( l_1 \) and to the upper-right of \( l_6 \). Additionally, Region VI and Region IV move away from \( l_7 \) in the opposite directions, therefore, we get that the two-period contract performs better than the period-by-period contract to the upper-left \( l_7 \).

To summarize, the optimal contract for the principal with limited inventory is presented in Proposition 7.