The Logic and Management of “Digital Co-op” in Search Advertising

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Abstract

Manufacturers and the retailers through which they sell their products often advertise against the same keywords in search advertising. This implies that these retailers increase advertising costs for manufacturers while also appropriating a part of their channel margins. In spite of this, it is observed that manufacturers directly or indirectly subsidize retailers’ advertising expenditures; this practice is known as “digital co-op” and essentially induces the retailers to advertise more against the manufacturers themselves. In this paper, we use a game theory model with two competing manufacturers and a common retailer to study the incentives of manufacturers to participate in digital co-op in search advertising, and provide guidance on how to manage it. A main insight that we obtain is that, because competing manufacturers are non-independent bidders, if they both have a strong incentive to win an ad slot, subsidizing the retailer to win the ad slot instead is a viable strategy to moderate their competition. We determine the subsidy rates and bids that manufacturers should use for different types of keywords reflecting different brand preferences of consumers, and the competitive responses and outcomes that they can expect. For instance, a manufacturer should subsidize the retailer’s ad spend for category keywords at moderate levels and for the competitor’s brand keywords at high levels, enabling the retailer to win in both cases. We also consider the case when the manufacturers cannot specify keyword-level subsidy rates and find that they may subsidize the retailer only if the relative search volume of category keywords is sufficiently high.

Keywords: search advertising; cooperative advertising; game theory; category keyword; brand keyword.

1 Introduction

As worldwide Internet use grows constantly, advertisers seeking to promote their products and services are turning increasingly toward search advertising. Compared with a 6% growth of the entire advertising industry, search advertising had a growth rate of over 12% and is expected to reach $118 billion globally in 2018. Top spenders on search advertising include both manufacturers and retailers. In fact, when a product related keyword is searched, often both manufacturers of a relevant product and retailers selling the manufacturers’ product advertise in response. This is interesting because the retailers, while taking a fraction of the channel margin of the manufacturers, increase the competition for ads for the manufacturers and also capture some of the demand from the searching consumers.

While the above practice of retailers’ advertising against manufacturers for the same products is arguably hurtful for the manufacturers in various ways, manufacturers in turn follow an intriguing practice — they subsidize retailers’ ad expenses which essentially induces the retailers to advertise more intensely against the manufacturers themselves. This practice of sharing the retailers’ ad spend is widely adopted and is commonly known as “digital co-op.”

Cooperative advertising is not new, and it prevails in traditional media—from the famous “Intel

Inside” campaign, to Bloomingdale’s flyers featuring many brands, to Verizon’s TV commercials on iPhone, to product displays in Walmart stores. The estimated annual spend of cooperative advertising in the United States range around $50 billion in 2011.

However, cooperative advertising in digital media is indeed quite new, though growing fast. Industry reporters observe that 69% of brands already offer digital marketing as a component of their co-op programs. Furthermore, there are calls for this practice to grow even faster: “In 2015, 39 percent of the major marketers wished their retail partners would spend more of the co-op ad dollars on mobile search, but only 22 percent local merchants said they use co-op money for mobile search.” However, this growth is impeded because of lack of clarity about how to manage these programs: “Many brands are not only struggling to keep up in this digital era, but also leaving more advertising dollars — about $14 billion — unused. Half of US brand marketers said that a lack of digital marketing understanding was a key obstacle to participating in a co-op program.” It is noteworthy that digital marketing platforms are providing tools to enable digital co-op. For instance, Google has recently developed a “New Manufacturer Center” to help manufacturers support their retailing partners to run Google shopping ads. Whenever users search for a product by a specific brand, or by the category attribute, Google helps to make a match between a keyword and a manufacturer or its retailers’ sites. According to one author’s first-hand experience, big online retailers, such as walmart.com, receive search advertising dollars from many manufacturers. Manufacturers, on the other hand, can set up a cooperative advertising (co-op) fund and sponsor their retailers on advertising expense.

Intuitively speaking, cooperative search advertising may have both pros and cons for manufacturers. On one hand, selling through retailers allows manufacturers to compete for a wider range of customers than their own sites could have access to. This is especially true for new or unknown brands, who can leverage the brand names of big retailers, like Amazon and Walmart.com to improve credibility and boost sales. On the other hand, a manufacturer may not want to sponsor its retailers on search advertising for several reasons. First, a manufacturer may have a higher margin via direct sales. For example, booking.com charges their partner hotels a basic commission rate of 15%; Expedia charges 10% base commission on hotels and substantial commissions on packages, car rentals, and activities. Second,
the retailer’s participation in the search advertising auctions will increase the competition and thus may cost the manufacturer more money to get displayed. Lastly, a consumer may choose other brands at the retailers’ sites, and consequently, by supporting the retailers to win, a manufacturer may divert demand to its competitors.

In this paper, we take these tradeoffs into consideration, and provide a competitive account of cooperative search advertising. Using a game theory model, we aim to understand what exactly is the benefit to a manufacturer of participating in cooperative advertising with a retailer in a competitive market. After we develop this understanding, we provide insights that can guide manufacturers and retailers on how to navigate digital co-op in search advertising.

We build a stylized model with two competing manufacturers selling through a common retailer who charges a commission fee to the manufacturers. A proportion of consumers buy from the retailer’s site and the rest buy from the manufacturers’ sites. Consumers search using keywords that can lie on a continuum from category keywords (indicating that the consumer is indifferent between manufacturers, e.g., “shoes”) to brand keywords (indicating that the consumer prefers one manufacturer, e.g., “Nike” and “Adidas”). The search engine runs a single-slot second-price auction and all players can bid for their ads to be placed in response to the keyword search. The winner, who gets to place its ad on the search results page, diverts towards itself a fraction of the demand from all its competitors. For subsidization, we consider a simple mechanism using which each manufacturer can share a fixed percentage of the retailer’s search ad spending. Such a mechanism has been widely adopted in previous literature on cooperative advertising in vertical settings (Dutta et al. 1995; Bergen and John 1997; Nagler 2006; Cao and Ke 2016) and is a reasonable approximation of industry practice.

Our analysis shows that the basic factor at play in this scenario is that when a manufacturer loses the auction for an ad slot, who it loses to matters. Specifically, losing to the other manufacturer is worse than losing to the retailer. Furthermore, competing intensely with the other manufacturer for the ad slot can be worse than losing to the retailer. Therefore, helping the retailer win can be advantageous, and this is the primary motivation for subsidizing its ad spend to make it more competitive in the auction. More specifically, we find that for category keywords both manufacturers sponsor the retailer to enable it to win the auction; in this case, the sponsorship improves not only the retailer’s profit, but also both manufacturers’ profits by softening the head-to-head competition between them. For brand keywords that are associated with a specific manufacturer, a competing manufacturer, whose direct bidding in
the auction is restricted by the auctioneer, may sponsor the retailer to steal demand from the focal manufacturer; in this case the sponsorship improves the sponsoring manufacturer’s profit at the cost of the focal manufacturer’s profit.

We also extend our analysis to a general framework in which consumers can search for a range of keywords lying between a category keyword and a brand keyword. Also, the search engine assigns different bidding weights for different bidders depending on the keyword being searched for. We show that our insights in the main analysis hold more generally. Specifically, for keywords closer to the category keyword, both manufacturers will sponsor the retailer to win to soften their head-to-head competition; for keywords closer to a brand keyword, the competing manufacturer who is less likely to be chosen will sponsor the retailer to win to steal demand via the retailer. The equilibrium outcome depends on the manufacturers’ relevance with the keyword of interest and the retailer’s commission fee.

Lastly, we consider the case where a manufacturer cannot specify different sponsorship rates for different keywords. This happens when it is difficult for the manufacturer to track the retailer’s bids at keyword level, or keyword-level contracts between the manufacturer and the retailer are too labor-intensive to administrate. Given that a manufacturer can only specify a uniform sponsorship rate that applies to all types of keywords, we find that it is still optimal for a manufacturer to sponsor the retailer for its search advertising spend, but the sponsorship is less likely to happen (in terms of a smaller parameter space).

This paper is related to three streams of literature. First, it contributes to the literature of cooperative advertising by incorporating brand competition into advertising decisions. Most previous literature focuses on how a single manufacturer uses cooperative advertising to coordinate vertical relationships with multiple downstream retailers [Berger 1972, Desai 1992, Bergen and John 1997, Kim and Staelin 1999]. In contrast, we consider two manufacturers’ “competitive” cooperative advertising strategies. The most related paper in this literature is Cao and Ke (2016), which investigates how a manufacturer cooperates with multiple retailers in search advertising, and finds that it is not always optimal for a manufacturer to cooperate with multiple retailers, and when deciding which one to cooperate with, the manufacturer should choose the one with the highest channel profit per click. Cao and Ke (2016) focuses on a monopolistic manufacturer’s optimal channel coordination strategy via cooperative advertising. In contrast, our paper considers multiple manufacturers and a common retailer, and provides a competitive account of the retailer’s role in cooperative search advertising.
Second, our paper is related to the work on common agency. Bernheim and Whinston (1985) shows that two manufacturers can achieve perfect collusion via a common marketing agency by each independently delegating its marketing effort to the retailer and contracting with the retailer on retail prices. The two manufacturers achieve collusion by essentially selling out themselves to the common agency. In the case of category keyword auctions, we also find that under some (but not all) conditions, the two manufacturers soften the competition between them via a common retailer. However, different from Bernheim and Whinston (1985), in our setup, only the retailer’s advertising decision is coordinated between the manufacturers and the retailer. The manufacturers do not delegate their own advertising decision to the retailer; instead, their sponsorship to the retailer’s ads, to some extent, serves as a commitment device that restrains them from competing with each other too hard. Furthermore, we show that the reason to sponsor a retailer’s ad spend is quite different for branded keywords. For similar reasons, our paper also differs from some recent empirical works that investigate collusive bidding behaviors (Decarolis et al. 2017, Decarolis and Rovigatti 2017), where competing bidders delegate their bidding decisions to a common marketing agency. Choi (1991) considers two manufacturers selling products via a common retailer using linear wholesale contracts. He studies how power structures and demand function forms influence the equilibrium retail prices and profits. We employ the same channel structure but instead focus on the channel coordination on advertising spending in search ad auctions; also we focus on the role of the retailer instead of the implications of the channel structure.

Lastly, this paper contributes to the growing stream of research on search advertising. Auction mechanisms for sponsored search and their equilibrium properties have been investigated extensively (Edelman et al. 2007, Varian 2007, Chen and He 2011, Athey and Ellison 2011, Zhu and Wilbur 2011, Sayedi et al. 2018, etc.). Competitive strategies in search advertising have also been studied extensively: the interaction between firms’ advertising auction and price competition (Xu et al. 2011), the interplay between organic and sponsored links (Katona and Sarvary 2010, Berman and Katona 2013), the bidding strategies of vertically differentiated firms (Jerath et al. 2011), the competitive poaching strategy (Sayedi et al. 2014, Desai et al. 2014), the impact of advertisers’ budget constraints on their own profits and the platform’s revenue (Lu et al. 2015, Shin 2015), etc. However, most previous studies assume that bidders are independent (i.e., the value of winning and of the position an advertiser obtains does not depend on the outcomes for others). In our setting, a major difference is that a manufacturer’s profit is affected by not only if it wins, but also whether the competing manufacturer or the retailer wins if it loses, so the
bidders are not independent from each other. We investigate the involved and interdependent channel interactions between competing manufacturers and a common retailer in search ad auctions. We also make an explicit and meaningful distinction between category and brand keywords in our model.

In the rest of the paper, we proceed as follows. In Section 2, we present the model. Then, in Section 3, we analyze two special cases — in Section 3.1, we analyze the case of a category keyword that is equally relevant to competing manufacturers and, in Section 3.2, we analyze the case of a brand keyword that has higher relevance to one of the manufacturers and the other manufacturer is not allowed to poach on this keyword directly. Analyzing these two special, “polar” cases helps to develop the basic insights for why, when and how much a manufacturer, facing a competing manufacturer, will subsidize its retailer’s ads even though they compete with its own ads. Following this, in Section 4, we present the analysis for a generalized keyword. In Section 5, we consider the case in which the manufacturers cannot customize their sponsorship rates for different keywords. In Section 6, we discuss managerial implications and conclude.

2 Model

We consider three players — two manufacturers $M_1$ and $M_2$, and a common retailer $R$. This is the simplest setting to exposit our key ideas. Each manufacturer (“he”) makes a product under his own brand, and the retailer (“she”) carries both manufacturers’ products. The two manufacturers and the retailer bid in a search ad auction on a keyword searched by a consumer (“he”).

First, we discuss the demand model. In absence of search ads, the consumer buys directly from one of the manufacturers with probability $\theta \in [0, 1]$ and from the retailer with probability $1 - \theta$. We assume that the consumer may have a preference for one brand over the other (and this can be indicated in the search query). For example, he may be an existing customer who has previously purchased from the brand, or a new customer who has just learned about the brand and decides to search for it with a buying intention. Even if he chooses to visit the retailer’s site, he is more likely to choose the brand he has already got exposed to. We model a consumer’s relative preference for one of the two manufacturers by a “loyalty” parameter $l \in [-1/2, 1/2]$. $l$ reflects the consumer’s preference for one brand over the other. We denote by $l_i$ the probability of brand $i$ being chosen given a consumer chooses between the two brands, and define $l_1 = \frac{1}{2} + l$ and $l_2 = \frac{1}{2} - l$, which satisfy that $l_1 + l_2 = 1$ and $l_i \in [0, 1]$ ($i = 1, 2$). $l = 0$ corresponds
to no brand preference, \( l \in (0, 1/2] \) corresponds to a preference for \( M_1 \), and \( l \in [-1/2, 0) \) corresponds to a preference for \( M_2 \). The larger \(|l|\) is the stronger consumers’ brand preference is; at the extreme, \( l = 1/2 \) (\( l = -1/2 \)) corresponds to the case that the consumer who searches for the keyword considers only \( M_1 \) (\( M_2 \)). We assume that the probability that the consumer clicks on the retailer’s link, given by \( \theta \), is independent of \( l \); this can be understood to imply that the consumer’s preferences over brands are independent from his preferences over channels. Figure 1(a) illustrates the demand distribution of keyword searchers without advertising.

The existence of search advertising has two effects: the stealing effect and the cannibalization effect. In particular, the winner of the search ad auction reduces the the probability that the consumer buys from a competitor by \( \alpha \in [0, 1] \) and gains this additional demand; this is the stealing effect. Moreover, the winner also induces the consumer who would buy from it in the case without search ads to purchase via its search ad with probability \( g \in [0, 1] \); this is the cannibalization effect, named so because the winner has to pay for some of its own demand that was originally coming for free (Blake et al. 2015, Simonov et al. 2018). As shown in Figures 1(b) and 1(c) if \( M_i \) wins the auction, his market share increases from \( \theta l_i \) to \( \theta l_i + (1 - \theta l_i)\alpha \), which implies that he pays for \( N_{M_i} \) clicks, where

\[
N_{M_i} = g\theta l_i + (1 - \theta l_i)\alpha. \tag{1}
\]

As shown by Figure 1(d) if the retailer wins, her market share increases from \( 1 - \theta \) to \( 1 - \theta + \alpha \theta \) and she pays for \( N_R \) clicks, where

\[
N_R = g(1 - \theta) + \alpha \theta. \tag{2}
\]

Note that \( N_R \) is independent of \( l_i \) but \( N_{M_i} \) is not.

Now, we discuss the ad selling mechanism. The search engine runs a single-slot second-price auction. We assume, as practiced by all major search engines, that the search engine weighs the bids of every player to reflect relevance between the search query and the player. We normalize the bid weight that the search engine assigns to the retailer to 1. We denote the bid weight assigned to \( M_i \) by \( w_i \), which can depend on \( l_i \). Given the bid weights, the auction operates as the following way. Suppose the retailer bids \( b_R \), and manufacturer \( i \) bids \( b_{M_i} \). In deciding who is the winner, each bid will be multiplied by the bidder’s bidding weight, and the winner will pay the minimum amount required to stay in that
\[ M_1 : (1 - \theta)l_1 \]
\[ M_2 : (1 - \theta)l_2 \]

(a) Without ad

\[ M_1 : (1 - \theta)(1 - \alpha)l_1 \]
\[ M_2 : (1 - \theta)(1 - \alpha)l_2 \]

(b) \( M_1 \) wins and places ad

\[ M_1 : (1 - \theta)l_1 + (1 - \theta_1)\alpha \]
\[ M_2 : \theta(1 - \alpha)l_2 \]

(c) \( M_2 \) wins and places ad

\[ M_1 : (1 - \theta)(1 - \alpha)l_1 \]
\[ M_2 : (1 - \theta)(1 - \alpha)l_2 \]

(d) Retailer wins and places ad

Figure 1: Demand model
position. In particular, if $w_i b_{M_i} > \max\{w_j b_{M_j}, b_R\}$, $M_i$ wins and pays $\frac{1}{w_i} \max\{w_j b_{M_j}, b_R\}$ per click; if $b_R \geq \max\{w_1 b_{M_1}, w_2 b_{M_2}\}$, then the retailer wins and pays $\max\{w_1 b_{M_1}, w_2 b_{M_2}\}$ per click.

We allow the manufacturers to subsidize some of the retailer’s advertising expense. We consider a game with the following timeline. First, the two manufacturers simultaneously decide the levels of their subsidies to the retailer. We consider a simple sponsorship mechanism, where manufacturer $M_i$ ($i \in \{1, 2\}$) contributes $\beta_i$ percentage of the retailer’s spending on search advertising. $\beta_i \in [0, 1]$ is the so-called sponsorship rate. This mechanism has been widely used, in both industry practice (Dutta et al. 1995, Nagler 2006) and analytical modeling (e.g., Bergen and John 1997, Aust and Buscher 2014, Cao and Ke 2016), for ad spend sharing in vertical relationships. Under the mechanism, the retailer only needs to pay $1 - \beta_1 - \beta_2$ percentage of her ads spending if she wins the keyword auction.\footnote{This assumes that each manufacturer pays the retailer as per its sponsorship rate for each search ad dollar that the retailer spends, without consideration for which manufacturer’s ad was served (or product highlighted) by the retailer. We can make the alternative assumption that a manufacturer pays the retailer only for its own ads (or product highlighted). This alternative formulation is identical in all respects, except that equilibrium $\beta_i$s are modified by a constant factor of $1/l_i$, and therefore all our insights continue to hold.} Second, given the sponsorship rates, two manufacturers and the retailer submit their bids simultaneously. Finally, the search engine runs the auction and demand realizes. If there are several bidders that submitted the same highest bid, the search engine will randomly pick one winner from them.

We consider a revenue sharing model between the retailer and manufacturers, which is a common practice between a retailing platform and its merchants. Specifically, for each product sold, the retailer charges the manufacturer a percentage $\phi \in [0, 1]$ of the transaction revenue as commission. Consistent with many real-world applications, $\phi$ is assumed to be the same for both manufacturers. We normalize the retail price of the product to 1 and assume that a consumer will pay the same retail price whether she buys directly from a manufacturer or from the retailer. Our focus is on manufacturers subsidizing retailer’s ad expenses and therefore, this assumption is conservative in the sense that it implies that a manufacturer earns a higher profit margin via direct sales compared with selling via the retailer, so it should make subsidization less attractive. Our focus being on search advertising, assuming the commission rate and the retail prices to be exogenously given is also a reasonable assumption for our purposes as in many cases there exist numerous other factors that go into deciding these quantities, and search advertising strategy may not directly influence commission rates and retail prices. We note that under the assumptions above, our revenue sharing model is equivalent to the wholesale model with exogenous and symmetric wholesale contracts and universal retail prices. The manufacturers’ production
costs are normalized to zero.

3 Two Special Cases: Category Keyword and Brand Keyword

In this section, we consider two special cases, namely those of a category keyword and a brand keyword. Analyzing these cases helps us to develop basic insights for why and how much competing manufacturers will subsidize the retailer’s ads.

3.1 Category Keyword

In this section, we consider the case of a category keyword, i.e., the consumer searching the keyword has equal preference for both manufacturers’ brands, which implies that $l = 0$, and $l_1 = l_2 = 1/2$. Since the keyword is equally relevant to both manufacturers, we assume equal bid weights $w_1 = w_2 = 1$ (which are also equal to the retailer’s bid weight).

We start our analysis with the benchmark case where manufacturers are not allowed to sponsor the retailer’s search advertising. This allows us to understand how the auction outcome is shaped by the vertical relationships between the two manufactures and the retailer. We then proceed to the scenario in which each manufacturer chooses the optimal sponsorship rate to support the retailer’s bidding. We explore under what conditions and to what extent the retailer gets subsidized, and whether sponsorship can benefit manufacturers.

3.1.1 Category Keyword: Without Sponsorship

Consider a bidding game where the manufacturers do not provide sponsorship to the retailer. We first derive the retailer’s equilibrium bid. If the retailer wins the auction by paying $p_R$ per search, she expects a profit of,

$$\pi_R(\text{win}) = (1 - \theta + \theta \alpha) \phi - N_R p_R.$$  (3)

If she loses, her expected profit is given by

$$\pi_R(\text{lose}) = (1 - \theta)(1 - \alpha) \phi.$$  (4)
By equating $\pi_R(\text{win})$ with $\pi_R(\text{lose})$ to solve $p_R$, we get the retailer’s maximum willingness to pay for getting the advertising slot. Denote the solution as $b^{ca}_R$, where the superscript $^{ca}$ denotes “category”. This will also be her equilibrium bid according to the locally envy-free equilibrium selection criterion characterized by [Edelman et al. (2007)], i.e.,

$$b^{ca}_R = \frac{\alpha}{N_R} \phi. \tag{5}$$

Next, we analyze the manufacturers’ bids. Different from most papers on search engine advertising, in our case manufacturers are non-independent bidders because a manufacturer’s bid depends on who he expects to win the auction if he himself loses. In fact, if a manufacturer loses, he earns higher profit when the retailer wins, compared with the case when the other manufacturer wins; this is because if the retailer wins, the manufacturer can still get a share of the revenue from the demand stolen by the retailer, but if the competing manufacturer wins then he does not get anything from the demand stolen by this manufacturer. Therefore, to calculate a manufacturer’s bid, we need to consider two cases respectively — when he anticipates one of the manufacturers to win and when he anticipates the retailer to win. Then, we identify the condition for each case to happen in equilibrium. We analyze $M_1$’s bidding decision, while $M_2$’s bidding decision can be obtained by symmetry.

**Case 1:** Suppose the retailer is expected to win the auction. Given the retailer as the winner, $M_1$’s profit is given by

$$\pi^{ca}_{M_1}(\text{lose}| R \text{ wins}) = \theta(1 - \alpha)l_1 + (1 - \phi)(1 - \theta + \theta \alpha)l_1$$

$$= \theta(1 - \alpha) \frac{1}{2} + (1 - \phi)(1 - \theta + \theta \alpha) \frac{1}{2} \tag{6}$$

Here, a manufacturer’s profit, upon losing the auction, is comprised of two parts — the amount he sells directly to consumers, and the amount he collects from the retailer. If $M_1$ deviates to overbid the retailer and becomes the winner instead, his profit will increase to,

$$\pi^{ca}_{M_1}(\text{win}) = [\theta l_1 + (1 - \theta l_1)\alpha] + (1 - \phi)(1 - \theta)(1 - \alpha)l_1 - N^{ca}_{M_1M}$$

$$= \left[ \frac{\theta}{2} + \left(1 - \frac{\theta}{2}\right) \alpha \right] + (1 - \phi)(1 - \alpha)(1 - \theta) \frac{1}{2} - N^{ca}_{M_1M}. \tag{7}$$

Here, $N^{ca}_M$ is equal to $N_{M_1}$ (from equation [1]) evaluated at $l = 0$, which gives $N^{ca}_M = g\theta/2 + (1 - \theta/2)\alpha$,
and $p_M$ is the per-click ad price $M_1$ pays upon winning. By equating $\pi_M^{ca}(\text{lose}|R \text{ wins})$ with $\pi_M^{ca}(\text{win})$ to solve $p_M$, we get $M_1$’s maximum willingness to pay to overbid the retailer. Denote the solution as $b_M^{ca}$, which will also be his equilibrium bid,

$$b_M^{ca} = \frac{\alpha}{N_M^{ca}} \frac{1 + \phi}{2}. \quad (8)$$

**Case 2:** Consider the other case that the retailer is expected to lose the auction. Suppose $M_2$ wins the auction. Then $M_1$’s will get a profit given by:

$$\pi_M^{ca}(\text{lose}|R \text{ loses}) = \frac{1}{2} (1 - \alpha) \left( \theta + (1 - \phi)(1 - \theta) \right). \quad (9)$$

By equating $\pi_M^{ca}(\text{win})$ in equation (7) with $\pi_M^{ca}(\text{lose}|R \text{ loses})$ to solve $p_M$, we get $M_1$’s equilibrium bid as,

$$b_M^{ca} = \frac{\alpha}{N_M^{ca}}. \quad (10)$$

Due to symmetry, $M_2$ will also bid $b_M^{ca}$.

Now, we combine the two cases together. Aligned with our intuition, $M_1$’s bid when he anticipates the retailer to win, $b_M^{ca}$, is strictly lower than his bid when one of the manufacturers is anticipated to win, $b_M^{ca}$. Furthermore, we know that if $b_R < a_M^{ca}$, one of the manufacturers wins in equilibrium. If $b_R > b_M^{ca}$, the retailer wins in equilibrium. If $b_M^{ca} < b_R < b_M^{ca}$, both cases can be an equilibrium. We select the equilibrium where the retailer wins in our main analysis. Put together, the retailer wins the auction in equilibrium when $b_R \geq b_M^{ca}$, and one of the manufacturers wins otherwise.

We note that a bidder’s equilibrium bid can be decomposed into the product of two terms—the *advertising effect* and the *revenue effect*. The former measures the total increase in sales divided by the number of clicks paid, while the latter is the profit margin. In particular, under our setting, any bidder—no matter a manufacturer or a retailer—sells to $\alpha$ more consumers by winning the auction compared with losing it. Therefore, the advertising effect for the retailer is $\alpha/N_R$, and for manufacturers is $\alpha/N_M^{ca}$ (irrespective of who wins in equilibrium). The revenue effect is $\phi$ for the retailer, whereas it is $(1 + \phi)/2$ for each manufacturer given the retailer winning in equilibrium, and 1 for each manufacturer given the other manufacturer winning in equilibrium. To facilitate the analysis, we define $\kappa^{ca}$ as the

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7The analysis in Section OA1.3 in the Online Appendix, shows that our result does not change qualitatively if we select the other case that one of the manufacturers wins in equilibrium.
ratio of the advertising effect for the manufacturer relative to that for the retailer. 

\[ \kappa^{ca} = \frac{\alpha}{N_M^{ca}} \left( \frac{N_R^{ca}}{N_M^{ca}} \right) = \frac{g(1 - \theta) + \alpha \theta}{g^2 + \alpha (1 - \frac{g^2}{2})}, \]  

which depends on the stealing effect \( \alpha \), the cannibalization effect \( g \), and the consumers’ channel preference parameter \( \theta \). For instance, \( \kappa^{ca} \) increases with \( \theta \) if and only if \( \alpha \geq g \). The intuition is as the following. As \( \theta \) increases, each manufacturer gets a larger market share, and consequently there are two countervailing forces at play: manufacturers will pay for more cannibalized clicks, but less stolen clicks upon winning. The manufacturers’ relative advertising effect \( \kappa^{ca} \) is higher if the stealing effect dominates the cannibalization effect, so that overall the manufacturers pay for less clicks upon winning.

In equilibrium, the retailer wins the category keyword auction if and only if \( b_R \geq b_M^{ca} \), which is equivalent to \( \kappa^{ca} \leq \frac{2 \phi}{1 + \phi} \). We obtain the following proposition.

**Proposition 1.** In a category keyword auction without sponsorship, in equilibrium, the retailer always bids \( \frac{\alpha}{N_R} \phi \). If \( \kappa^{ca} \leq \frac{2 \phi}{1 + \phi} \), each manufacturer bids \( \frac{\alpha}{N_M} \frac{1 + \phi}{2} \) and the retailer wins the auction; otherwise, each manufacturer bids \( \frac{\alpha}{N_M} \) and one of the manufacturers wins the auction.

The equilibrium outcome is illustrated in Figure 2(a) for a representative set of parameter values. The retailer wins the ad auction when her advertising effect is relatively stronger than the manufacturers’ (small \( \kappa^{ca} \)) or when when her commission fee is relatively high (large \( \phi \)). We also plot equilibrium bids and profits in Figures 2(b) and 2(c), respectively. For small \( \kappa^{ca} \) the retailer gains significantly from
winning the auction and bids high to win. Note that in this region each manufacturer obtains half of the traffic that goes to the retailer at a margin of $1 - \phi$. As $\kappa^{ca}$ increases, the gain from winning for the retailer decreases and she reduces her bid, and for $\kappa^{ca} > \frac{2\phi}{1+\phi}$ one of the manufacturers wins. In this case, the losing manufacturer does not obtain any of the additional traffic that goes to the winning manufacturer, which intensifies each manufacturer’s incentive to win the auction. This is reflected in the discrete upward jump in the manufacturers’ bids (and the discrete downward jump in their profits) at $\kappa^{ca} = \frac{2\phi}{1+\phi}$. Essentially, we find that, in a category keyword auction, the retailer’s winning may benefit manufacturers because it softens the competition between the two manufacturers. This insight is at the heart of the manufacturers’ incentive to subsidize the retailer’s ad expenses, which we analyze next.

3.1.2 Category Keyword: With Sponsorship

We now consider the bidding game where manufacturers can provide sponsorship to the retailer. Given the two manufacturers’ sponsorship rates as $\beta_1$ and $\beta_2$ respectively, the retailer’s profit if she wins the auction will be,

$$\pi_R(\text{win}|\beta_1, \beta_2) = (1 - \theta + \theta\alpha)\phi - p_R N_R (1 - \beta_1 - \beta_2).$$

(12)

On the other hand, if she loses, her expected profit will be $\pi_R(\text{lose})$ as in equation (4). By equating $\pi_R(\text{win}|\beta_1, \beta_2)$ with $\pi_R(\text{lose})$, we can solve $p_R$ as the retailer’s maximum willingness to pay to win the auction, which will become her bid given $\beta_1$ and $\beta_2$:

$$b_R(\beta_1, \beta_2) = \frac{1}{1 - \beta_1 - \beta_2} \frac{\alpha N_R^{ca} \phi}{R}. $$

(13)

Comparing to equation (5), we note that the manufacturers’ sponsorship leads to a higher bid by the retailer. The manufacturers’ bidding strategies are cumbersome to compute but follow the logic of the analysis in the previous section; we relegate the detailed analysis to Section A1.1 in the Appendix, and characterize the equilibrium using the following proposition.

Proposition 2. In a category keyword auction with sponsorship:

- When $\kappa^{ca} \leq \frac{2\phi}{1+\phi}$, the equilibrium sponsorship rates are $\beta_1^{ca} = \beta_2^{ca} = 0$; both manufacturers bid $b_M^{ca}$, and the retailer bids $b_R$ and wins the auction.

- When $\frac{2\phi}{1+\phi} < \kappa^{ca} \leq \frac{1+\phi^2}{1+\phi}$, the retailer is sponsored to win by both manufacturers or by only one
of them. The equilibrium sponsorship rates are any $\beta_1^{ca} > 0$ and $\beta_2^{ca} > 0$ that satisfy $\beta_1^{ca} + \beta_2^{ca} = 1 - \frac{1}{\kappa^{ca}} \frac{2\phi}{1+\phi}$, or $\beta_1^{ca} = \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 1 \cdot \frac{4\phi}{2\kappa^{ca}}}}{2\kappa^{ca}}$ and $\beta_2^{ca} = 0$. Manufacturer $i$ bids $b_M^{ca}$ if $\beta_i = 0$ and bids zero if $\beta_i > 0$, and the retailer bids no lower than $b_M^{ca}$ and wins the auction.

- When $\kappa^{ca} > \frac{1+\phi^2}{1+\phi}$, there are two equilibria. In the first equilibrium, the equilibrium sponsorship rates are any $\beta_1^{ca} > 0$ and $\beta_2^{ca} > 0$ that satisfy $\beta_1^{ca} + \beta_2^{ca} = 1 - \frac{1}{\kappa^{ca}} \frac{2\phi}{1+\phi}$; both manufacturers bid zero, and the retailer bids $b_M^{ca}$ and wins the auction. In the second equilibrium, the equilibrium sponsorship rates $\beta_1^{ca} = \beta_2^{ca} = 0$; both manufacturers bid $b_M^{ca}$, the retailer bids $b_R$, and one of the manufacturers wins the auction.

Compared with the case without sponsorship, both manufacturers’ bids are either lower or the same, and the retailer may increase her bid. Both manufacturers and the retailer get higher profit with sponsorship.

Figure 3(a) presents, for a representative set of parameter values, the equilibrium outcome with respect to $\kappa^{ca}$ and $\phi$ under the manufacturers’ equilibrium sponsorship rates. We find that it is an equilibrium for the manufacturers to sponsor the retailer when $\kappa^{ca}$ is not very small, and it is a unique equilibrium for the manufacturers to sponsor the retailer when $\kappa^{ca}$ is intermediate. Correspondingly, it is always an equilibrium for the retailer to win under any parameter setting, and it is a unique equilibrium for the retailer to win when $\kappa^{ca}$ is not very large. The intuition is as follows.

When the manufacturers’ advertising effect is relatively weak compared with the retailer’s, i.e., $\kappa^{ca} \leq \frac{2\phi}{1+\phi}$, the retailer wins without manufacturers’ sponsorship, and there is no incentive for the manufacturers to sponsor the retailer. When the manufacturers’ relative advertising effect is intermediate, i.e. $\frac{2\phi}{1+\phi} < \kappa^{ca} \leq \frac{1+\phi^2}{1+\phi}$, the retailer will lose the auction without manufacturers’ sponsorship. To avoid head-to-head competition, each manufacturer is willing to sponsor the retailer to win, no matter whether his competitor provides sponsorship or not. Note that the sum of the sponsorship rates is higher when only one manufacturer sponsors the retailer to win (given by $\frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \cdot \frac{2\phi}{2\kappa^{ca}}}}{2\kappa^{ca}}$), compared with the case when both manufacturers sponsor (given by $1 - \frac{1}{\kappa^{ca}} \frac{2\phi}{1+\phi}$). This is because if only one of the manufacturers is sponsoring, the other manufacturer that does not sponsor will submit a positive bid. Consequently, the sponsoring manufacturer needs to propose a higher sponsorship rate to help the retailer to win. Lastly, when the manufacturers’ relative advertising effect is strong, i.e., $\kappa^{ca} > \frac{1+\phi^2}{1+\phi}$, similarly, the retailer will lose the auction without manufacturers’ sponsorship. To avoid head-to-head competition, each manufacturer is willing to sponsor the retailer to win, only if his competitor also provides sponsorship.
When both manufacturers provide positive sponsorship rates, they will optimally bid zero to minimize the amount of their sponsorship. In contrast, if one manufacturer does not provide sponsorship, he will submit a high bid, which means a high price for the retailer to pay to win the auction. As a result, it will be too expensive for the other manufacturer to provide sponsorship to the retailer alone. In this case, both manufacturers will provide no sponsorship and compete aggressively. Therefore, there are two possible equilibria in this case—either both manufacturers sponsor or none of them sponsor. In the following discussion, we select the conservative equilibrium where none of the two manufacturers sponsor the retailer.

From Proposition 2, we can see that the equilibrium only pins down the sum of the sponsorship rates from the two manufacturers. We define $\beta^{ca} \equiv \beta_1^{ca} + \beta_2^{ca}$. Figure 3(b) plots $\beta^{ca}$ against $\kappa^{ca}$. When $\kappa^{ca} \leq 2\phi / (1 + \phi)$, the retailer wins without sponsorship. When $\kappa^{ca} \geq 1 + \phi^2 / (1 + \phi)$, the manufacturer wins. In between, the retailer wins with sponsorship or manufacturer wins.
When \( \frac{2\phi}{1+\phi} < \kappa^{ca} \leq \frac{1+\phi^2}{1+\phi} \), the manufacturers provide no sponsorship to the retailer and \( \beta^{ca} = 0 \). When \( \frac{2\phi}{1+\phi} < \kappa^{ca} \leq \frac{1+\phi^2}{1+\phi} \), the retailer wins the auction with sponsorship from both manufactures or only one of them. In this case, \( \beta^{ca} \) increases with \( \kappa^{ca} \), as the manufacturers need to provide more sponsorship to the retailer when their relative advertising effect is stronger. When \( \kappa^{ca} > \frac{1+\phi^2}{1+\phi} \), we select the equilibrium with \( \beta^{ca} = 0 \).

Figures 3(c) and 3(d) present the equilibrium bids and profits respectively, under some parameter setting. To understand the figure, we again consider three cases. In the first case with \( \kappa^{ca} \leq \frac{2\phi}{1+\phi} \), we find that the bids and profits in the case with sponsorship are exactly the same as that in the benchmark case without sponsorship. As a result, the dark lines overlap with the light ones. In the second case with \( \frac{2\phi}{1+\phi} < \kappa^{ca} \leq \frac{1+\phi^2}{1+\phi} \), the manufacturers bid zero and provide sponsorship to the retailer, who bids \( b^{ca}_R \), which is higher than her bid without sponsorship, \( b^{ca}_R \). Therefore, we see the dark lines of bids are above the light ones. Correspondingly, we find that both the manufacturers and the retailer’s profits get higher — the dark lines of profits are above the light ones too. Sponsorship benefits both manufacturers and the retailer — the retailer receives extra money for advertisement spending and faces lower competing bids, and the manufacturers use sponsorship as a coordination tool to avoid head-to-head competition. In the third case with \( \kappa^{ca} > \frac{1+\phi^2}{1+\phi} \), under our conservative equilibrium selection rule, neither manufacturer sponsors the retailer, and we go back to the benchmark case without sponsorship.

So far, we have characterized the equilibrium outcomes in terms of the combined measure \( \kappa^{ca} \). In the following corollary, we state how the equilibrium outcomes depend on the model primitives, specifically, \( \theta \), \( \alpha \) and \( \phi \).

**Corollary 1.** In a category keyword auction, manufacturers may subsidize the retailer’s search advertising when: (1) for a small \( \alpha \), \( \theta \) is not too large, or (2) for a large \( \alpha \), \( \theta \) is not too small, or (3) \( \phi \) is small.

### 3.2 Brand Keyword

In this section, we consider the case of a brand keyword, i.e., the consumer searching the keyword has a preference for one of the manufacturers’ brands, which implies that \( l \neq 0 \). We explicitly assume that the search engine only allows the manufacturer whose brand keyword gets searched for, and the retailer, to bid in the brand keyword auction. In other words, a manufacturer cannot bid for his competitor’s

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8For sake of simplicity, we only plot the equilibrium that \( \beta^{ca}_1 > 0 \) and \( \beta^{ca}_2 > 0 \). The other cases with \( \beta^{ca}_1 = 0 \) or \( \beta^{ca}_2 = 0 \) are similar and not presented here.
brand keyword directly, that is, direct “poaching” at the search engine is not allowed. In terms of the model parameters, this means that if \( M_i \)'s brand keyword is searched, then \( w_i = 1 \) and \( w_{3-i} = 0 \) (the bid of the focal manufacturer is taken as is, and the bid of the competing is weighted to zero). Making this assumption helps us to cleanly deliver some key insights, and is a stylized version of the common practice in industry wherein the search engine penalizes poachers by assigning them low bidding weights (such as relevance scores on Google; Sayedi et al. 2014). We will relax this assumption in Section 4 by allowing both manufacturers to bid for a brand keyword with arbitrary bidding weights. The competing manufacturer, though not allowed to bid for the search ad directly, can sponsor the retailer to bid so as to compete for the keyword indirectly.

3.2.1 Brand Keyword: Without Sponsorship

We consider a search ad auction for \( M_1 \)'s brand keyword. Only \( M_1 \) and \( R \) are allowed to bid in the auction. First, we observe that retailer \( R \)'s profit does not depend on \( l \), as discussed previously. Consequently, the retailer bids the same as in the category keyword auctions, i.e., it bids \( b_R \). The manufacturer’s bid, however, depends on \( l \). If \( M_1 \) wins the auction by paying \( p_{M_1} \) per click, his profit is,

\[
\pi_{M_1}^{br} \text{(win)} = \left( \frac{1}{2} + l \right) \theta + \left[ 1 - \left( \frac{1}{2} + l \right) \theta \right] \alpha + \left( \frac{1}{2} + l \right) (1 - \theta)(1 - \alpha)(1 - \phi) - p_{M_1} N_{M}^{br},
\]

(14)

where \( N_{M}^{br} = g \theta \left( \frac{1}{2} + l \right) + ((1 - \theta \left( \frac{1}{2} + l \right)) \alpha \), and superscript \( br \) denotes “brand”. If he loses to the retailer, we have his profit as,

\[
\pi_{M_1}^{br} \text{(lose|R wins)} = \left( \frac{1}{2} + l \right) \theta(1 - \alpha) + \left( \frac{1}{2} + l \right) (1 - \theta + \theta \alpha)(1 - \phi).
\]

(15)

By equating \( \pi_{M_1}^{br} \text{(win|R wins)} \) with \( \pi_{M_1}^{br} \text{(lose)} \), we can solve for \( p_{M_1} \). We denote the solution by \( b_{M_1}^{br} \), which is \( M_1 \)'s equilibrium bid.

\[
b_{M_1}^{br} = \frac{\alpha}{N_{M}^{br}} \left( \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right).
\]

(16)

We can see that with only one manufacturer and one retailer in the auction, we can pin down a unique equilibrium with envy-free equilibrium selection rule. Similar to the category keyword case, we define
the manufacturer’s advertising effect relative to the retailer’s as the following,

\[ \kappa^{br} = \frac{\alpha/N_M}{\alpha/N_R} = \frac{N_R}{N_M} \frac{g(1 - \theta) + \alpha \theta}{g \left( \frac{1}{2} + l \right) \theta + \alpha \left( 1 - \left( \frac{1}{2} + l \right) \theta \right)} . \] (17)

\( \kappa^{br} \) increases with \( \theta \) if and only if \( \alpha \geq g \), \( \kappa^{br} \) increases with \( \alpha \) if and only if \( \theta \geq 2/(3 + 2l) \), and \( \kappa^{br} \) increases with \( g \) if and only if \( \theta \leq 2/(3 + 2l) \). The intuition for the relationship is similar with the case of category keywords. We also find that \( \kappa^{br} \) increases with \( l \) if and only if \( \alpha \geq g \). The intuition is that the cannibalization effect increases with the brand loyalty \( l \) while the stealing effect decreases with \( l \). With \( \alpha \geq g \), the stealing effect dominates the cannibalization effect, so the manufacturer will need to pay less clicks to win when \( l \) is large, i.e., his advertising effect relative to the retailer’s, \( \kappa^{br} \) is larger.

Comparing \( b_R \) with \( b^{br}_{M_1} \), we find that a manufacturer wins his brand keyword auction if and only if \( \kappa^{br} > \frac{\phi}{\frac{1}{2} - l + (\frac{1}{2} + l) \phi} \). We present this result in Proposition 3.

**Proposition 3.** In a brand keyword auction without sponsorship, in equilibrium, the retailer bids \( \frac{\alpha}{N_R} \phi \), and the manufacturer bids \( \frac{\alpha}{N_M} \left[ \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right] \). If \( \kappa^{br} > \frac{\phi}{\frac{1}{2} - l + (\frac{1}{2} + l) \phi} \), the manufacturer wins the auction, and otherwise, the retailer wins.

Figure 4(a) illustrates, for a representative set of parameter values, the equilibrium outcome for brand keywords; this looks similar to Figure 2(a) for category keywords. We plot the equilibrium bids and profits in Figures 4(b) and 4(c), respectively. The retailer wins the auction when \( \kappa^{br} \) is relatively small; otherwise, \( M_1 \) wins the auction. We find that there are no jumps in the bids and profits of the
retailer as well as the focal brand’s manufacturer, $M_1$. However, the profit of the competing manufacturer $M_2$ drops by $(\frac{1}{2} - \alpha)(1 - \phi)$ at the point when $M_1$ beats the retailer to win the auction. This is because $M_2$ gets more traffic from the retailer when the retailer wins the auction. Note that there is no direct competition between the two manufacturers to be softened here; rather, manufacturers benefit from the retailer’s winning because they can steal each other’s brand keyword searchers via the retailer.

An interesting finding here is that the manufacturer is less likely to win his brand keyword auction when his brand loyalty, $l$, is large. The intuition is that with higher brand loyalty, the manufacturer gets more demand in his direct sales channel whether he wins or loses the auction. However, the manufacturer will get relatively more demand from the retailer’s channel, when the retailer wins. Therefore, as the loyalty $l$ gets larger, the manufacturer is willing to bid lower and let the retailer to win.

To summarize, in both category keyword and brand keyword auctions, manufacturers can benefit from the retailer’s winning — for category keywords, the benefit arises from softened inter-brand competition, while for brand keywords, the benefit arises from stealing competitors’ brand keyword searchers via the retailer.

### 3.2.2 Brand Keyword: With Sponsorship

Intuitively speaking, a manufacturer may not want to sponsor the retailer for his own brand keyword auctions, but he may want to sponsor the retailer for his competitor’s brand keyword auctions. This is because if the retailer wins his own brand keyword auctions: (1) the manufacturer has a lower margin via indirect sales, (2) the retailer will divert demand to his competitor, and (3) the manufacturer may need to subsidize the retailer’s advertising spending. Yet the manufacturer may want to sponsor the retailer in his competitor’s keyword auctions to steal shares indirectly. In turn, his competitor has the same incentive and may sponsor for the focal brand’s manufacturer’s keyword. The focal manufacturer thus needs to pay a higher ad price to win his brand keyword auctions.

We denote the sponsorship rate of the manufacturer of the focal brand by $\beta_{own}^{br}$, and the sponsorship rate of the competing manufacturer by $\beta_{comp}^{br}$. We provide the details of the solution of the equilibrium in Section A1.2 in the Appendix and characterize the equilibrium in the following proposition.

**Proposition 4.** In a brand keyword auction with sponsorship, in equilibrium, manufacturer $M_1$ will never sponsor the retailer on his own brand keyword auction, $\beta_{own}^{br} = 0$, and will always bid $b_{M_1}^{br}$. 
• When $\kappa^{br} \leq \frac{\phi}{2-I+(\frac{1}{2})I} \phi$, the competing manufacturer $M_2$’s equilibrium sponsorship rate $\beta^{br}_{\text{comp}} = 0$, and the retailer bids $b^{br}_R$ and wins the auction.

• When $\frac{\phi}{2-I+(\frac{1}{2})I} \phi < \kappa^{br} \leq 1$, the competing manufacturer $M_2$’s equilibrium sponsorship rate $\beta^{br}_{\text{comp}} = 1 - \frac{\phi}{2-I+(\frac{1}{2})I} \phi \frac{1}{\kappa^{br}}$, and the retailer bids $b^{br}_{M_1}$ and wins the auction.

• When $\kappa^{br} > 1$, the competing manufacturer $M_2$’s equilibrium sponsorship rate $\beta^{br}_{\text{comp}} = 0$, and the retailer bids $b^{br}_R$ and loses the auction.

Compared with the case without sponsorship, the manufacturer’s bid remains the same, and the retailer may increase her bid; the retailer’s profit with sponsorship is exactly the same, manufacturer $M_1$’s profit with sponsorship is lower, and manufacturer $M_2$’s profit with sponsorship is higher.

Figure 5(a) illustrates the equilibrium outcome for a representative set of parameter values. In line with our intuition, we find that in equilibrium, a manufacturer never subsidizes the retailer in his own brand keyword auctions, but he may subsidize the retailer in his competitor’s brand keyword auctions. Compared with the case of category keyword auctions where it is always an equilibrium for the retailer to be the winner, we find that in brand keyword auctions, the retailer will lose the auction when $\kappa^{br} > 1$, because she gets no sponsorship in this case. In fact, in $M_1$’s brand keyword auctions, $M_1$ always bids $b^{br}_{M_1}$, which makes $M_2$’s sponsorship very expensive when $\kappa^{br}$ is relatively large. As a result, $M_2$ does not find it profitable to sponsor the retailer when $\kappa^{br}$ is large, and in this case, $M_1$ will win the auction. This is in contrast with the category keyword auctions, where both manufacturers can bid zero and use sponsorship as a coordination tool.

Figure 5(b) plots the competing manufacturer $M_2$’s equilibrium sponsorship rate, $\beta^{br}_{\text{comp}}$ as a function of $\kappa^{br}$. We can see that $\beta^{br}_{\text{comp}}$ increases with $\kappa^{br}$ first and then decreases with $\kappa^{br}$. The intuition is that, as $\kappa^{br}$ gets larger, the retailer is less willing to win the auction, so $M_2$ needs to provide higher sponsorship to push the retailer to win. However, as $\kappa^{br}$ gets very large and passes a threshold, $M_2$ finds it unprofitable to sponsor the retailer and would rather prefer the retailer to lose the auction.

Figures 5(c) and 5(d) present the manufacturers’ and retailer’s bids and profits. We can see that compared with the scenario without sponsorship, the retailer bids higher with sponsorship, and consequently the focal brand’s manufacturer’s profit gets lower and the competing brand’s manufacturer’s profit gets higher with sponsorship.
Figure 5: Equilibrium outcomes and other quantities in a brand keyword auction with sponsorship for $\alpha = 0.5$, $g = 0$ and $l = 0.1$. In figure (a), the non-focal manufacturer (whose keyword was not searched) sponsors in the shaded region and no manufacturer sponsors in the unshaded regions. In figures (c) and (d), the dark colored lines represent the case with sponsorship, and the light colored lines represent the benchmark case without sponsorship.

So far, we have characterized the equilibrium outcomes in terms of the combined measure $\kappa^{br}$. In the following corollary, we state how the equilibrium outcomes depend on the model primitives, specifically, $\theta$, $\alpha$, $l$ and $\phi$.

**Corollary 2.** In a brand keyword auction, the competing manufacturer subsidizes the retailer’s search advertising when: (1) both $\alpha$ and $\theta$ are intermediate, or (2) $l$ is small, or (3) $\phi$ is small.

The analysis in this section of the polar cases of a category keyword and a brand keyword with direct poaching prohibited provides the interesting insight that in both cases there is incentive for the manufacturers to subsidize the retailer’s ad spend. This enables the retailer to advertise more and, while this means greater competition for ad space for the manufacturers as well as more sharing of commissions
with the retailer, it also has advantages—in the category keyword case it softens direct competition between the manufacturers, and in the brand keyword case it enables more effective poaching of the competing manufacturer’s customers. In the next section, we study a scenario with a general keyword.

4 General Keyword

In this section, we solve the full model described in Section 2. A keyword (equivalently, the consumer who searches) is characterized by a parameter \( l \in \left[-\frac{1}{2}, \frac{1}{2}\right] \). This encompasses a continuum of keywords between category keywords (e.g., “shoes”, for which \( l = 0 \)) and brand keywords (e.g., “Nike” and “Adidas”, for which we can arbitrarily assign \( l = -\frac{1}{2} \) and \( \frac{1}{2} \), respectively); for instance, “running shoes,” which leans towards Nike will have \( l \in (-\frac{1}{2}, 0) \) and “tennis shoes,” which leans towards Adidas will have \( l \in (0, \frac{1}{2}) \). Both the manufacturers and the retailer are allowed to bid for the keyword, with the search engine assigning different bidding weights (e.g., quality and relevance scores from Google) to different bidders, depending on the keyword being searched for. This analysis provides a fuller picture of digital co-op in search advertising.

4.1 General Keyword: Without Sponsorship

We first solve the general bidding game in absence of sponsorship. Recall that if \( M_i \) wins the auction, he pays for \( N_{M_i} \equiv g\theta l_i + \alpha(1-\theta l_i) \) clicks, while if the retailer wins, she pays for \( N_R \equiv g(1-\theta) + \alpha\theta \) clicks. The manufacturer \( M_i \)’s advertising effect relative to the retailer’s is defined as \( \kappa_i \equiv N_R/N_{M_i} \).

In Section [OAI.1] in the Online Appendix, we solve the three players’ bidding game. Similar to the previous analysis, we find that it is the retailer’s weakly dominant strategy to always bid \( b_R = \frac{\alpha}{N_R} \phi \).

The manufacturers’ bidding strategies are more complicated as they depend on whether a manufacturer expects the other manufacturer or the retailer to win when he loses the auction. In some cases, multiple equilibria can occur, and we adopt the same equilibrium selection rule as in Section 3.1 to select the retailer to be the winner. In Section [OAI.3] in the Online Appendix, we show that if we select one of the manufacturers to be the winner instead, our result holds qualitatively. The equilibrium outcome is summarized by the following proposition. Without loss of generality, we assume that \( \kappa_i w_i \geq \kappa_j w_j \).

**Proposition 5.** In a general keyword auction without sponsorship, in equilibrium, the retailer wins the ad auction if \( \kappa_i w_i \leq \frac{\phi}{l_j+l_i}\phi \) for \( i = 1, 2 \). Otherwise, if \( \kappa_i w_i > \frac{\phi}{l_j+l_i}\phi \) and \( \kappa_j w_j \leq \phi \), \( M_i \) wins, and the
retailer ranks in the second position; if \( \kappa_i w_i > \frac{\phi}{l_i + l_i \phi} \) and \( \phi < \kappa_j w_j \leq \kappa_i w_i \), \( M_i \) wins, and \( M_j \) ranks in the second position.

Figure 6(a) illustrates the equilibrium outcome in terms of \( \kappa_i w_i \) while keeping \( l \) fixed. The results are intuitive. The larger \( \kappa_i w_i \) is, the more competitive \( M_i \) gets. Therefore, the retailer wins when both \( \kappa_1 w_1 \) and \( \kappa_2 w_2 \) are relatively small. When \( \kappa_i w_i \) is relatively large but \( \kappa_j w_j \) is relatively small, \( M_1 \) beats the retailer to win; on the other hand, when both \( \kappa_1 w_1 \) and \( \kappa_2 w_2 \) are relatively large, the two manufacturers compete with each other directly in the auction, with the winning position awarded to the one with higher \( \kappa_i w_i \).

In Figure 6(b) we present the equilibrium outcome in terms of \( l \), the keyword type being searched for, and \( \phi \), the commission fee charged by the retailer. To understand the equilibrium outcome, first consider how \( \phi \) influences the equilibrium for a fixed \( l \). The more-relevant manufacturer wins when \( \phi \) is small, and the retailer wins when \( \phi \) is large. This is because as \( \phi \) increases, the retailer charges a higher commission fee from indirect selling, therefore she has more incentive to bid higher to increase her profit. Next, consider a fixed \( \phi \) at a relatively small value. The more-relevant manufacturer (the one with higher \( \kappa_i w_i \)) wins the ad auction, with either the retailer or the less-relevant manufacturer ranking in the second position depending on \( |l| \). Considering a large \( \phi \), the retailer wins for small \( |l| \) and the more-relevant manufacturer wins for a large \( |l| \).

4.2 General Keyword: With Sponsorship

In this section, we allow the manufacturers to subsidize the ad spend of the retailer through sponsorship rates \( \beta_i \). We relegate the details to Section OA1.2 in the Online Appendix, and provide the solution in the following proposition. To simplify notation, we define:

\[
\tilde{\beta} \equiv 1 - \min_{i=1,2} \left\{ \frac{1}{\kappa_i w_i} \frac{\phi}{l_{3-i} + l_i \phi} \right\},
\]

\[
\tilde{\beta}_i \equiv \frac{\kappa_j w_j (l_i + l_j \phi) - (l_j + l_i \phi) + \sqrt{(\kappa_j w_j (l_i + l_j \phi) + l_j + l_i \phi)^2 - 4\phi \kappa_i w_i (l_j + l_i \phi)}}{2\kappa_j w_j (l_i + l_j \phi)}.
\]

Proposition 6. In a general keyword auction with sponsorship:

- When \( \kappa_i w_i \leq \frac{\phi}{l_i + l_i \phi} \) for \( i = 1, 2 \), the equilibrium sponsorship rates are \( \beta_i = \beta_j = 0 \), and the retailer wins the auction.
Figure 6: Equilibrium outcomes in the general keyword setup without sponsorship (figures (a) and (b)) and with sponsorship (figures (c), (d), (e) and (f)). In figures (c) and (d), both manufacturers sponsor in the dark shaded region, one manufacturer sponsors in the light shaded regions, and no manufacturer sponsors in the unshaded regions. For figures (a) and (c), \( l \) is fixed at 0. For figures (b) and (d), \( w_1 = 1, w_2 = 1 - 2l \) for \( l \geq 0 \), and \( w_1 = 1 + 2l, w_2 = 1 \) for \( l < 0 \).
• When $\kappa_i w_i > \frac{\phi}{l_j+l_i \phi}$ and $\frac{l_j(1-\phi^2)}{\kappa_i w_i - (l_i + l_j \phi)} < \kappa_j w_j \leq \kappa_i w_i$, the retailer is sponsored to win by both manufacturers or by either one of them. In the equilibrium where both manufacturers sponsor, the equilibrium sponsorship rates are any $\beta_i > 0$ and $\beta_j > 0$ that satisfy $\beta_i + \beta_j = \tilde{\beta}$. In the equilibrium where only $M_i$ sponsors, the equilibrium sponsorship rates are $\beta_i = \tilde{\beta}_i$ and $\beta_j = 0$. In the equilibrium where only $M_j$ sponsors, the equilibrium sponsorship rates are $\beta_i = 0$ and $\beta_j = \max \left\{ 1 - \frac{l_j(1-\phi)}{\kappa_i w_i l_j + l_i \phi}, \tilde{\beta}_j \right\}$.

• When $1 \geq \kappa_i w_i > \frac{\phi}{l_j+l_i \phi}$, $\kappa_j w_j < \frac{l_j(1-\phi^2)}{\kappa_i w_i - (l_i + l_j \phi)}$, and $\kappa_i w_i < \frac{l_j(1-\phi)}{(l_j+l_i \phi)(1-\frac{\phi}{\kappa_j w_j})}$, the retailer is sponsored to win by the less-relevant manufacturer $M_j$. The equilibrium sponsorship rates are $\beta_i = 0$ and $\beta_j = \max \left\{ 1 - \frac{\phi}{\kappa_j w_j} l_j + l_i \phi, \tilde{\beta}_j \right\}$.

• When $\kappa_j w_j \leq \kappa_i w_i$ and $\kappa_j w_i > \kappa_i w_i \frac{l_j(1-\phi)}{(l_j+l_i \phi)(1-\frac{\phi}{\kappa_j w_j})}$, or $\kappa_i w_i > 1$, the more-relevant manufacturer $M_i$ wins, and neither manufacturer sponsors.

Figure 6(c) illustrates the equilibrium outcome in terms of $\kappa_1 w_1$ and $\kappa_2 w_2$. When both $\kappa_i w_i$ and $\kappa_j w_j$ are relatively small, the retailer bids high relative to the manufacturers, and manufacturers do not need to sponsor the retailer ($\beta_i = \beta_j = 0$). When both $\kappa_i w_i$ and $\kappa_j w_j$ are relatively large, the direct competition between the two manufacturers is strong. In equilibrium, the retailer will be sponsored to win by either one or both manufacturers. If both manufacturers sponsor, the sum of the sponsorship rates will be just high enough to ensure the retailer as the winner, and both manufacturers will bid zero so that they pay zero subsidy to the retailer. If one of the manufacturers does not sponsor and bids positively, the other manufacturer still has the incentive to sponsor the retailer to win alone to avoid the strong direct competition. When $\kappa_i w_i$ is relatively large but $\kappa_j w_j$ is relatively small, if the less-relevant manufacturer $M_j$ does not sponsor, then the more-relevant manufacturer $M_i$ will not sponsor either. This is because, in this region, the direct competition between manufacturers becomes weaker and it is more profitable for $M_i$ to play an “offensive” strategy by overbidding the retailer than playing a “defensive” strategy by supporting the retailer’s winning. Meanwhile if the more-relevant manufacturer
$M_i$ does not sponsor, then the less-relevant manufacturer $M_j$, who gets penalized in direct bidding, still has the incentive to sponsor the retailer’s winning to steal consumers via the retailer. Therefore, in this case, it is an equilibrium for the less-relevant manufacturer to sponsor alone, but not an equilibrium for the more-relevant manufacturer to sponsor alone. When $\kappa_i w_i$ is very large, it becomes too costly for $M_j$ to sponsor the retailer to win alone, therefore, if $M_i$ is not sponsoring, $M_j$ will not sponsor either. In this case, it is an equilibrium for neither manufacturer to sponsor.

The above analysis also gives us a good understanding on how the insight from the previous sections can be generalized to any keyword. For keywords closer to the category keywords where the direct competition between manufacturers is strong, both manufacturers have incentive to sponsor the retailer to win to avoid head-to-head competition. For keywords closer to the brand keyword where the direct competition is weak, the less-relevant manufacturer has incentive to sponsor the retailer to win to poach indirectly.

Figure 6(d) presents the equilibrium outcome in terms of $l$ and $\phi$. To understand some key patterns, we first consider $\phi$ fixed at a small value. For keywords closer to the category keyword (i.e., small $|l|$), the retailer gets sponsored from either one or both manufacturers. Consumers searching for these keywords have similar preferences towards the two brands. The best scenario for the manufacturers is to coordinate to sponsor the retailer to win so that they can soften their direct competition. For keywords with an intermediate leaning toward one manufacturer’s brand (i.e., intermediate $|l|$), the retailer gets sponsored only from the less-relevant manufacturer. The intuition behind is that the less-relevant manufacturer gets penalized for direct bidding for those consumers, so that it becomes more economical to for him to poach indirectly via the retailer. For keywords with a strong leaning towards one manufacturer (large $|l|$), neither manufacturer will sponsor. This is because consumers searching for those keywords are very loyal to one brand and are unlikely to be “poached” indirectly. Next, if we fix $l$ and vary $\phi$, as $\phi$ increases, it is less likely for the retailer to get sponsored, since the retailer has the incentive to bid high relative to the manufacturers, and thus is more likely to win the ad auction even without sponsorship.

Figures 6(e) and (f) plot the sponsorship rates for the two manufacturers, and show that determining these rates is not straightforward. Figure 6(e) shows how the sponsorship rates vary with $\kappa_1 w_1$ for a fixed value of $\kappa_2 w_2$ (fixed at the value indicated by the point A on the y-axis of Figure 6(c)). $\kappa_1$ is the benefit that $M_1$ obtains from winning the ad slot and $w_1$ is the weight that the search engine uses for its bid; therefore, intuitively, a larger $\kappa_1 w_1$ implies that $M_1$ will be more effective in winning the auction.
(without sponsorship). Given this interpretation, we can understand how the sponsorship rate for $M_1$ varies in Figure 6(e). For very small $\kappa_1 w_1$, $M_1$ has a small chance of winning the auction, so sponsoring the retailer is not very effective and therefore the rate is small. As this grows, sponsoring the retailer can actually be effective in helping it win the auction, while $M_2$ is not sponsoring the retailer, and so the sponsorship rate increases. As $\kappa_1 w_1$ increases further, sponsoring the retailer is effective but $M_2$ is also sponsoring it, so $M_1$ can reduce its sponsorship rate to a medium value and still enable the retailer to win the ad auction. As $\kappa_1 w_1$ increases further, $M_1$ would like to win the auction for itself and $M_2$ sponsors at a high rate, so $M_1$ stops sponsoring.

Figure 6(f) shows how the sponsorship rates vary as $l$ varies from $-1/2$ to $1/2$ for a fixed value of $\phi$ (fixed at the value indicated by the point B on the y-axis of Figure 6(d)). Varying $l$ spans the continuum of keywords from being equally relevant to both manufacturers’ brands (small $|l|$) to being relevant only for one manufacturer’s brand (large $|l|$). To have the discussion about sponsorship rates and outcomes at an intuitive level, we classify keywords into three types: “category keywords” (e.g., “shoes”), “strongly branded keywords” (e.g., “Nike” and “Adidas”) and “weakly branded keywords” (e.g., “running shoes,” which leans towards Nike and “tennis shoes,” which leans towards Adidas). For category keywords, both manufacturers sponsor the retailer, the retailer wins the ad auction, and both manufacturers benefit from this sponsorship. Since both manufacturers sponsor, the sponsorship rate is at a moderate level, which is sufficient to enable the retailer to win the auction. For weakly branded keywords, only the less-relevant manufacturer sponsors the retailer, the retailer wins the ad auction, the less-relevant manufacturer benefits from the retailer’s winning whereas the more-relevant manufacturer is hurt from the retailer’s winning. This sponsorship is at a high level as only this manufacturer is sponsoring and the sponsorship rate should be sufficiently high to enable the retailer to win the auction. Finally, for strongly branded keywords, neither manufacturer sponsors the retailer, and the more-relevant manufacturer wins the auction.

5 Uniform Sponsorship Rate for All Keywords

In the analysis until now, we have assumed that manufacturers can specify different sponsorship rates to the retailer for different types of keywords. However, to implement this keyword-level sponsorship scheme in practice, the manufacturers need to track the retailer’s bids at the keyword level. Moreover, it
may involve a fair amount of labor to administrate contracts that specify different sponsorship rates for different keywords. Due to these considerations, we consider an alternative case in this section, where a manufacturer can only specify one sponsorship rate for the retailer that applies to all types of keywords. Given the sponsorship rate, the retailer decides how much to bid for different keywords. We assume that a random consumer has a probability of $q$ to search by a category keyword, and a probability of $(1-q)/2$ to search by each manufacturer’s brand keyword (here, we follow the brand keyword framework of Section 3.2).

To solve the problem, note that after manufacturers specify their sponsorship rates in the first stage, all players bid optimally conditional on a specific keyword in the same way as in the previous analysis. What is different here is that when a manufacturer chooses his sponsorship rate in the first stage, he will choose it optimally by maximizing his expected profit over all the keywords consumers are expected to search for.

Unfortunately, it is very complicated and tedious to solve the equilibrium sponsorship rates under a general parameter setting. To simplify the analysis, we restrict ourselves to a parameter subspace by assuming $0 < \phi < \min \left\{ 1, \frac{\alpha - (\alpha + \theta)}{\alpha} \right\}$ (i.e., the retailer’s commission rate is not too high) and $l < \min \left\{ \frac{1}{2}, \frac{(g-2\alpha\phi)(1-\theta)-g\phi(1-2\theta)+2\alpha\theta(1-2\theta)}{2g(1-\theta-\phi(1-2\theta))} \right\}$ (i.e., the consumer’s loyalty parameter is not too large). One can show that when $1/2 \leq g/\alpha \leq 2$, $\phi \leq 0.3$ is a sufficient condition for the two conditions, and thus it can be argued that the assumptions are not too restrictive for practical purposes. We leave the detailed analysis to Section OA2 in the Online Appendix and summarize the equilibrium outcome in the following proposition.

**Proposition 7.** When each manufacturer uses a uniform sponsorship rate across all keywords, in equilibrium, the two manufacturers sponsor the retailer to win both category keyword and brand keyword auctions at a total sponsorship rate of $\beta_1 + \beta_2 = 1 - \frac{1}{\kappa^{ca}} \frac{2\phi}{1 + \phi}$ with $\beta_1 > 0$ and $\beta_2 > 0$ when: (1) $\kappa^{ca} > \frac{2\phi}{1 + \phi}$ and \( \frac{\phi}{\frac{1}{2} - l + \left(\frac{1}{2} + l\right)\phi} < \kappa^{br} < \sqrt{\frac{\phi}{\frac{1}{2} - l + \left(\frac{1}{2} + l\right)\phi}}, \) or (2) $\frac{2\phi}{1 + \phi} < \kappa^{ca} < \frac{1 + \phi^2}{1 + \phi}$, $\kappa^{br} < \frac{\phi}{\frac{1}{2} - l + \left(\frac{1}{2} + l\right)\phi}$, and $q > q_1$, or (3) $\frac{2\phi}{1 + \phi} < \kappa^{ca} < \frac{1 + \phi^2}{1 + \phi}, \kappa^{br} > \sqrt{\frac{\phi}{\frac{1}{2} - l + \left(\frac{1}{2} + l\right)\phi}}$, and $q > q_2$.\(^{11}\) Otherwise, the manufacturers do not sponsor the

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\(^{11}\)These two conditions come from the following considerations. The minimal cumulative sponsorship rate to support the retailer to win a category keyword auction, $1 - \frac{\phi}{\frac{1}{2} - l + \left(\frac{1}{2} + l\right)\phi}$, is higher than the minimal sponsorship rate to support the retailer to win a brand keyword auction, $1 - \frac{\phi}{\frac{1}{2} - l + \left(\frac{1}{2} + l\right)\phi} \kappa^{br}$. Also, $\kappa^{ca} \leq \frac{2\phi}{1 + \phi}$ implies $\kappa^{br} < \frac{\phi}{\frac{1}{2} - l + \left(\frac{1}{2} + l\right)\phi}$ (that is, $\kappa^{br} < \frac{\phi}{\frac{1}{2} - l + \left(\frac{1}{2} + l\right)\phi}$) is a subset of $\kappa^{ca} \leq \frac{2\phi}{1 + \phi}$.

\(^{12}\)\(q_1 \equiv \frac{\kappa^{ca} - 1 + \sqrt{\left(\kappa^{ca} + 1\right)^2 - 4 \frac{2\phi}{1 - \phi} \left(\frac{1}{2} + l + \left(\frac{1}{2} + l\right)\phi\right)\phi}}{\frac{1}{2} + l + \left(\frac{1}{2} + l\right)\phi} \) and \(q_2 \equiv \frac{\phi \left(1 - \frac{\phi}{\frac{1}{2} - l + \left(\frac{1}{2} + l\right)\phi}\right) + \frac{\kappa^{ca} - 1 + \sqrt{\left(\kappa^{ca} + 1\right)^2 - 4 \frac{2\phi}{1 + \phi} \left(\frac{1}{2} + l + \left(\frac{1}{2} + l\right)\phi\right)\phi}}{2 \frac{1 + \phi}{\frac{1}{2} - l + \left(\frac{1}{2} + l\right)\phi} \left(\frac{1}{2} + l + \left(\frac{1}{2} + l\right)\phi\right)}}{\frac{1}{2} - l + \left(\frac{1}{2} + l\right)\phi} \)
Figure 7: Equilibrium outcomes when manufacturers sponsor the retailer at a uniform rate across category and brand keyword auctions, under the parameter setting that $\theta = 0.8, \alpha = 0.2, g = 0.5$ and $l = 0.5$.

We illustrate the intuition behind Proposition 7 with the representative plot in Figure 7. In Region I, where $\phi$ is large, the retailer wins both category and brand keyword auctions without sponsorship. When $\phi$ becomes smaller, the retailer can still win category keyword auctions without sponsorship but she cannot win brand keyword auctions without sponsorship. In this case, manufacturers trade off the benefit from sponsoring the retailer to win category keyword auctions, and the losses from over compensating the retailer for brand keyword auctions. In Region II, where $\phi$ is small and $q$ is large, the gains from sponsoring the retailer to win category keyword auctions dominate, so that manufacturers will sponsor the retailer at a cumulative rate of $1 - \frac{1}{\kappa \alpha} \frac{2\phi}{1 + \phi}$. In Region III, where both $\phi$ and $q$ are small, the losses from over compensating the retailer for brand keyword auctions dominate, so the manufacturers do not sponsor the retailer. The main takeaway from this analysis is that even when the manufacturers cannot specify keyword-level sponsorship rates, it is still optimal them to sponsor the retailer for her search advertising spend.
6 Conclusions and Discussion

In this paper, we study the interesting and growing practice of “digital co-op” in search advertising under which a manufacturer subsidizes the ad spend of a retailer selling its product, even though this enables the retailer to compete more effectively in ad auctions with the manufacturer himself, all while the retailer takes a share of the channel margin. Using a game theoretic model, we provide a competitive account of a common retailer’s role in search advertising, which differs for different types of keywords. For category keywords, a common retailer serves as an intermediary for cooperation by softening the manufacturers’ head-to-head competition in direct bidding. By sponsoring the retailer to win the search ad auction, both manufacturers commit to bid lower, and their profits as well as the retailer’s profit are higher with sponsorship, compared with the case without sponsorship. For brand keywords, a common retailer serves as an intermediary for competition by facilitating manufacturers to steal loyal customers from their competitors. In this case, a manufacturer’s sponsorship of the retailer’s ad spend increases his own profit while reducing his competitor’s profit. Our findings can be generalized to incorporate keywords of a more general nature, or to accommodate to the setting where a manufacturer cannot customize his sponsorship rates for different types of keywords.

Our research provides valuable guidance to a manufacturer, who is facing competition, on why he should participate in a sponsored search digital co-op program with a retailer and how it should manage this program. The basic insight is that when the manufacturer has a strong incentive to win the ad slot for a keyword, but it is not easy to win the ad slot because a competitor’s high bidding or stronger market position, sponsoring the retailer is an optimal strategy. More specifically, if keyword-level sponsorship is possible for the manufacturer, the following provide guidance on what to do and what outcomes to expect:

- For category keywords relevant to all manufacturers, sponsor the retailer at a moderate level and also expect competing manufacturers to sponsor the retailer.
- For brand keywords moderately relevant to a competitor, sponsor the retailer at a high level to poach customers indirectly and do not expect the more-relevant manufacturer to sponsor the retailer.
- For brand keywords moderately relevant to yourself, do not sponsor the retailer but expect competitors to sponsor the retailer as they want to poach indirectly.
• For brand keywords highly relevant to yourself or to another manufacturer, do not sponsor the retailer and do not expect competitors to sponsor the retailer either.

If keyword level sponsorship is not possible and one sponsorship rate has to be determined across all keywords, the sponsored search digital co-op program is still informed by the above rules of thumb as building blocks. Specifically, the manufacturer should take an estimation of the relative search volumes of the different types of keywords discussed above and choose a sponsorship rate to reflect the strategies associated with the types of keywords with higher volumes. Finally, if a retailer appropriates a large fraction of the channel margin, a manufacturer should not sponsor the retailer’s ad spend. In this case, if the keyword is a strongly branded keyword, the manufacturer will win the ad auction; otherwise, the retailer should be allowed to win the ad auction.

In formulating our model, we have made a number of assumptions. Some of them are inessential and have been made to facilitate tractability of the analysis or to simplify exposition, while others are crucial for our findings to go through. We discuss their roles and implications here. First, we only consider search ad auctions of one slot. Our main findings should carry through if we consider more than two manufacturers and a common retailer bidding in a position auction with multiple slots. Second, we do not explicitly model consumers’ sequential search behaviors over search ads, nor do we explicitly model their brand and channel choice decisions. This is because we only consider search ad auctions of one slot, and the prices of the products under consideration are assumed to be the same across all selling channels. Third, we assume that the retail prices as well as wholesale contracts are exogenously given, which isolate cooperative advertising as the sole coordination tool between the retailer and manufacturers. This is a reasonable assumption given that there are other major selling channels besides search advertising. That being said, we would like to point out that our findings depend on this assumption, and therefore may not always hold if this assumption is relaxed. Lastly, we only study a particular channel structure in our paper — two manufacturers advertising via one common retailer. This is the simplest setting that enables us to provide a competitive account for the retailer’s role in search advertising. It is interesting to investigate more complex channel structures with multiple manufacturers and multiple retailers. We leave this pursuit for future research.
References


### Appendix

#### A1 Derivation of Equilibrium Sponsorship Rates

**A1.1 Proof of Proposition 2 for Category Keyword Auctions**

We first consider the case that the two manufacturers’ sponsorship rates $\beta_1 > 0$ and $\beta_2 > 0$. When the retailer wins, if we follow the envy-free equilibrium selection rule, as a manufacturer bids higher, the retailer needs to pay more upon winning, and consequently the manufacturer needs to pay more to the retailer to sponsor her ads. Therefore, when a manufacturer provides sponsorship to the retailer, his bid has a direct impact on his profit even conditioning on his position — the envy-free equilibrium selection rule, which is based on the standard second-price auction in which a bidder’s bid does not influence its payoff given its position, thus does not apply here. However, in the spirit of the envy-free equilibrium, we will select the equilibrium with the highest possible bids among all Nash equilibria. We call this the
maximum bidding rule, and we will show that this equilibrium selection rule allows us to to pin down a unique equilibrium. Denote manufacturer $i$’s bid as $b_{M_i}^a(\beta_1, \beta_2)$ for $i = 1, 2$. In the following, we first identify the conditions for Nash equilibria.

**Case 1:** Consider the case when the retailer wins in equilibrium, i.e., $b_{R}^a(\beta_1, \beta_2) \geq b_{M_i}^a(\beta_1, \beta_2)$ for $i = 1, 2$. Manufacturer $i$’s profit upon the retailer’s winning is,

$$
\pi_{M_i}^a(\text{lose}|\text{R wins}) = \theta (1 - \alpha) + \frac{1 - \phi}{2} (1 - \theta + \theta \alpha) - \beta_i \max \{ b_{M_1}^a(\beta_1, \beta_2), b_{M_2}^a(\beta_1, \beta_2) \} N_{R}^a. \tag{A1}
$$

If $M_i$ deviates to win the auction, his profit will be,

$$
\pi_{M_i}^a(\text{win}) = \theta (1 - \alpha) + \frac{1 - \phi}{2} (1 - \theta)(1 - \alpha) - b_{R}^a(\beta_1, \beta_2) N_{M_i}^a. \tag{A2}
$$

This deviation is unprofitable if and only if $\pi_{M_i}^a(\text{lose}|\text{R wins}) \geq \pi_{M_i}^a(\text{win})$ for $i = 1, 2$, i.e.,

$$
\max \{ \beta_1, \beta_2 \} \max \{ b_{M_1}^a(\beta_1, \beta_2), b_{M_2}^a(\beta_1, \beta_2) \} N_{R}^a \leq b_{R}^a(\beta_1, \beta_2) N_{M_i}^a - \alpha \frac{1 + \phi}{2}. \tag{A3}
$$

Note that when the two manufactures expect the retailer to win, they should bid the same in equilibrium; otherwise, the manufacturer who bids higher can strictly improve his profit by reducing his bid. That is, $b_{M_i}^a(\beta_1, \beta_2) = b_{M_i}^a(\beta_1, \beta_2)$. We define

$$
b_{M}^a(\beta_1, \beta_2) \equiv \min \left\{ \frac{b_{R}^a(\beta_1, \beta_2) N_{M}^a - \alpha \frac{1 + \phi}{2}}{\max \{ \beta_1, \beta_2 \} N_{R}^a}, b_{R}^a(\beta_1, \beta_2) \right\}. \tag{A4}
$$

Then, we can write down the Nash equilibrium condition that guarantees the retailer to win in equation (A3) as the following,

$$
b_{M_1}^a(\beta_1, \beta_2) = b_{M_2}^a(\beta_1, \beta_2) \leq b_{M}^a(\beta_1, \beta_2). \tag{A5}
$$

Equation (A5) establishes the bounds on manufacturers’ equilibrium bids when the retailer wins. In particular, manufacturers can bid as low as 0, or as high as $b_{M}^a(\beta_1, \beta_2)$ in equilibrium. The former case is the “best” scenario for manufacturers because they pay zero subsidy to the retailer. The latter case is the “worst” scenario in which manufacturers pays the highest possible subsidies. While any bid that lies in between is a Nash equilibrium given $\beta_1$ and $\beta_2$, we apply our maximum bidding rule and assume that manufacturers bid at the highest possible level, i.e., $b_{M_i}^a(\beta_1, \beta_2) = b_{M_i}^a(\beta_1, \beta_2)$ for $i = 1, 2$. Notice that
this is also the most conservative equilibrium in the sense that manufacturers have the least incentive to sponsor the retailer. Later we will show that given the retailer as the winner, manufacturers will choose $\beta_1 > 0$ and $\beta_2 > 0$ optimally so that the upper bound on the equilibrium bid $b^{ca}_{M_1}(\beta_1, \beta_2)$ is driven down to 0. In that case, manufacturers can only bid 0 in equilibrium.

Notice that by choosing $b^{ca}_{M_1}(\beta_1, \beta_2) = b^{ca}_{M}(\beta_1, \beta_2)$, equation (A5) is satisfied, but this does not mean that it is always a Nash equilibrium for the retailer to win. In fact, equation (A5) also requires that $b^{ca}_{R}(\beta_1, \beta_2) \geq b^{ca}_{M_1}(\beta_1, \beta_2) \geq 0$, which implies that

$$b^{ca}_{R}(\beta_1, \beta_2) \geq b^{ca}_{M},$$

where $b^{ca}_{M}$ is defined in equation (8). Given $b^{ca}_{M_1}(\beta_1, \beta_2) = b^{ca}_{M}(\beta_1, \beta_2)$, equation (A6) is essentially the Nash equilibrium condition that guarantees the retailer to be the winner.

**Case 2:** When $b^{ca}_{R}(\beta_1, \beta_2) < b^{ca}_{M_1}$, one of the manufacturers wins in equilibrium. In this case, it is straightforward to show that the two manufacturers will again compete head-to-head for the ad position in the auction, and their equilibrium bids $b^{ca}_{M_1}(\beta_1, \beta_2) = \overline{b^{ca}_{M}}$ for $i = \{1, 2\}$, where $\overline{b^{ca}_{M}}$ is defined in equation (10).

Note that the analysis above depends on the premise that $\beta_i > 0$ for $i = 1, 2$. In the case that $\beta_i = 0$, we will go back to the case without sponsorship, and the manufacturer $i$ will optimally bid $b^{ca}_{M}$ when he expects the retailer to win (and bid $\overline{b^{ca}_{M}}$ when he expects the retailer to lose). Therefore there is a jump in manufacturer $i$’s bid when his sponsorship rate $\beta_i$ goes from zero to any positive percentage. This is worthy of further clarification: when $\beta_i = 0$, the manufacturer’s bid does not impact his own profit directly, and he bids $b^{ca}_{M}$ so as to gain potential profit that is outside the explicit specification of the bidding game — as Varian (2011) points out, if the retailer failed to attend the auction, by bidding $b^{ca}_{M}$, manufacturer $i$ would get at least his current equilibrium profit; on the other hand, when $\beta_i > 0$, the manufacturer’s bid will impact his own profit directly, so he bids $b^{ca}_{M}(\beta_1, \beta_2)$ to maximize profit.

So far, we have completely solved the bidding game given manufacturers’ sponsorship rates $\beta_1$ and $\beta_2$. Next, we investigate the two manufacturers’ decisions on the sponsorship rates.

We first derive $M_i$’s best-response on the sponsorship rate given $\beta_i > 0$. In fact, given the optimal
bidding strategy, manufacturer $i$’s profit can be written down as,

$$\pi_i(\beta_1, \beta_2) = \begin{cases} 
\frac{\theta}{2}(1-\alpha) + \frac{1-\phi}{2}(1-\theta + \theta\alpha) - \beta_i b^c_M(\beta_1, \beta_2) N^c_R, & b^c_R(\beta_1, \beta_2) \geq b^c_M \text{ and } \beta_j > 0, \\
\frac{\theta}{2}(1-\alpha) + \frac{1-\phi}{2}(1-\theta)(1-\alpha), & b^c_R(\beta_1, \beta_2) < b^c_M \text{ and } \beta_j > 0.
\end{cases}$$

for $j \neq i$ and $i, j \in \{1, 2\}$. To maximize $\pi_i(\beta_1, \beta_2)$ with respect to $\beta_i$, we have the following response function of sponsorship rate for manufacturer $i$ given $\beta_j$.

$$\beta_i^*(\beta_j > 0) = \begin{cases} 
1 - \frac{1}{\kappa^c a} \frac{2\phi}{1+\phi} - \beta_j, & \kappa^c a > \frac{2\phi}{1+\phi}, \text{ and } 0 < \beta_j < 1 - \frac{1}{\kappa^c a} \frac{2\phi}{1+\phi}, \\
0, & \kappa^c a \leq \frac{2\phi}{1+\phi}, \text{ and } \beta_j > 0, \\
0, & \kappa^c a > \frac{2\phi}{1+\phi}, \text{ and } \beta_j \geq 1 - \frac{1}{\kappa^c a} \frac{2\phi}{1+\phi}.
\end{cases}$$

It is noteworthy that when $\beta_j > 0$, the best-response from manufacturer $i$ is to set $\beta_i = 1 - \frac{1}{\kappa^c a} \frac{2\phi}{1+\phi} - \beta_j$, so that the highest-possible equilibrium bid is 0.

Now we proceed to find out $M_i$’s best response for his sponsorship rate given $\beta_j = 0$. We first derive the bidding outcome for $\beta_i > 0$ and $\beta_j = 0$. Consider the case where the retailer wins in equilibrium. In this scenario, we have the equilibrium bids $b_R > b_M_j \geq b_M_i$. Here, $M_i$ bids the lowest among the three bidders in order to minimize his expense on sponsorship fee. We now find the condition for $M_j$ not to deviate to win over the retailer. For $M_j$, if the retailer is expected to win, upon losing to the retailer, $M_j$’s profit comes as,

$$\pi_M_j(\text{lose|R wins}) = \frac{\theta}{2}(1-\alpha) + \frac{1-\phi}{2}(1-\theta + \theta\alpha).$$

If deviating to win, $M_j$’s profit upon winning is,

$$\pi_M_j(\text{win}) = \frac{\theta}{2} + (1-\frac{\theta}{2}) \alpha + \frac{1-\phi}{2}(1-\theta)(1-\alpha) - \frac{1}{1-\beta\frac{\alpha\phi}{N^c_R M^c}} N^c_M,$$

$$= \frac{\theta}{2} + (1-\frac{\theta}{2}) \alpha + \frac{1-\phi}{2}(1-\theta)(1-\alpha) - \frac{\alpha\phi}{1-\beta\frac{\alpha\phi}{N^c_M}}.$$

To ensure the retailer’s winning, we need $\pi_M_j(\text{lose|R wins}) \geq \pi_M_j(\text{win})$, which comes as,

$$\beta_i \geq 1 - \frac{1}{\kappa^c a} \frac{2\phi}{1+\phi}.$$
For $M_i$, if the retailer is expected to win, upon losing to the retailer, $M_i$’s profit comes as,

$$\pi_{M_i}(\text{lose} | R \text{ wins}) = \frac{\theta}{2} (1 - \alpha) + \frac{1 - \phi}{2} (1 - \theta + \theta \alpha) - \beta_i \frac{\alpha}{N_{M_i}} \frac{1 + \phi}{2} N_R,$$

$$= \frac{\theta}{2} (1 - \alpha) + \frac{1 - \phi}{2} (1 - \theta + \theta \alpha) - \alpha \beta_i \kappa^{ca} \frac{1 + \phi}{2}.$$

If deviating to win and paying for the retailer’s bid, $M_i$ profit upon winning is,

$$\pi_{M_i}(\text{win}) = \frac{\theta}{2} + (1 - \frac{\theta}{2}) \alpha + \frac{1 - \phi}{2} (1 - \theta)(1 - \alpha) - \frac{1}{1 - \beta_i} \frac{\alpha \phi}{N_R} N_M,$$

$$= \frac{\theta}{2} + (1 - \frac{\theta}{2}) \alpha + \frac{1 - \phi}{2} (1 - \theta)(1 - \alpha) - \frac{1}{1 - \beta_i} \frac{\alpha \phi}{\kappa^{ca}}.$$

To ensure $M_i$ will not deviate to win, we need $\pi_{M_i}(\text{lose} | R \text{ wins}) \geq \pi_{M_i}(\text{win})$, which comes as,

$$1 + \frac{\phi}{2} (1 + \beta_i \kappa^{ca}) \leq \frac{1}{1 - \beta_i} \frac{\phi}{\kappa^{ca}}.$$

This is equivalent to,

$$\beta_i \geq \begin{cases} 0, & \kappa^{ca} \leq \frac{2 \phi}{1 + \phi}, \\
\frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2 \phi}{1 + \phi}}}{2 \kappa^{ca}}, & \kappa^{ca} > \frac{2 \phi}{1 + \phi}. \end{cases}$$

Now let’s combine the two conditions above for both $M_i$ and $M_j$ to underbid the retailer. Notice that when $\kappa^{ca} \geq \frac{2 \phi}{1 + \phi}$, we have $\frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2 \phi}{1 + \phi}}}{2 \kappa^{ca}} > 1 - \frac{1}{\kappa^{ca}} \frac{2 \phi}{1 + \phi}$. As a result, given $\beta_j = 0$, for the retailer to win in equilibrium, we need,

$$\beta_i \geq \begin{cases} 0, & \kappa^{ca} \leq \frac{2 \phi}{1 + \phi}, \\
\frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2 \phi}{1 + \phi}}}{2 \kappa^{ca}}, & \kappa^{ca} > \frac{2 \phi}{1 + \phi}. \end{cases}$$

When $\kappa^{ca} < \frac{2 \phi}{1 + \phi}$, regardless of the sponsorship rate from $M_j$, we have the retailer wins in equilibrium.

When $\kappa^{ca} > \frac{2 \phi}{1 + \phi}$, if $\beta_i < \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2 \phi}{1 + \phi}}}{2 \kappa^{ca}}$, either $M_i$ or $M_j$ will deviate to win over the retailer, which again results in the head-to-head competition between manufacturers.

At this moment we are ready to derive $M_i$’s best-response function on the sponsorship rate given $\beta_j = 0$. It is easy to see that when $\kappa^{ca} \leq \frac{2 \phi}{1 + \phi}$, $\beta_i^* = 0$. When $\kappa^{ca} > \frac{2 \phi}{1 + \phi}$, $M_i$ will either sponsor at $\beta_i = \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2 \phi}{1 + \phi}}}{2 \kappa^{ca}}$ or at $\beta_i = 0$. That is, $M_i$ will sponsor the retailer to win if doing so indeed
increases his profit, compared with not sponsoring and competing with \( M_j \). This requires

\[
\frac{\theta}{2}(1 - \alpha) + \frac{1 - \phi}{2}(1 - \theta + \theta \alpha) - \alpha \beta_i \kappa^{ca} \frac{1 + \phi}{2} > \frac{1}{2}(1 - \alpha)(\theta + (1 - \phi)(1 - \theta)),
\]

which translates into,

\[
\beta_i = \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4\frac{\phi}{1+\phi}}}{2\kappa^{ca}} < \frac{1}{\kappa^{ca}} \frac{1-\phi}{(1+\phi)},
\]

i.e.,

\[
\frac{2\phi}{1+\phi} \leq \kappa^{ca} < \frac{1+\phi^2}{1+\phi}.
\]

To summarize, given \( \beta_j = 0 \), we present \( M_i \)'s best response on his sponsorship rate as follows,

\[
\beta^*_i(\beta_j = 0) = \begin{cases} 
\frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4\frac{\phi}{1+\phi}}}{2\kappa^{ca}}, & \frac{2\phi}{1+\phi} \leq \kappa^{ca} < \frac{1+\phi^2}{1+\phi}, \\
0, & \text{otherwise}.
\end{cases}
\]

Combined together the cases when \( \beta_j > 0 \) and \( \beta_i = 0 \), we have \( M_i \)'s best-response function on the sponsorship rate given \( \beta_i \geq 0 \) as,

\[
\beta^*_i = \begin{cases} 
1 - \frac{1}{\kappa^{ca}} \frac{2\phi}{1+\phi} - \beta_j, & \kappa^{ca} \geq \frac{2\phi}{1+\phi}, \text{and } 0 < \beta_j < 1 - \frac{1}{\kappa^{ca}} \frac{2\phi}{1+\phi}, \\
\frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4\frac{\phi}{1+\phi}}}{2\kappa^{ca}}, & \frac{2\phi}{1+\phi} \leq \kappa^{ca} < \frac{1+\phi^2}{1+\phi}, \beta_j = 0, \\
0, & \kappa^{ca} \geq \frac{2\phi}{1+\phi}, \text{and } \beta_j \geq 1 - \frac{1}{\kappa^{ca}} \frac{2\phi}{1+\phi}, \\
0, & \kappa^{ca} < \frac{2\phi}{1+\phi}, \text{and } \beta_j = 0, \\
0, & \kappa^{ca} \geq \frac{1+\phi^2}{1+\phi}, \text{and } \beta_j = 0.
\end{cases}
\]

By intersecting the two response functions \( \beta^*_i(\beta_j) \) and \( \beta^*_j(\beta_i) \), we have the equilibrium sponsorship rates as,

\[
(\beta^*_1, \beta^*_2) \in \begin{cases} 
\{(0, 0)\}, & \kappa^{ca} \leq \frac{2\phi}{1+\phi}, \\
\{(\beta_1, \beta_2) \mid \beta_1 + \beta_2 = 1 - \frac{1}{\kappa^{ca}} \frac{2\phi}{1+\phi} \text{ and } \beta_1 > 0, \beta_2 > 0\} \cup \{(0, \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4\frac{\phi}{1+\phi}}}{2\kappa^{ca}})\} \cup \{(0, \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4\frac{\phi}{1+\phi}}}{2\kappa^{ca}})\}, & \frac{2\phi}{1+\phi} < \kappa^{ca} \leq \frac{1+\phi^2}{1+\phi}, \\
\{(\beta_1, \beta_2) \mid \beta_1 + \beta_2 = 1 - \frac{1}{\kappa^{ca}} \frac{2\phi}{1+\phi} \text{ and } \beta_1 > 0, \beta_2 > 0\} \cup \{(0, 0)\}, & \kappa^{ca} > \frac{1+\phi^2}{1+\phi}.
\end{cases}
\]

It is straightforward to show that, when sponsorship happens, each player makes weakly greater profit with sponsorship than without.
A1.2 Proof of Proposition 4 for Brand Keyword Auctions

Given $M_1$’s sponsorship rate $\beta_{own}$, and the competing manufacturer, $M_2$’s sponsorship rate $\beta_{comp}$, we can calculate the retailer’s maximum willingness to pay to win the auction, which will become her equilibrium bid,

$$ b_{br}^R(\beta_{own}, \beta_{comp}) = \frac{1}{1 - \beta_{own} - \beta_{comp}} \frac{\alpha \phi}{N_{br}}. \quad (A7) $$

Now we consider $M_1$’s bidding strategy. We denote his bid as $b_{br}^{M_1}(\beta_{own}, \beta_{comp})$. We first consider the case that the retailer will win the auction, i.e., $b_{br}^R(\beta_{own}, \beta_{comp}) \geq b_{br}^{M_1}(\beta_{own}, \beta_{comp})$. Similar to the category keyword case, we will show below that the manufacturer’s bidding strategy differs on whether $\beta_{own} > 0$ or $\beta_{own} = 0$.

In particular, given $\beta_{own} > 0$, the manufacturer’s profit is,

$$ \pi_{br}^{M_1}(\text{lose}) = \theta l_1 (1 - \alpha) + l_1 (1 - \theta + \theta \alpha)(1 - \phi) - \beta_{own} b_{br}^{M_1}(\beta_{own}, \beta_{comp}) N_{br}^R. \quad (A8) $$

For the manufacturer to maximize profit $\pi_{br}^{M_1}(\text{lose})$ with respect to $b_{br}^{M_1}(\beta_{own}, \beta_{comp})$, we should have,

$$ b_{br}^{M_1}(\beta_{own}, \beta_{comp}) = 0, \text{ for } \beta_{own} > 0. \quad (A9) $$

If the manufacturer deviates to win the auction, his profit will be,

$$ \pi_{br}^{M_1}(\text{win}) = \theta l_1 + [1 - \theta l_1] \alpha + l_1 (1 - \theta)(1 - \alpha)(1 - \phi) - b_{br}^R(\beta_{own}, \beta_{comp}) N_{br}^M. \quad (A10) $$

To ensure that it is unprofitable for the manufacturer to deviate to win the auction, we need $\pi_{br}^{M_1}(\text{lose}) \geq \pi_{br}^{M_1}(\text{win})$, i.e.,

$$ b_{br}^R(\beta_{own}, \beta_{comp}) \geq \frac{\alpha}{N_{br}^M} [l_2 + l_1 \phi] = b_{br}^{M_1}. \quad (A11) $$

In the other case that $\beta_{own} = 0$, we will go back to the case without sponsorship, and the manufacturer’s bid will be

$$ b_{br}^{M_1}(\beta_{own}, \beta_{comp}) = b_{br}^{M_1}, \text{ for } \beta_{own} = 0. \quad (A12) $$

Therefore similar to the case of category keywords, there is a jump in manufacturer $i$’s bid when his sponsorship rate $\beta_{own}$ goes from zero to any positive percentage.
On the other hand, when $b_R^{br}(\beta_{own}, \beta_{comp}) < b_M^{br}$, it is easy to show that the Nash equilibrium conditions in equation (A11) will fail, and thus, the retailer will lose the auction. In this case, the manufacturer will choose $\beta_{own} = 0$ and bid $b_M^{br}$ and win the auction.

So far, we have completely solved the bidding game given two manufacturers’ sponsorship rates $\beta_{own}$ and $\beta_{comp}$. To summarize, the retailer bids $b_R^{br}(\beta_{own}, \beta_{comp})$ and wins the auction if and only if $b_R^{br}(\beta_{own}, \beta_{comp}) \geq b_M^{br}$ and $\beta_{own} > 0$; otherwise he bids $b_M^{br}$.

Given the optimal bidding strategy, manufacturer $M_1$’s profit can be written down as,

$$
\pi_{M_1}(\beta_{own}, \beta_{comp}) = \begin{cases} 
  l_1(1 - \alpha)\frac{\theta}{2} + l_1(1 - \theta + \theta\alpha)(1 - \phi), & b_R^{br}(\beta_{own}, \beta_{comp}) \geq b_M^{br}, \\
  l_1\frac{\theta}{2} + l_1(1 - \theta)(1 - \alpha)(1 - \phi) + \left[1 - l_1\frac{\theta}{2}\right] \alpha - b_R^{br}(\beta_{own}, \beta_{comp})N_M^{br}, & \text{otherwise}.
\end{cases}
$$

To maximize $\pi_{M_1}(\beta_{own}, \beta_{comp})$ with respect to $\beta_{own}$, we have the following response function of the sponsorship rate for $M_1$ given $\beta_{comp}$.

$$
\beta_{own}(\beta_{comp}) = \begin{cases} 
  \geq 0, & \beta_{comp} \geq 1 - \frac{\phi}{l_2 + t_1\phi^{br}}, \\
  = 0, & \text{otherwise}.
\end{cases}
$$

It can be seen that $\beta_{own} = 0$ is always an equilibrium, implying the focal manufacturer does not want to deviate to sponsor the retailer bidding for his own brand keyword. If $\beta_{comp} \geq 1 - \frac{2-2\phi}{2(1+2\phi)}\frac{1}{k^{br}}$, the focal manufacturer is indifferent towards sponsoring or not, so we have that $\beta_{own}^{br} = 0$.

The competing manufacturer $M_2$’s profit function is

$$
\pi_{M_2}(\beta_{own}, \beta_{comp}) = \begin{cases} 
  l_2(1 - \alpha)\frac{\theta}{2} + l_2(1 - \theta + \theta\alpha)(1 - \phi), & b_R^{br}(\beta_{own}, \beta_{comp}) \geq b_M^{br} \text{ and } \beta_{own} > 0, \\
  l_2(1 - \alpha)\frac{\theta}{2} + l_2(1 - \theta + \theta\alpha)(1 - \phi) - \beta_{comp}b_M^{br}N_M^{br}, & b_R^{br}(\beta_{own}, \beta_{comp}) \geq b_M^{br} \text{ and } \beta_{own} = 0, \\
  l_2(1 - \alpha)\frac{\theta}{2} + l_2(1 - \theta)(1 - \alpha)(1 - \phi), & b_R^{br}(\beta_{own}, \beta_{comp}) < b_M^{br}.
\end{cases}
$$

To maximize $\pi_{M_2}(\beta_{own}, \beta_{comp})$ with respect to $\beta_{comp}$, we have the following response function of spon-
sorship rate for the manufacturer given $\beta_{own}$.

\[
\beta_{comp}^{br}(\beta_{own}) \begin{cases} 
\geq 1 - \frac{\phi}{l_2 + l_1} \frac{1}{k^{br}} - \beta_{own}, & \beta_{own} > 0, \\
= 1 - \frac{\phi}{l_2 + l_1} \frac{1}{k^{br}}, & \beta_{own} = 0 \text{ and } \kappa^{br} \leq 1, \\
< 1 - \frac{\phi}{l_2 + l_1} \frac{1}{k^{br}}, & \beta_{own} = 0 \text{ and } \kappa^{br} > 1.
\end{cases}
\]

Since $\beta_{own}^{br} = 0$, we get $\beta_{comp}^{br} = 1 - \frac{\phi}{l_2 + l_1} \frac{1}{k^{br}}$ when $\frac{\phi}{l_2 + l_1} < \kappa^{br} \leq 1$, and $\beta_{comp}^{br} = 0$ otherwise.

It is straightforward to show that for $\frac{\phi}{l_2 + l_1} < \kappa^{br} \leq 1$, i.e., when $M_2$ sponsors $R$: (1) $R$’s profit is weakly greater than its profit without sponsorship, (2) $M_1$’s profit with sponsorship is weakly lesser than its profit without sponsorship, and (3) $M_2$’s profit with sponsorship is weakly greater than its profit without sponsorship.
Online Appendix for “The Logic and Management of “Digital Co-op” in Search Advertising”

OA1 Equilibrium Analysis for the General Setup

OA1.1 Proof of Proposition 5 for the Case without Sponsorship

We denote the original profit of manufacturer $i$ without search ads as $\Pi^0_{M_i} \equiv l_i \theta + l_i (1 - \theta)(1 - \phi)$, and denote the original profit of the retailer without search ads as $\Pi^0_{R} \equiv (1 - \theta)\phi$. Conditional on the bids $b_R, b_{M_1}$ and $b_{M_2}$, the three players’ expected profits $\Pi_R, \Pi_{M_1}$ and $\Pi_{M_2}$ are summarized in Table OA1.

<table>
<thead>
<tr>
<th>Winner</th>
<th>$\Pi_{M_1}$</th>
<th>$\Pi_{M_2}$</th>
<th>$\Pi_{R}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R$</td>
<td>$(1 - \alpha)\Pi^0_{M_1} + \alpha l_1 (1 - \phi)$</td>
<td>$(1 - \alpha)\Pi^0_{M_2} + \alpha l_2 (1 - \phi)$</td>
<td>$(1 - \alpha)\Pi^0_{R} + \alpha \phi - \max{w_1 b_{M_1}, w_2 b_{M_2}} N_R$</td>
</tr>
<tr>
<td>$M_1$</td>
<td>$(1 - \alpha)\Pi^0_{M_1} + \alpha - \frac{\alpha \phi}{\pi} \max{w_2 b_{M_2}, b_R} N_{M_1}$</td>
<td>$(1 - \alpha)\Pi^0_{M_2}$</td>
<td>$(1 - \alpha)\Pi^0_{R}$</td>
</tr>
<tr>
<td>$M_2$</td>
<td>$(1 - \alpha)\Pi^0_{M_1}$</td>
<td>$(1 - \alpha)\Pi^0_{M_2} + \alpha - \frac{\alpha \phi}{\pi} \max{w_1 b_{M_1}, b_R} N_{M_2}$</td>
<td>$(1 - \alpha)\Pi^0_{R}$</td>
</tr>
</tbody>
</table>

Table OA1: Profits in a general auction.

We observe that, compared with losing the auction, the retailer can increase her revenue by $\alpha \phi$ upon winning the auction. It is easy to show that bidding $b_R = \frac{\alpha \phi}{N_R}$ is still the weakly-dominant strategy for the retailer, and will become her equilibrium bid. As for manufacturers’ bids, since all three players are allowed to bid now, we solve them by numerating all the possible rankings of the three players in the auction.

**Case 1:** Suppose that the retailer wins, i.e., $b_R \geq \max\{w_1 b_{M_1}, w_2 b_{M_2}\}$. To guarantee it is an equilibrium, we need to ensure that given the retailer’s $b_R = \frac{\alpha \phi}{N_R}$, neither $M_1$ nor $M_2$ would like to deviate to win over the retailer.

From Table OA1 when the retailer wins, $M_i$ earns a profit of

$$\Pi_{M_i}(\text{lose} | R \text{ wins}) = (1 - \alpha)\Pi^0_{M_i} + \alpha l_i (1 - \phi).$$

If $M_i$ deviates to overbid the retailer instead, he has to pay the minimum amount required to win over the retailer, $\frac{b_R}{\pi_i}$. Accordingly, his profit upon winning is,

$$\Pi_{M_i}(\text{win}) = (1 - \alpha)\Pi^0_{M_i} + \alpha - \frac{b_R}{\pi_i} N_{M_i}.$$

To ensure that it is unprofitable for $M_i$ to deviate to win the auction, we need $\pi_{M_i}(\text{lose} | R \text{ wins}) \geq \pi_{M_i}(\text{win})$, i.e.,

$$\alpha l_i (1 - \phi) \geq \alpha - \frac{b_R}{\pi_i} N_{M_i}.$$

This simplifies into,

$$\kappa_i w_i \leq \frac{\phi}{l_j + l_i \phi}.$$

Therefore, when $\kappa_i w_i \leq \frac{\phi}{l_j + l_i \phi}$ for $i = 1, 2$, the retailer wins the auction.
**Case 2:** Suppose $M_i$ wins, and the retailer’s bid is ranked the second highest, i.e., $w_i b_{M_i} > b_R \geq w_j b_{M_j}$. We need to ensure that it is not profitable for $M_i$ to underbid the retailer, and not profitable for $M_j$ to overbid the retailer nor $M_i$.

$M_i$’s profit upon winning is $\Pi_{M_i}(\text{win})$. If $M_i$ deviates to lower his bid to let the retailer become the winner instead, $M_i$’s profit becomes $\Pi_{M_i}(\text{lose}|R \text{ wins})$. To ensure that it is unprofitable for $M_i$ to deviate to lose the auction, we need $\pi_{M_i}(\text{win}) > \pi_{M_i}(\text{lose}|R \text{ wins})$, which leads to

$$\kappa_i w_i > \frac{\phi}{l_j + l_i \phi}.$$ 

Next, we need to ensure that $M_j$’s bid is the lowest. We first determine $M_j$’s equilibrium bid. In fact, $M_j$’s profit when $M_i$ wins comes as,

$$\Pi_{M_j}(\text{lose}|M_i \text{ wins}) = (1 - \alpha)\Pi^0_{M_i}.$$ 

By paying $p_M$ and winning the auction, To overbid $M_i$’s profit will be,

$$\Pi_{M_j}(\text{win}) = (1 - \alpha)\Pi^0_{M_j} + \alpha - p_M N_{M_j}.$$ 

By equating $\Pi_{M_j}(\text{lose}|M_i \text{ wins})$ and $\Pi_{M_j}(\text{win})$ to solve $M_i$’s maximum willingness to pay, $p_M$, which will also be his equilibrium bid.

$$b_{M_j} = \frac{\alpha}{N_{M_j}},$$

which does not depend on his or $M_i$’s bidding weights. To ensure that $M_j$ is below the retailer, we have that $b_R \geq w_j b_{M_j}$, which can be simplified as,

$$\kappa_j w_j \leq \phi.$$ 

To summarize, we have that when $\kappa_i w_i > \frac{\phi}{l_j + l_i \phi}$ and $\kappa_j w_j \leq \phi$, $M_i$ overbids the retailer, who then overbids $M_j$.

**Case 3:** Now still consider the last case when $M_i$ wins and $M_j$’s bid is ranked the second, i.e., $w_i b_{M_i} \geq w_j b_{M_j} > b_R$. Similar to Case 2, we can show that $M_j$’s equilibrium bid is $b_{M_j} = \frac{\alpha}{N_{M_j}}$. $M_i$’s profit upon winning is,

$$\Pi_{M_i}(\text{win}) = (1 - \alpha)\Pi^0_{M_i} + \alpha - \frac{w_j b_j}{w_i} N_{M_i}.$$ 

If $M_i$ deviates to lose and $M_j$ becomes the winner, $M_i$ gets,

$$\Pi_{M_i}(\text{lose}|M_j \text{ wins}) = (1 - \alpha)\Pi^0_{M_i}.$$ 

From $\pi_{M_i}(\text{win}) \geq \pi_{M_i}(\text{lose}|M_j \text{ wins})$, we have that,

$$\kappa_j w_j \leq \kappa_i w_i.$$ 

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Finally, to ensure that this is indeed the case when $M_j$ overbids the retailer, i.e., $w_j b_{M_j} > b_R$, we need
\[ \kappa_j w_j > \phi. \]

To summarize, when $\phi < \kappa_j w_j \leq \kappa_i w_i$, $M_i$ overbids $M_j$, who then overbids the retailer.

We find that the conditions for the retailer winning (Case 1) and manufacturer winning (Case 3) are not mutually exclusive. Both of them can be in equilibrium when $\phi < \kappa_i w_i \leq \frac{\phi}{\gamma_{j+1} \phi}$ and $\phi < \kappa_j w_j \leq \frac{\phi}{\gamma_{j+1} \phi}$. This is similar to the multi-equilibria problem we encounter in the category keyword auction. Similarly, we select the retailer’s winning as the equilibrium outcome. The following table summarizes the equilibrium outcome.

<table>
<thead>
<tr>
<th>Condition</th>
<th>$b_{M_1}$</th>
<th>$b_{M_2}$</th>
<th>$b_R$</th>
<th>Outcome</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa_1 w_1 \leq \frac{\alpha}{N_M + \phi}, \kappa_2 w_2 \leq \frac{\alpha}{N_M + \phi}$</td>
<td>$\min{\frac{1}{w_1} \phi, \frac{1}{w_2} \phi}$</td>
<td>$\min{\frac{1}{w_1} \phi, \frac{1}{w_2} \phi}$</td>
<td>$\frac{\phi}{N_R}$</td>
<td>\begin{tabular} {l} $R$ wins; \end{tabular}</td>
</tr>
<tr>
<td>$\kappa_1 w_1 &gt; \frac{\alpha}{N_M + \phi}, \kappa_2 w_2 \leq \phi$</td>
<td>$\frac{\alpha}{N_M}$</td>
<td>$\frac{\alpha}{N_M}$</td>
<td>$\frac{\phi}{N_R}$</td>
<td>$M_1$ wins; $R$ ranks 2nd</td>
</tr>
<tr>
<td>$\kappa_1 w_1 &gt; \frac{\alpha}{N_M + \phi}, \phi \leq \kappa_2 w_2 \leq \kappa_1 w_1$</td>
<td>$\frac{\alpha}{N_M}$</td>
<td>$\frac{\alpha}{N_M}$</td>
<td>$\frac{\phi}{N_R}$</td>
<td>$M_1$ wins; $M_2$ ranks 2nd</td>
</tr>
<tr>
<td>$\kappa_2 w_2 &gt; \frac{\alpha}{N_M + \phi}, \phi \leq \kappa_1 w_1 \leq \kappa_2 w_2$</td>
<td>$\frac{\alpha}{N_M}$</td>
<td>$\frac{\alpha}{N_M}$</td>
<td>$\frac{\phi}{N_R}$</td>
<td>$M_2$ wins; $M_1$ ranks 2nd</td>
</tr>
<tr>
<td>$\kappa_2 w_2 &gt; \frac{\alpha}{N_M + \phi}, \kappa_1 w_1 \leq \phi$</td>
<td>$\frac{\alpha}{N_M}$</td>
<td>$\frac{\alpha}{N_M}$</td>
<td>$\frac{\phi}{N_R}$</td>
<td>$M_2$ wins; $R$ ranks 2nd</td>
</tr>
</tbody>
</table>

Table OA2: Equilibrium outcome under general setup without sponsorship.

### OA1.2 Proof of Proposition 6 for the Case with Sponsorship

Given the two manufacturers’ sponsorship rates $\beta_1$ and $\beta_2$, one can show that the retailer always bids

\[ b_R(\beta_1, \beta_2) = \frac{1}{1 - \beta_1 - \beta_2} \frac{\alpha \phi}{N_R}. \]

Similar with the analysis of category keywords, each manufacturer’s bidding strategy depends on whether he provides zero or positive sponsorship to the retailer. We first consider the case where both manufacturers provide positive sponsorship, with $\beta_1 > 0$ and $\beta_2 > 0$. In this case, it is not a Nash equilibrium for a manufacturer to follow the envy-free equilibrium selection rule, since his bid has a direct impact on his profit when the retailer wins. It is assumed that $\kappa_i w_i \geq \kappa_j w_j$ without loss of generality. Depending on the relative positions of the three bidders, we have three cases below. We will identify the Nash equilibrium condition for each case. We denote $M_i$’s bid as $b_{M_i}(\beta_1, \beta_2)$ for $i = 1, 2$.

**Case 1:** Suppose the retailer wins the auction, i.e., $b_R(\beta_1, \beta_2) \geq \max\{w_1 b_{M_1}(\beta_1, \beta_2), w_2 b_{M_2}(\beta_1, \beta_2)\}$. Similar with the case of category keywords, we know that the two manufacturers’ weight-adjusted bids must be equal, i.e., $w_1 b_{M_1}(\beta_1, \beta_2) = w_2 b_{M_2}(\beta_1, \beta_2)$. Otherwise, the manufacturer with higher weight-adjusted bid can strictly increase his profit by lowering his bid. We define $\tilde{b}_M(\beta_1, \beta_2) \equiv w_1 b_{M_1}(\beta_1, \beta_2) = w_2 b_{M_2}(\beta_1, \beta_2)$.

$M_i$’s profit is,

\[ \pi_{M_i}^{\text{lose}}(\text{R wins}) = (1 - \alpha) \Pi_{M_i}^0 + \alpha l_i (1 - \phi) - \beta_i \tilde{b}_M(\beta_1, \beta_2) N_{M_i}. \]
If $M_i$ deviates to win the auction, his profit will be,

$$\pi_{M_i}(\text{win}) = (1 - \alpha)\Pi_{M_i}^0 + \alpha - \frac{b_R}{w_i}N_{M_i}.$$ 

This deviation is unprofitable if and only if $\pi_{M_i}(\text{lose} | \text{R wins}) \geq \pi_{M_i}(\text{win})$, i.e.,

$$\beta_i b_M(\beta_1, \beta_2) \leq \frac{1}{1 - \beta_1 - \beta_2} \frac{\alpha \phi}{w_i \kappa_i} - \alpha (l_i + l_j \phi), \text{ for } i \neq j = 1, 2.$$ 

By defining

$$\tilde{b}_M(\beta_1, \beta_2) \equiv \min_{i=1,2} \left\{ \frac{1}{\beta_i} \left[ \frac{1}{1 - \beta_1 - \beta_2} \frac{\alpha \phi}{w_i \kappa_i} - \alpha (l_{3-i} + l_i \phi) \right] \right\},$$

we can rewrite the above inequality as

$$\tilde{b}_M(\beta_1, \beta_2) \leq \tilde{b}_M(\beta_1, \beta_2),$$

which is the Nash equilibrium condition for the retailer to be the winner. To make sure the righthand side of the above equation is non-negative, we need the following condition,

$$\beta_i + \beta_j \geq 1 - \min_{i=1,2} \left\{ \frac{1}{\kappa_i w_i l_{3-i} + l_i \phi} \right\} \equiv \tilde{\beta}.$$

To uniquely pin down the two manufacturers’ equilibrium bids, we apply the maximal bidding rule as introduced in Section 3.1, and we have that,

$$b_{M_i}(\beta_1, \beta_2) = \frac{1}{w_i} \tilde{b}_M(\beta_1, \beta_2), \text{ when } \beta_i + \beta_j \geq \tilde{\beta}.$$ 

**Case 2:** Next consider the case when $M_i$ wins, and the retailer’s bid is ranked the second highest, i.e., $w_i b_M(\beta_1, \beta_2) > b_R(\beta_1, \beta_2) \geq w_j b_M(\beta_1, \beta_2)$. In this case, $M_i$’s profit upon winning is

$$\pi_{M_i}(\text{win}) = (1 - \alpha)\Pi_{M_i}^0 + \alpha - \frac{b_R}{w_i}N_{M_i}.$$ 

If $M_i$ deviates to lose and the retailer wins instead, for the manufacturer to maximize profit with respect to $b_{M_i}(\beta_1, \beta_2)$, we should have,

$$b_{M_i}(\beta_1, \beta_2) = 0.$$ 

Although $M_i$ optimally drops his bid to zero if deviating, the retailer still needs to pay a positive price to stay above $M_j$, given by $w_j b_{M_j}(\beta_1, \beta_2) N_R$. Thus, $M_i$ needs to share this expense with the retailer, even if he deviates to lose. His profit upon losing then comes as,

$$\pi_{M_i}(\text{lose} | \text{R wins}) = (1 - \alpha)\Pi_{M_i}^0 + \alpha l_i (1 - \phi) - \beta_i w_j b_{M_j}(\beta_1, \beta_2) N_R.$$ 

To figure out $\pi_{M_i}(\text{lose} | \text{R wins})$, we need to further find out $M_j$’s bid when anticipating $M_i$ to be the
winner. The result is that \( M_j \) will bid according to the envy-free equilibrium selection rule,

\[
b_{M_j}(\beta_1, \beta_2) = \frac{\alpha}{N_{M_j}},
\]

which is the same as in the case without sponsorship. This is because, when \( M_i \) wins, the retailer pays nothing in equilibrium, and therefore, \( M_j \)’s profit will not be affected by sponsorship. Substituting \( b_{M_j}(\beta_1, \beta_2) \) into the condition that \( \pi_{M_i}(\text{win}) \geq \pi_{M_i}(\text{lose}| R \text{ wins}) \), we get,

\[
\beta_i + \beta_j \leq 1 - \frac{1}{\kappa_i w_i} \frac{\phi}{l_j + l_i \phi + \beta_i \kappa_j w_j}.
\]

Furthermore, to make sure that the retailer overbids \( M_j \), we should have \( b_R(\beta_1, \beta_2) > b_{M_j}(\beta_1, \beta_2) \), which is equivalent to,

\[
\beta_i + \beta_j > 1 - \frac{\phi}{\kappa_j w_j}.
\]

To summarize, the equilibrium condition is that,

\[
1 - \frac{\phi}{\kappa_j w_j} < \beta_i + \beta_j \leq 1 - \frac{1}{\kappa_i w_i} \frac{\phi}{l_j + l_i \phi + \beta_i \kappa_j w_j}.
\]

Now let’s compare the parameter space specified above with that in Case 1. Indeed, the two parameters space may overlap with each other, implying both cases can be an equilibrium under certain condition. Similarly, we select the equilibrium where the retailer is the winner when both cases can be an equilibrium. Based on this equilibrium selection rule, we need to revise the equilibrium condition for Case 2 as the following,

\[
1 - \frac{\phi}{\kappa_j w_j} < \beta_i + \beta_j \leq \min \left\{ 1 - \frac{1}{\kappa_i w_i} \frac{\phi}{l_j + l_i \phi + \beta_i \kappa_j w_j}, \tilde{\beta} \right\}.
\]

We further simplify the above condition. First, we need \( \tilde{\beta} > 1 - \frac{\phi}{\kappa_j w_j} \) to ensure that the condition can hold for some parameter space. This implies that \( \kappa_i w_i (l_j + l_i \phi) > \kappa_j w_j \), under which condition, \( \tilde{\beta} = 1 - \frac{1}{\kappa_i w_i} \frac{\phi}{l_j + l_i \phi + \beta_i \kappa_j w_j} \). This implies that the equilibrium condition for Case 2 can be equivalently expressed as the following,

\[
1 - \frac{\phi}{\kappa_j w_j} < \beta_i + \beta_j < \tilde{\beta}, \kappa_j w_j < \kappa_i w_i (l_j + l_i \phi).
\]

Also, we need \( \kappa_i w_i > \frac{\phi}{l_j + l_i \phi} \) or \( \kappa_j w_j > \frac{\phi}{l_i + l_j \phi} \) to ensure \( \tilde{\beta} \geq 0 \).

**Case 3:** Now consider the case when \( M_i \) overbids \( M_j \), and \( M_j \) overbids the retailer, i.e., \( w_i b_{M_i}(\beta_1, \beta_2) \geq w_j b_{M_j}(\beta_1, \beta_2) > b_R(\beta_1, \beta_2) \). Similar to the benchmark case without sponsorship, we can show that it is unprofitable for \( M_i \) to deviate to lose if,

\[
w_i \kappa_i \geq w_j \kappa_j.
\]
To ensure \( w_j b_{M_j}(\beta_1, \beta_2) > b_R(\beta_1, \beta_2) \), we have,

\[
\beta_i + \beta_j < 1 - \frac{\phi}{\kappa_j w_j}.
\]

Again we need to rule out the parameter space that meets both the above two conditions and the equilibrium condition for Case 1. The result is that the equilibrium condition for Case 3 is that,

\[
\beta_i + \beta_j < \min \left\{ 1 - \frac{\phi}{\kappa_j w_j}, \tilde{\beta} \right\}, \text{ and } \phi < \kappa_j w_j \leq \kappa_i w_i,
\]

The condition \( \phi < \kappa_j w_j \) is added to ensure that the above parameter space is non-empty. Also, we need \( \kappa_i w_i > e \frac{\phi}{l_j + l_i \phi} \) or \( \kappa_j w_j > \frac{\phi}{l_i + l_j \phi} \) to ensure \( \tilde{\beta} \geq 0 \).

So far, we have fully characterized the equilibrium bids given both manufacturers sponsoring, given \( \beta_i > 0 \) and \( \beta_j > 0 \). To summarize, without loss of generality under the assumption of \( \kappa_j w_j \leq \kappa_i w_i \), we have that,

- If \( \kappa_j w_j \leq \kappa_i w_i \leq \frac{\phi}{l_j + l_i \phi} \), the retailer wins regardless of what the sponsorship rates are.
- If \( \phi < \kappa_i w_i (l_j + l_i \phi) \leq \kappa_j w_j \leq \kappa_i w_i \),
  - when \( \beta_i + \beta_j < \tilde{\beta} \), \( M_i \) wins while \( M_j \) ranks in the second position in the auction;
  - when \( \beta_i + \beta_j \geq \tilde{\beta} \), the retailer wins.
- If \( \phi \leq \kappa_j w_j < \kappa_i w_i (l_j + l_i \phi) \),
  - when \( \beta_i + \beta_j < 1 - \frac{\phi}{\kappa_j w_j} \), \( M_i \) wins while \( M_j \) ranks in the second position in the auction;
  - when \( 1 - \frac{\phi}{\kappa_j w_j} \leq \beta_i + \beta_j < \tilde{\beta} \), \( M_i \) wins while \( R \) ranks in the second position in the auction;
  - when \( \beta_i + \beta_j \geq \tilde{\beta} \), the retailer wins.
- If \( \kappa_j w_j < \phi < \kappa_i w_i (l_j + l_i \phi) \),
  - when \( \beta_i + \beta_j < \tilde{\beta} \), \( M_i \) wins while \( R \) ranks in the second position in the auction;
  - when \( \beta_i + \beta_j \geq \tilde{\beta} \), the retailer wins.

The above analysis is based on the condition that \( \beta_i > 0 \) and \( \beta_j > 0 \). Now we consider the case when \( \beta_i > 0 \) and \( \beta_j = 0 \). The case when \( \beta_i = 0 \) and \( \beta_j > 0 \) will be considered later. We first find the condition for the retailer to win in equilibrium, which requires that neither \( M_i \) nor \( M_j \) will deviate to win over the retailer. Under this scenario, we have \( b_R \geq b_{M_j} \geq b_{M_i} \), where the sponsoring manufacturer \( M_i \), bids the lowest among the three bidders to minimize his expense on sponsorship fee. For \( M_j \), if the retailer is expected to win, upon losing to the retailer, \( M_j \)'s profit comes as,

\[
\pi_{M_j}(\text{lose}| R \text{ wins}) = (1 - \alpha)\Pi_{M_j}^0 + \alpha l_j (1 - \phi).
\]
If deviating to win, \( M_j \)’ profit upon winning is,

\[
\pi_{M_j}(\text{win}) = (1 - \alpha)\Pi_{M_j}^0 + \alpha - \frac{b_R}{w_j} N_{M_j},
\]

\[
= (1 - \alpha)\Pi_{M_j}^0 + \alpha - \alpha \frac{1}{1 - \beta_i \kappa_j w_j} \phi.
\]

To ensure the retailer’s winning, we need \( \pi_{M_j}(\text{lose}|R \text{ wins}) \geq \pi_{M_j}(\text{win}) \), which comes as,

\[
\beta_i \geq 1 - \frac{1}{\kappa_j w_j \frac{l_i}{l_i} + l_j \phi}.
\]

For \( M_i \), if the retailer is expected to win, upon losing to the retailer, \( M_i \)’s profit comes as,

\[
\pi_{M_i}(\text{lose}|R \text{ wins}) = (1 - \alpha)\Pi_{M_i}^0 + \alpha l_i (1 - \phi) - \beta_i w_j \alpha \frac{l_i + l_j \phi}{N_{M_j}} N_R,
\]

\[
= (1 - \alpha)\Pi_{M_i}^0 + \alpha l_i (1 - \phi) - \alpha \beta_i w_j \kappa_j (l_i + l_j \phi).
\]

If deviating to win and paying for the retailer’s bid, \( M_i \)'s profit upon winning is,

\[
\pi_{M_i}(\text{win}) = (1 - \alpha)\Pi_{M_i}^0 + \alpha - \frac{1}{w_i} \frac{1}{1 - \beta_i N_R} N_{M_i}.
\]

To ensure \( M_i \) will not deviate to win, we need \( \pi_{M_i}(\text{lose}|R \text{ wins}) \geq \pi_{M_i}(\text{win}) \), which comes as,

\[
(l_j + l_i \phi) + \beta_i w_j \kappa_j (l_i + l_j \phi) \leq \frac{1}{1 - \beta_i \kappa_i w_i} \phi,
\]

\[i.e., \quad \kappa_j w_j \left( l_i + l_j \phi \right) \beta_i^2 + \left( l_j + l_i \phi - \kappa_j w_j (l_i + l_j \phi) \right) \beta_i + \left( \frac{\phi}{\kappa_i w_i} - (l_j + l_i \phi) \right) > 0.
\]

This is equivalent to,

\[
\beta_i \geq \left\{ \begin{array}{ll}
0, & \kappa_i w_i (l_j + l_i \phi) \leq \phi, \\
\frac{\kappa_j w_j (l_i + l_j \phi)(l_i + l_i \phi) - (\kappa_j w_j (l_i + l_j \phi) + l_i + l_i \phi)^2 - 4 \phi (\kappa_j w_j (l_i + l_j \phi))}{2 \kappa_j w_j (l_i + l_j \phi)} & \equiv \beta_i, \\
\alpha, & \kappa_i w_i (l_j + l_i \phi) > \phi.
\end{array} \right.
\]

In arriving at the first line, we apply the result that under the assumption of \( \kappa_j w_j \leq \kappa_i w_i \), if \( \kappa_i w_i (l_j + l_i \phi) \leq \phi \), we also have \( \kappa_j w_j (l_i + l_j \phi) \leq \phi \leq l_i + l_j \phi \).

Now we combine the conditions required for both \( M_i \) and \( M_j \) to underbid the retailer. Given \( \kappa_j w_j \leq \kappa_i w_i \), when \( \kappa_i w_i (l_j + l_i \phi) \leq \phi \), both conditions require \( \beta_i \geq 0 \) to ensure the retailer’s winning. This aligns with our previous analysis that the retailer will win herself even without sponsorship in this region. When \( \kappa_j w_j (l_i + l_j \phi) < \phi < \kappa_i w_i (l_j + l_i \phi) \), we have \( 1 - \frac{1}{\kappa_j w_j (l_i + l_j \phi)} < 0 < \beta_i \), thus to ensure the retailer’s winning, we need \( \beta_i > \beta_i \). When \( \kappa_j w_j (l_i + l_j \phi) > \phi \) and \( \kappa_i w_i (l_j + l_i \phi) > \phi \), we have \( \beta_j > 1 - \frac{1}{\kappa_j w_j (l_i + l_j \phi)} \) (this requires \( \kappa_j w_j (l_i + l_j \phi) = \phi > (\kappa_j w_j - 1)(l_j + l_i \phi) \), which is true under in this case), thus to ensure the retailer’s winning, we again need \( \beta_i > \beta_i \). We summarize the condition for the
retailer to win in equilibrium with \( M_i \)'s sponsorship alone as,

\[
\beta_i \geq \begin{cases} 
0, & \kappa_i w_i(l_j + l_i \phi) \leq \phi, \\
\tilde{\beta}_i, & \kappa_i w_i(l_j + l_i \phi) > \phi.
\end{cases}
\]

Notice that when \( \kappa_i w_i(l_j + l_i \phi) > \phi \), we can prove that \( \tilde{\beta}_i > \tilde{\beta} \). This implies that it requires a higher sponsorship rate for \( M_i \) to sponsor the retailer to win alone, compared with the case when both manufacturers provide sponsorship.

When the above condition is not met, that is, when \( \kappa_i w_i(l_j + l_i \phi) > \phi \) and \( \beta_i < \tilde{\beta}_i \), \( M_i \) will deviate to win over the retailer herself, paying either the retailer’s bid \( b_R = \frac{1}{1-\beta_i} \frac{\phi}{N_R} \) or \( M_j \)'s weighted bid \( w_j b_{M_j} = w_j \frac{\phi}{N_{M_j}} \), depending on whether \( \beta_i < 1 - \frac{\phi}{\kappa_i w_j} \) or \( 1 - \frac{\phi}{\kappa_j w_j} \leq \beta_i < \tilde{\beta}_i \).

To summarize, under the assumption of \( \kappa_j w_j \leq \kappa_i w_i \), given \( \beta_i > 0, \beta_j = 0 \), we have that,

- If \( \kappa_i w_i(l_j + l_i \phi) \leq \phi \), the retailer wins regardless of the sponsorship rate from \( M_i \);
- If \( \phi < \kappa_i w_i(l_j + l_i \phi) \) and \( \phi \leq \kappa_j w_j \),
  - when \( \beta_i < 1 - \frac{\phi}{\kappa_j w_j} \), \( M_i \) wins while \( M_j \) ranks in the second position in the auction;
  - when \( 1 - \frac{\phi}{\kappa_j w_j} \leq \beta_i < \tilde{\beta}_i \), \( M_i \) wins while \( R \) ranks in the second position in the auction;
  - when \( \beta_i \geq \tilde{\beta}_i \), the retailer wins.
- If \( \kappa_j w_j < \phi < \kappa_i w_i(l_j + l_i \phi) \),
  - when \( \beta_i < \tilde{\beta}_i \), \( M_i \) wins while \( R \) ranks in the second position in the auction;
  - when \( \beta_i \geq \tilde{\beta}_i \), the retailer wins.

Finally, let’s consider the case when \( \beta_j > 0, \beta_i = 0 \). Similarly, the minimal sponsorship rate \( \beta_j \) to make it incentive compatible for \( M_i \) and \( M_j \) not to deviate to win over the retailer comes as,

\[
\beta_j \geq 1 - \frac{1}{\kappa_i w_i l_j + l_i \phi},
\]

\[
\beta_j \geq \frac{\kappa_i w_i(l_j + l_i \phi) - (l_i + l_j \phi) + \sqrt{(\kappa_i w_i(l_j + l_i \phi) + l_i + l_j \phi)^2 - 4\phi \kappa_i w_i (l_i + l_j \phi)}}{2\kappa_i w_i(l_j + l_i \phi)} \equiv \tilde{\beta}_j.
\]

Given \( \kappa_j w_j \leq \kappa_i w_i \), this can be further simplified into,

\[
\beta_i \geq \begin{cases} 
0, & \kappa_i w_i(l_j + l_i \phi) \leq \phi, \\
\max \left\{ 1 - \frac{\phi}{\kappa_i w_i l_j + l_i \phi}, \tilde{\beta}_j \right\}, & \kappa_i w_i(l_j + l_i \phi) > \phi.
\end{cases}
\]

Notice that here we do not further simplify \( \max \left\{ 1 - \frac{\phi}{\kappa_i w_i l_j + l_i \phi}, \tilde{\beta}_j \right\} \), since keeping it as a whole will facilitate derivation of the equilibrium sponsorship rates later.

When \( \beta_i \) does not meet the above condition, that is, when \( \kappa_i w_i(l_j + l_i \phi) > \phi \) and \( \beta_i < \max \left\{ 1 - \frac{\phi}{\kappa_i w_i l_j + l_i \phi}, \tilde{\beta}_j \right\} \), \( M_i \) wins the auction. Upon \( M_i \)'s winning, \( R \)'s bid ranks in the second position if \( \beta_i \geq 1 - \frac{\phi}{\kappa_j w_j} \), while \( M_j \)'s bid ranks in the second position if \( \beta_j < 1 - \frac{\phi}{\kappa_j w_j} \).
To summarize, under the assumption of $\kappa_j w_j \leq \kappa_i w_i$, given $\beta_i = 0, \beta_j > 0$, we have that,

- If $\kappa_i w_i (l_j + l_i \phi) \leq \phi$, the retailer wins regardless of the sponsorship rate from $M_j$;
- If $\kappa_i w_i (l_j + l_i \phi) > \phi$,
  
  - when $\beta_j < 1 - \frac{\phi}{\kappa_j w_j}$ and $\beta_j < \max\{1 - \frac{1}{\kappa_i w_i l_j + l_i \phi}, \tilde{\beta}_j\}$, $M_i$ wins while $M_j$ wins in the second position in the auction;
  - when $1 - \frac{\phi}{\kappa_j w_j} \leq \beta_j < \max\{1 - \frac{1}{\kappa_i w_i l_j + l_i \phi}, \tilde{\beta}_j\}$, $M_i$ wins while $R$ ranks in the second position in the auction;
  - when $\beta_j \geq \max\{1 - \frac{1}{\kappa_i w_i l_j + l_i \phi}, \tilde{\beta}_j\}$, the retailer wins.

We now write down both manufacturers’ profit functions given $\beta_i \geq 0, \beta_j \geq 0$. To summarize, without loss of generality, given $\kappa_i w_i \geq \kappa_j w_j$.

$$
\pi_{M_i}(\beta_i, \beta_j) =
\begin{cases}
(1 - \alpha)\Pi_{M_i} + \alpha l_i (1 - \phi) - \alpha \beta w_j \kappa_j (l_i + l_j \phi), & \kappa_j w_j \leq \kappa_i w_i \leq \frac{\phi}{l_j + l_i \phi}, \text{ or,} \\
\phi < \kappa_i w_i (l_j + l_i \phi), \beta_i \geq \bar{\beta}_i, \beta_j = 0, \text{ or,} \\
\phi < \kappa_i w_i (l_j + l_i \phi), \beta_i = 0, \beta_j \geq \max\{1 - \frac{1}{\kappa_i w_i l_j + l_i \phi}, \tilde{\beta}_j\}, \\
\end{cases}
$$

$$
\pi_{M_j}(\beta_i, \beta_j) =
\begin{cases}
(1 - \alpha)\Pi_{M_j} + \alpha l_j (1 - \phi) - \alpha \beta w_j \kappa_j (l_i + l_j \phi), & \kappa_j w_j \leq \kappa_i w_i \leq \frac{\phi}{l_j + l_i \phi}, \text{ or,} \\
\phi < \kappa_i w_i (l_j + l_i \phi), \beta_i \geq \bar{\beta}_i, \beta_j = 0, \text{ or,} \\
\phi < \kappa_i w_i (l_j + l_i \phi), \beta_i = \beta_j = \beta \geq \max\{1 - \frac{1}{\kappa_i w_i l_j + l_i \phi}, \tilde{\beta}_j\}, \\
\end{cases}
$$

Now we move on to the next step to derive the optimal sponsorship rates in the general setup. We first show that if both manufacturers sponsor the retailer at a positive rate (i.e., $\beta_i > 0, \beta_j > 0$), given the retailer as the winner, manufacturers will choose their sponsorship rates satisfying $\beta_i + \beta_j = \tilde{\beta}$, so that the upper bound on the equilibrium bid, $\tilde{b}_M(\beta_1, \beta_2)$ is driven down to 0 and manufacturers bid $\tilde{b}_M(\beta_1, \beta_2) = 0$ in equilibrium. This is the best scenario for manufacturers where the retailer is ensured to win, but manufacturers pay zero subsidy to the retailer. This observation greatly simplifies our analysis.
Now we derive $M_i$’s best-response function on his sponsorship rate. We solve the problem by considering $\beta_j = 0$ and $\beta_j > 0$ separately, since $M_j$’s bid may be different depending on whether he is sponsoring or not,

- First consider the boundary case given $\beta_j = 0$. From $M_i$’s profit functions given above, it is easy to see that when $\kappa_i w_i (l_i + l_i \phi) \leq \phi$, $M_i$’s best response is not to sponsor, i.e., $\beta^*_i = 0$. When $\kappa_i w_i (l_i + l_i \phi) > \phi$, $M_i$’s best response is either to sponsor the retailer just high enough at $\beta_i = \tilde{\beta}_i$ to ensure her winning, or not sponsor and win the auction himself, depending on which gives him a higher profit.

- Consider the parameter space with $\kappa_i w_i (l_i + l_i \phi) > \phi$, and $\kappa_j w_j > \phi$. In this case, if $M_i$ does not sponsor the retailer, we come back to the case without sponsorship, where $M_i$ will win the auction, with $M_j$’s bid ranked in the second position. $M_i$’s profit upon winning is then $(1 - \alpha)\Pi^0_{M_i} + \alpha - \frac{\phi w_i \kappa_j}{\kappa_i \kappa_j}$. To ensure $M_i$ having a higher profit from sponsoring the retailer to win, we then need the following condition,

\[ l_j + l_i \phi + \tilde{\beta}_i \kappa_j w_j (l_i + l_i \phi) < \frac{\kappa_j w_j}{\kappa_i w_i}. \]

This translates into the following condition,

\[ l_j + l_i \phi + (1 - \frac{\phi}{\kappa_j w_j}) \kappa_j w_j (l_i + l_i \phi) < \frac{\kappa_j w_j}{\kappa_i w_i}, \]

\[ i.e., \kappa_j w_j > \frac{l_j (1 - \phi^2)}{\kappa_i w_i - (l_i + l_i \phi)}. \]

- Then consider the parameter space with $\kappa_i w_i (l_i + l_i \phi) > \phi$, and $\kappa_j w_j < \phi$. In this case, if $M_i$ does not sponsor the retailer, then $M_i$ will win the auction, with $R$’s bid ranking in the second position. $M_i$’s profit upon winning is then $(1 - \alpha)\Pi^0_{M_i} + \alpha - \frac{\phi \kappa_i w_i}{\kappa_i \kappa_j}$. To ensure the above profit level is lower than $M_i$’s profit when sponsoring the retailer to win, we need the following condition,

\[ (l_j + l_i \phi) + \beta \kappa_j w_j (l_i + l_i \phi) < \frac{\phi}{\kappa_i w_i}, \]  \hspace{1cm} (OA1)

However, this cannot be true in the parameter space we are considering where $\kappa_i w_i (l_j + l_i \phi) > \phi$. That is to say, given $\kappa_i w_i (l_i + l_i \phi) > \phi$, and $\kappa_j w_j < \phi$, $M_i$ has no incentive to sponsor the retailer to win in this scenario.

- To summarize, assuming $\kappa_i w_i \geq \kappa_j w_j$, given $\beta_j = 0$, $M_i$ sponsors the retailer to win alone when $\phi < \kappa_i w_i (l_j + l_i \phi)$, $\kappa_j w_j > \frac{l_j (1 - \phi^2)}{\kappa_i w_i - (l_i + l_i \phi)}$.

- Given $\beta_j > 0$, in a similar way, we can find that $M_i$ sponsors the retailer at $\tilde{\beta} = \beta_j$ if $\phi < \kappa_i w_i (l_j + l_i \phi) < \kappa_j w_j$, $0 < \beta_j < \tilde{\beta}$ and 0 otherwise.
Combined together, given \( \beta_j \geq 0 \), \( M_i \)'s best-response function comes as,

\[
\beta_i^*(\beta_j) = \begin{cases} 
\tilde{\beta} - \beta_j, & \phi < \kappa_i wi (l_j + li \phi) < \kappa_j w_j, 0 < \beta_j < \tilde{\beta}, \\
\tilde{\beta} i, & \phi < \kappa_i wi (l_j + li \phi), \kappa_j w_j > \frac{l_j (1 - \phi^2)}{\kappa_i wi (l_j + li \phi)}, \beta_j = 0, \\
0, & \text{otherwise}. 
\end{cases}
\]

Similarly, we derive \( M_j \)'s best-response function given \( M_i \)'s sponsorship rate.

- When \( \beta_i = 0 \), if \( \kappa_i wi (l_j + li \phi) \leq \phi \), \( M_j \)'s best response is not to sponsor the retailer. If \( \kappa_i wi (l_j + li \phi) > \phi \), \( M_j \) will sponsor the retailer at \( \max\{1 - \frac{1}{\kappa_i wi (l_j + li \phi)}, \tilde{\beta} \} \), if doing so indeed increases his profit than not sponsoring and letting \( M_i \) win. This requires (1 - \( \alpha \))\( \Pi^0_{M_j} + \alpha l_j (1 - \phi) - \alpha \beta j \kappa_i wi (l_j + li \phi) > (1 - \alpha) \Pi^0_{M_j} \), which translates into,

\[
\beta_j < \frac{l_j (1 - \phi)}{(l_j + li \phi) \kappa_i wi}
\]

This requires both \( 1 - \frac{1}{\kappa_i wi (l_j + li \phi)} \) and \( \tilde{\beta} \) to be smaller than \( \frac{l_j (1 - \phi)}{(l_j + li \phi) \kappa_i wi} \). To ensure \( 1 - \frac{1}{\kappa_i wi (l_j + li \phi)} < \frac{l_j (1 - \phi)}{(l_j + li \phi) \kappa_i wi} \), we need,

\[
\kappa_i wi < 1
\]

The other condition required, \( \tilde{\beta} < \frac{l_j (1 - \phi)}{(l_j + li \phi) \kappa_i wi} \), is equivalent to that the maximal sponsorship allowed for \( M_j \) to make a profit \( \beta_j = \frac{l_j (1 - \phi)}{(l_j + li \phi) \kappa_i wi} > \tilde{\beta} \). This simplifies into,

\[
\kappa_i wi \leq \frac{l_j (1 - \phi)}{(l_j + li \phi)(1 - \frac{\phi}{\kappa_j w_j})}
\]

To summarize, assuming \( \kappa_i wi \geq \kappa_j w_j \), given \( \beta_i = 0 \), \( M_j \) will sponsor the retailer to win when \( \kappa_i wi (l_j + li \phi) > \phi, \kappa_i wi \leq \frac{l_j (1 - \phi)}{(l_j + li \phi)(1 - \frac{\phi}{\kappa_j w_j})}, \kappa_i wi < 1 \).

- Given \( \beta_i > 0 \), it is easy to find that \( M_j \) will sponsor the retailer at \( \tilde{\beta} - \beta_i \) if \( \kappa_i wi (l_j + li \phi) > \phi, 0 < \beta_i < \tilde{\beta} \). Otherwise, \( M_j \) will not sponsor the retailer.

Combined together, \( M_j \)'s best-response function given \( \beta_i \geq 0 \) is,

\[
\beta_j^*(\beta_i) = \begin{cases} 
\tilde{\beta} - \beta_i, & \kappa_i wi (l_j + li \phi) > \phi, 0 < \beta_i < \tilde{\beta}, \\
\max\{1 - \frac{1}{\kappa_i wi (l_j + li \phi)}, \tilde{\beta} \}, & \kappa_i wi (l_j + li \phi) > \phi, \kappa_i wi \leq \frac{l_j (1 - \phi)}{(l_j + li \phi)(1 - \frac{\phi}{\kappa_j w_j})}, \kappa_i wi \leq 1, \\
0, & \text{o.w.}
\end{cases}
\]

Intersecting the two best-response functions, given \( \kappa_i wi \geq \kappa_j w_j \), we get the following equilibrium
sponsorship rates,

\[(\beta^*_i, \beta^*_j) \in \begin{cases} 
(\beta_i > 0, \beta_j > 0 | \beta_i + \beta_j = \tilde{\beta}), & \kappa_i w_i (l_j + l_i \phi) > \phi, \kappa_j w_j > \kappa_i w_i (l_j + l_i \phi), \\
(\tilde{\beta}_i, 0), & \kappa_i w_i (l_j + l_i \phi) > \phi, \frac{l_j (1 - \phi^2)}{\kappa_i w_i (l_j + l_i \phi)} < \kappa_j w_j \leq \kappa_i w_i, \\
\left\{0, \max\left\{1 - \frac{\phi}{\kappa_i w_i (l_j + l_i \phi)}, \tilde{\beta}_j\right\}\right\}, & \kappa_j w_i (l_j + l_i \phi) > \phi, \kappa_i w_i \leq 1, \kappa_j w_j \leq \kappa_i w_i, \\
\{(0, 0)\}, & \text{otherwise}. 
\end{cases}\]

The solution for the other case when \(\kappa_j w_j > \kappa_i w_i\) can be derived symmetrically. Combined together, we have the overall solution as presented by Proposition 6.

### OA1.3 Alternative Equilibrium Bid Selections

In the auctions of category keywords as well as the general keywords, we have pointed out that sometimes, the retailer’s winning and manufacturers’ head-to-head competition can both be an equilibrium. Whenever this happens, we select the retailer’s winning as the equilibrium outcome. Here, we would like to show that if we select the alternative equilibrium outcome, our main insight holds even in a larger parameter space.

We sketch the proof graphically. Without sponsorship, the two equilibria mentioned above co-exist in the double shaded area in Figure 1(a) (the area corresponds to the case when \(\kappa_i w_i < \frac{1}{l_j + l_i \phi}\) and \(\phi < \kappa_j w_j < \kappa_i w_i\)). Under this region, if we select the manufacturers’ winning as the equilibrium outcome instead, it would then be in manufacturers’ interest to sponsor the retailer to win within this area, as Figure 1(b) demonstrates. Consequently, under the alternative equilibrium selection, the retailer gets sponsored to win even in a larger parameter space, so our results hold even more true.

![Equilibrium outcome with alternative equilibrium bid selection](image-url)
OA2   Equilibrium Analysis for case of Uniform Sponsorship Rate

We first derive $M_i$’s best response on the sponsorship rate given $M_j$’s sponsorship rate $\beta_j$. Given the optimal bidding strategy, we first write down manufacturer $i$’s profit from category keyword auctions, $M_i$’s brand keyword auctions, and $M_j$’s brand keyword auctions, respectively. Denote superscript $br_i$ as $M_i$’s brand keyword auctions, and superscript $br_j$ as $M_j$’s brand keyword auctions.

$$\pi^ca_{M_i}(\beta_i, \beta_j) = \left\{ \begin{array}{ll}
\frac{\theta}{2}(1-\alpha) + \frac{1-\phi}{2}(1-\theta + \theta \alpha) - \beta_i b^ca_{M_i}(\beta_1, \beta_2) N^ca_R, & b^ca_R(\beta_i, \beta_j) \geq b^ca_M \mbox{ and } \beta_j > 0, \\
\frac{\phi}{2}(1-\alpha) + \frac{1-\phi}{2}(1-\theta + \theta \alpha) - \beta_i b^ca_{M_i} N^ca_R, & \kappa^ca > \frac{2\phi}{1+\phi}, \beta_j > \frac{\kappa^ca}{\kappa^ca - 1 + \sqrt{(\kappa^ca + 1)^2 - 1 + \frac{2\phi}{1+\phi}}}, \mbox{ and } \beta_j = 0, \\
\frac{\theta}{2}(1-\alpha) + \frac{1-\phi}{2}(1-\theta)(1-\alpha), & \kappa^ca > \frac{2\phi}{1+\phi}, \beta_j < \frac{\kappa^ca}{\kappa^ca + 1 + \sqrt{(\kappa^ca + 1)^2 - 1 + \frac{2\phi}{1+\phi}}}, \mbox{ and } \beta_j > 0,
\end{array} \right. $$

$$\pi^br_j(\beta_i, \beta_j) = \left\{ \begin{array}{ll}
\frac{1}{2} (1-\alpha) + \frac{1}{2} (1-\theta + \theta \alpha)(1-\phi), & b^br_j(\beta_i, \beta_j) \geq b^br_M, \\
\frac{1}{2} (1-\alpha) - \frac{1}{2} (1-\theta + \theta \alpha)(1-\phi), & \beta_i b^br_{M_i} N^br_R, \beta_j > 0, \\
\frac{1}{2} (1-\alpha) + \frac{1}{2} (1-\theta)(1-\alpha)(1-\phi), & \beta_j = 0,
\end{array} \right. $$

$$\pi^br_j(\beta_i, \beta_j) = \left\{ \begin{array}{ll}
\frac{1}{2} (1-\alpha) \beta_i \frac{1}{2} + \frac{1}{2} (1-\theta + \theta \alpha)(1-\phi), & b^br_j(\beta_i, \beta_j) \geq b^br_M \mbox{ and } \beta_j > 0, \\
\frac{1}{2} (1-\alpha) \beta_i \frac{1}{2} + \frac{1}{2} (1-\theta + \theta \alpha)(1-\phi), & \beta_j = 0, \\
\frac{1}{2} (1-\alpha) \beta_i \frac{1}{2} + \frac{1}{2} (1-\theta)(1-\alpha)(1-\phi), & \beta_j = 0.
\end{array} \right. $$

Averaging across all keywords, manufacturer $i$’s expected profit is,

$$\pi_M(\beta_i, \beta_j) = q\pi^ca_{M_i}(\beta_i, \beta_j) + \frac{1- q}{2} \pi^br_j(\beta_i, \beta_j) + \frac{1- q}{2} \pi^br_j(\beta_i, \beta_j)$$

To simplify notations, in this appendix, we denote the minimal total sponsorship rate to ensure the retailer to win category keyword auctions, $1 - \frac{1}{\kappa^ca} \frac{2\phi}{1+\phi}$, as $\beta^ca$, and denote the minimal total sponsorship rate to ensure the retailer to win brand keyword auctions, $1 - \frac{1}{\kappa^br} \frac{2\phi}{1+\phi}$, as $\beta^br$. To maximize $\pi_M(\beta_1, \beta_2)$ with respect to $\beta_i$, we have the following response function of sponsorship rate for manufacturer $i$ given $\beta_j > 0$ as,

$$\beta^*_i(\beta_j > 0) = \left\{ \begin{array}{ll}
(\beta^ca - \beta_j)^+, & \kappa^ca > \frac{2\phi}{1+\phi}, \\
0, & \kappa^ca \leq \frac{2\phi}{1+\phi}.
\end{array} \right. $$

The solving process for $M_i$’s best response given $\beta_j > 0$ is as follows.

- Consider the case when $\kappa^ca \leq \frac{2\phi}{1+\phi}$. In this case we will also have $\kappa^br \leq \frac{1}{\kappa^br} \frac{2\phi}{1+\phi}$ according to the assumption. Since the retailer can win by herself even without $M_i$ sponsoring, we have $\beta_i = 0$.

- Next, consider the case when $\kappa^ca > \frac{2\phi}{1+\phi}$. We discuss the following sub-cases respectively.
– If $\kappa^{ca} > \frac{2\phi}{1+\phi}$, and $\kappa^{br} < \frac{2-2\phi}{2(1+2\phi)}$, since the retailer is going to win brand keyword auctions anyway, $M_i$ will make sure the retailer can further win category keyword auctions by sponsoring at $(\beta^{ca} - \beta_j)^+$. Note that this will not affect his profit in brand keyword cases since both $M_i$ and $M_j$ will bid 0 in their own brand keyword auctions when the retailer is expected to win.

– If $\kappa^{ca} > \frac{2\phi}{1+\phi}$, and $\frac{\phi}{\frac{1}{2} - l + (\frac{1}{2} + l)\phi} < \kappa^{br} \leq 1$, first suppose $\beta_j < \beta^{br}$, and then if $M_i$ sponsors at $(\beta^{ca} - \beta_j)$, compared with not sponsoring, $M_i$ increases profit in category keyword auctions by

\[
\left( \theta (1 - \alpha) + \frac{1-\phi}{2} (1 - \theta + \theta \alpha) \right) - \left( \theta (1 - \alpha) + \frac{1-\phi}{2} (1 - \theta)(1 - \alpha) \right) = \alpha (1 - \phi).
\]

Since $\beta^{ca} > \beta^{br}$, sponsoring at $\beta^{ca}$ will also make the retailer win $M_i$’s brand keyword auction. Thus $M_i$’s profit in his own brand keyword auctions is decreased by,

\[
\left[ (\frac{1}{2} + l) (1 - \alpha) \theta + (\frac{1}{2} + l) (1 - \theta)(1 - \phi) \right] \alpha - b^{br}_R(0, \beta_j)N^{br}_M
\]

\[
- \left[ (\frac{1}{2} + l) (1 - \alpha) \theta + (\frac{1}{2} + l) (1 - \theta + \theta \alpha)(1 - \phi) \right] = \alpha \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi - \frac{1}{\kappa^{br}} \frac{\alpha}{1 - \beta_j} \phi;
\]

Similarly, since the retailer will also win $M_j$’s brand keyword auctions, $M_i$’s profit in $M_j$’s keyword auctions is increased by,

\[
\left[ (\frac{1}{2} - l) (1 - \alpha) \theta + (\frac{1}{2} - l) (1 - \theta + \theta \alpha)(1 - \phi) \right]
\]

\[
- \left[ (\frac{1}{2} - l) (1 - \alpha) \theta + (\frac{1}{2} - l) (1 - \theta)(1 - \alpha)(1 - \phi) \right] = \alpha \left( \frac{1}{2} - l \right) (1 - \phi);
\]

Combined, $M_i$’s expected gain by sponsoring the retailer to win comes as,

\[
q \times \alpha \frac{1 - \phi}{2} - \frac{1-q}{2} \times \left[ \alpha \left( \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right) - \frac{1}{\kappa^{br}} \frac{\alpha}{1 - \beta_j} \phi \right] + \frac{1-q}{2} \times \alpha \left( \frac{1}{2} - l \right) (1 - \phi) = q \times \alpha \frac{1 - \phi}{2} + \frac{1-q}{2} \times \alpha \phi \left[ \frac{1}{\kappa^{br}} \frac{1}{1 - \beta_j} - 1 \right];
\]

Since $\kappa^{br} < 1$ and $\beta^{br} < 1$, the above amount is positive, implying $M_i$ has a net gain by sponsoring the retailer to win at $\beta^{ca} - \beta_j$.

Notice that it suffices to compare $M_i$ sponsoring at $\beta^{ca} - \beta_j$ versus not sponsoring as the
optimal response. This is because sponsoring at $\beta^{ca} - \beta_j$ weakly dominates $\beta^{br} - \beta_j$ for $M_i$; manufactures bid zero in their own branded keyword auctions when the retailer is expected to win; thus increasing sponsoring rate from $\beta^{br}$ to $\beta^{ca}$ will only improve $M_i$’s profit in category keyword auctions, without affecting its profit in brand keyword auctions.

In the other case when $\beta^{br} \leq \beta_j < \beta^{ca}$, it is obvious that $M_i$ will sponsor at $\beta^{ca} - \beta_j$ since this would only improve this profit compared with not sponsoring. As a summary, when $\kappa^{ca} > \frac{2\phi}{1+\phi}$, and $\frac{\phi}{2-l+(\frac{1}{2}+t)\phi} < \kappa^{br} \leq 1$, $M_i$’s best sponsor is to set $\beta_i^* = (\beta^{ca} - \beta_j)^+$.

- Finally, if $\kappa^{ca} > \frac{1+\phi^2}{1+\phi}$ and $\kappa^{br} > 1$, first suppose $\beta_j < \beta^{br}$. If $M_i$ sponsors at $\beta^{ca} - \beta_j$, compared with not sponsoring, his expected gain in profit, $q \times \alpha \left( \frac{1}{2} - \frac{\phi}{2-l+(\frac{1}{2}+t)\phi} \right)$, is positive since we have $\beta_j < \beta^{br} < 1 - \frac{\phi}{\kappa^{br}}$ under this scenario. Similarly it suffices to consider $M_i$ sponsoring at $\beta^{ca} - \beta_j$ since this is his optimal choice if he decides to sponsor the retailer to win. When $\kappa^{ca} > \frac{1+\phi^2}{1+\phi}$, $\kappa^{br} > 1$, and $\beta^{br} < \beta^{ca}$, it is obvious that $M_i$ will sponsor at $\beta^{ca} - \beta_j$ since this would only improve this profit compared with not sponsoring. Therefore, when $\kappa^{ca} > \frac{1+\phi^2}{1+\phi}$, $\kappa^{br} > 1$, $M_i$’s best sponsor is also to set $\beta_i^* = (\beta^{ca} - \beta_j)^+$.

Next consider the best response function of sponsorship rate for manufacturer $i$ given $\beta_j = 0$. The solution is,

\[
\beta_i^*(\beta_j = 0) = \begin{cases} 
\frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{\phi}{1+\phi}}}{2\kappa^{ca}}, & 2\frac{\phi}{1+\phi} < \kappa^{ca} < \frac{1+\phi^2}{1+\phi}, \text{ and } \kappa^{br} < \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}, q \geq q_1 \\
\beta^{br}, & \frac{2\phi}{1+\phi} < \kappa^{ca} < \frac{1+\phi^2}{1+\phi}, \text{ and } \kappa^{br} > \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}, q \geq q_1 \\
0, & \frac{2\phi}{1+\phi} < \kappa^{ca} < \frac{1+\phi^2}{1+\phi}, \text{ and } \kappa^{br} > \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}, q \geq q_2 \\
\text{or } 2\frac{\phi}{1+\phi} < \kappa^{ca} < \frac{1+\phi^2}{1+\phi}, \kappa^{br} > \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}, q \geq q_2, \text{ and } \kappa^{ca} > \frac{1+\phi^2}{1+\phi}, \frac{\phi}{2-l+(\frac{1}{2}+t)\phi} < \kappa^{br} < \sqrt{\frac{\phi}{2-l+(\frac{1}{2}+t)\phi}}, q < q_3 \\
\text{or } 2\frac{\phi}{1+\phi} < \kappa^{ca} < \frac{1+\phi^2}{1+\phi}, \kappa^{br} < \sqrt{\frac{\phi}{2-l+(\frac{1}{2}+t)\phi}}, q < q_3, \text{ and } \kappa^{ca} > \frac{2\phi}{1+\phi} < \frac{\phi}{2-l+(\frac{1}{2}+t)\phi} \\
\text{or } \frac{2\phi}{1+\phi} < \kappa^{ca} < \frac{1+\phi^2}{1+\phi}, \kappa^{br} < \sqrt{\frac{\phi}{2-l+(\frac{1}{2}+t)\phi}}, q < q_3, \text{ and } \kappa^{ca} > \frac{1+\phi^2}{1+\phi}, \frac{\phi}{2-l+(\frac{1}{2}+t)\phi} < \kappa^{br} < \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}, q < q_3 \\
0, & \frac{2\phi}{1+\phi} < \kappa^{ca} < \frac{1+\phi^2}{1+\phi}, \text{ and } \kappa^{br} > \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}, q < q_1 \\
\text{or } 2\frac{\phi}{1+\phi} < \kappa^{ca} < \frac{1+\phi^2}{1+\phi}, \kappa^{br} > \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}, q < q_1, \text{ and } \kappa^{ca} > \frac{1+\phi^2}{1+\phi}, \frac{\phi}{2-l+(\frac{1}{2}+t)\phi} < \kappa^{br} < \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}, q < q_2 \\
\text{or } \frac{2\phi}{1+\phi} < \kappa^{ca} < \frac{1+\phi^2}{1+\phi}, \kappa^{br} > \sqrt{\frac{\phi}{2-l+(\frac{1}{2}+t)\phi}}, q < q_2, \text{ and } \kappa^{ca} > \frac{1+\phi^2}{1+\phi}, \frac{\phi}{2-l+(\frac{1}{2}+t)\phi} < \kappa^{br} < \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}, q < q_3
\end{cases}
\]

where $q_1 = \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{\phi}{1+\phi}}}{2\kappa^{ca}} \frac{\frac{\phi}{2-l+(\frac{1}{2}+t)\phi}}{\frac{\phi}{2-l+(\frac{1}{2}+t)\phi} - \kappa^{br}} \frac{1}{\frac{1}{2} - \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}}$, $q_2 = \frac{1}{\frac{1}{2} - \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}} \left\{ \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{\phi}{1+\phi}}}{2\kappa^{ca}} \frac{\frac{\phi}{2-l+(\frac{1}{2}+t)\phi}}{\frac{\phi}{2-l+(\frac{1}{2}+t)\phi} - \kappa^{br}} \frac{1}{\frac{1}{2} - \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}} \right\}$, and $q_3 = \frac{1}{\frac{1}{2} - \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}} \left\{ \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{\phi}{1+\phi}}}{2\kappa^{ca}} \frac{\frac{\phi}{2-l+(\frac{1}{2}+t)\phi}}{\frac{\phi}{2-l+(\frac{1}{2}+t)\phi} - \kappa^{br}} \frac{1}{\frac{1}{2} - \frac{\phi}{2-l+(\frac{1}{2}+t)\phi}} \right\}$.

Below are the details for the solving process.
• If $\kappa^{ca} \leq \frac{2\phi}{1+\phi}$, we also have $\kappa^{br} \leq \frac{\phi}{1-I-(\frac{1}{4}+1)\phi}$ by assumption. Since the retailer can win by herself even without $M_i$ sponsoring, it is obvious that $\beta_i = 0$.

• Next, consider the case when $\frac{2\phi}{1+\phi} < \kappa^{ca} < \frac{1+\phi^2}{1+\phi}$. We discuss the following subcases respectively.

  – Consider the case where $\frac{2\phi}{1+\phi} < \kappa^{ca} < \frac{1+\phi^2}{1+\phi}$, and $\kappa^{br} < \frac{\phi}{1-I-(\frac{1}{4}+1)\phi}$. In this region, the retailer wins brand keyword auctions regardless of $M_i$’s sponsorship. If $M_i$ further sponsors at $\beta_i = \frac{\kappa^{ca}-1+\sqrt{(\kappa^{ca}+1)^2-4\frac{2\phi}{1+\phi}}}{2\kappa^{ca}}$, this will lead the retailer to win category keywords also. Compared with not sponsoring, $M_i$ increases his profit at category keyword auctions by

$$
\left(\frac{\theta}{2}(1-\alpha) + \frac{1-\phi}{2}(1-\theta+\theta\alpha)\right) - \frac{\kappa^{ca}-1+\sqrt{(\kappa^{ca}+1)^2-4\frac{2\phi}{1+\phi}}}{2\kappa^{ca}} b^{br}_M N^{ca}_R
$$

However, sponsoring or not, the retailer will win $M_i$’s brand keywords. Thus sponsoring at $\frac{\kappa^{ca}-1+\sqrt{(\kappa^{ca}+1)^2-4\frac{2\phi}{1+\phi}}}{2\kappa^{ca}}$ will not change $M_i$’s profit in his own brand keyword auctions. However, since $M_j$ bids positively at $M_j$’s brand keyword auctions when $M_j$ is not sponsoring, $M_i$ needs to pay sponsorship fee if sponsoring. Therefore, we have $M_i$’s profit in $M_j$’s keyword auctions is decreased by,

$$
\frac{\kappa^{ca}-1+\sqrt{(\kappa^{ca}+1)^2-4\frac{2\phi}{1+\phi}}}{2\kappa^{ca}} \times b^{br}_M \times N^{br}_R
$$

Combined, $M_i$’s net gain by sponsoring the retailer at $\beta^{br}$ compared with not sponsoring is,

$$
q \times \alpha \left(\frac{1-\phi}{2} - \frac{1+\phi}{4} \left(\kappa^{ca} - \frac{1+\sqrt{(\kappa^{ca}+1)^2-4\frac{2\phi}{1+\phi}}}{1+\phi}\right)\right)
$$

$$
- \frac{1-q}{2} \times \frac{\kappa^{ca}-1+\sqrt{(\kappa^{ca}+1)^2-4\frac{2\phi}{1+\phi}}}{2} \alpha \left[\frac{1}{2} - l + \left(\frac{1}{2} + l\right) \phi\right],
$$

$$
= q\alpha \left[\frac{1-\phi}{2} - \frac{\kappa^{ca}-1+\sqrt{(\kappa^{ca}+1)^2-4\frac{2\phi}{1+\phi}}}{4} \left[\frac{1}{2} + l + \left(\frac{1}{2} - l\right) \phi\right]\right]
$$

$$
- \alpha \left(\kappa^{ca} - \frac{1+\sqrt{(\kappa^{ca}+1)^2-4\frac{2\phi}{1+\phi}}}{4} \left[\frac{1}{2} - l + \left(\frac{1}{2} + l\right) \phi\right]\right)
$$

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Consider the case where \( q > q_1 \), where \( q_1 = \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2 \phi}{1 + \phi} \left( \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right)}}{1 - \phi - \frac{2 \phi}{1 + \phi} \left( \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right)} \). The above amount is positive if and only if \( \kappa^{ca} < \frac{1 + \phi^2}{1 + \phi} \) and \( \frac{\phi}{2 - l + \left( \frac{1}{2} + l \right) \phi} < \kappa^{br} < \frac{\phi}{\sqrt{2 - l + \left( \frac{1}{2} + l \right) \phi}} \). First, if \( M_i \) sponsors at \( \beta_i = \beta^{br} < \beta^{ca} \), this does not affect its profit in category keyword auctions (since two manufactures still compete head to head). However, if not sponsoring, \( M_i \) wins its own brand keyword auctions. If sponsoring, the retailer wins \( M_i \)'s brand keyword auctions instead. Comparing with not sponsoring, \( M_i \)'s profit in his own brand keyword auctions is decreased by,

\[
\left[ \left( \frac{1}{2} + l \right) \theta + \left( \frac{1}{2} + l \right) (1 - \theta)(1 - \alpha)(1 - \phi) + \left[ 1 - \left( \frac{1}{2} + l \right) \theta \right] \right] \alpha - b^{br}_R(0, 0) N^{br}_M
\]

\[- \left[ \left( \frac{1}{2} + l \right) (1 - \alpha) \theta + \left( \frac{1}{2} + l \right) (1 - \theta + \theta \alpha)(1 - \phi) \right],
\]

\[= \alpha \left[ \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right] - \frac{1}{\kappa^{br} \alpha \phi}.
\]

Meanwhile, \( M_i \)'s profit in \( M_j \)'s keyword auctions is increased by,

\[
\left[ \left( \frac{1}{2} - l \right) (1 - \alpha) \frac{\theta}{2} + \left( \frac{1}{2} - l \right) (1 - \theta + \theta \alpha)(1 - \phi) - \beta^{br} b^{br}_M N^{br}_R \right]
\]

\[- \left[ \left( \frac{1}{2} - l \right) (1 - \alpha) \frac{\theta}{2} + \left( \frac{1}{2} - l \right) (1 - \theta)(1 - \alpha)(1 - \phi) \right],
\]

\[= \alpha \left[ \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right] \left( 1 - \kappa^{br} \right).
\]

Combined, \( M_i \)'s net gain by sponsoring the retailer at \( \beta^{br} \) compared with not sponsoring is,

\[- \frac{1 - q}{2} \times \left[ \alpha \left[ \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right] - \frac{1}{\kappa^{br} \alpha \phi} \right] + \frac{1 - q}{2} \times \left[ \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right] \left( 1 - \kappa^{br} \right)
\]

\[= \alpha \frac{1 - q}{2} \times \left[ \frac{\phi}{\kappa^{br} \phi} - \kappa^{br} \left( \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right) \right];
\]

The above amount is positive if and only if \( \kappa^{br} < \frac{\phi}{\sqrt{\frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi}} \).

Next consider the other option for \( M_i \) to sponsor at \( \beta_i = \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2 \phi}{1 + \phi} \left( \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right)}}{2 \kappa^{ca}} \). Compared
with not sponsoring, $M_i$ increases his profit at category keyword auctions by,

$$
\left(\frac{\theta}{2}(1-\alpha) + \frac{1-\phi}{2}(1-\theta + \theta\alpha)\right) - \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4\frac{2\phi}{1+\phi}}}{2\kappa^{ca}}b^{ca}_{M}N^{ca}_{R}
$$

$$
- \left(\frac{\theta}{2}(1-\alpha) + \frac{1-\phi}{2}(1-\theta)(1-\alpha)\right)
$$

$$
= \alpha \left(\frac{1-\phi}{2} - \frac{1+\phi}{4}\left(\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4\frac{2\phi}{1+\phi}}\right)\right) > 0;
$$

Since $\frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4\frac{2\phi}{1+\phi}}}{2\kappa^{ca}}$ is also higher than $\beta^{br}$, it will make the retailer win $M_i$'s brand keyword auctions. Comparing with not sponsoring, $M_i$'s profit in his own brand keyword auctions is decreased by,

$$
\left[\left(\frac{1}{2} + l\right)\theta + \left(\frac{1}{2} + l\right)(1-\theta)(1-\alpha)(1-\phi) + \left[1 - \left(\frac{1}{2} + l\right)\theta\right]\alpha - b^{br}_{R}(0,0)N^{br}_{M}\right]
$$

$$
- \left[\left(\frac{1}{2} + l\right)(1-\alpha)\theta + \left(\frac{1}{2} + l\right)(1-\theta + \theta\alpha)(1-\phi)\right],
$$

$$
= \alpha \left[\frac{1}{2} - l + \left(\frac{1}{2} + l\right)\phi\right] - \frac{1}{\kappa^{br}}\alpha\phi.
$$

Meanwhile, $M_i$'s profit in $M_j$'s keyword auctions is increased by,

$$
\left[\left(\frac{1}{2} - l\right)(1-\alpha)\theta + \left(\frac{1}{2} - l\right)(1-\theta + \theta\alpha)(1-\phi) - \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4\frac{2\phi}{1+\phi}}}{2\kappa^{ca}}b^{br}_{M}N^{br}_{R}\right]
$$

$$
- \left[\left(\frac{1}{2} - l\right)(1-\alpha)\theta + \left(\frac{1}{2} - l\right)(1-\theta)(1-\alpha)(1-\phi)\right],
$$

$$
= \alpha \left(\frac{1}{2} - l\right)(1-\phi) - \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4\frac{2\phi}{1+\phi}}}{2}\alpha \left[\frac{1}{2} - l + \left(\frac{1}{2} + l\right)\phi\right].
$$

Combined, $M_i$'s net gain by sponsoring the retailer at $\beta_i = \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4\frac{2\phi}{1+\phi}}}{2\kappa^{ca}}$ compared
with not sponsoring is,

\[
q \times \alpha \left( \frac{1 - \phi}{2} - \frac{1 + \phi}{4} \left( \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1 + \phi}}}{2} \right) \right) - \frac{1 - q}{2} \times \left[ \alpha \left( \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right) - \frac{1}{\kappa^{br}} \alpha \phi \right] + \frac{1 - q}{2} \times \left( \alpha \left( \frac{1}{2} - l \right) (1 - \phi) - \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1 + \phi}}}{2} \alpha \left[ \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right] \right)
\]

\[
= \frac{\alpha}{2} \left[ \phi \left( \frac{1}{\kappa^{br}} - 1 \right) - \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1 + \phi}}}{2} \left[ \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right] \right] + \frac{q \alpha}{2} \left[ 1 - \frac{\phi}{\kappa^{br}} - \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1 + \phi}}}{2} \left[ \frac{1}{2} + l + \left( \frac{1}{2} - l \right) \phi \right] \right]
\]

; 

Comparing the above three choices, \( M_i \) will sponsor at \( \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1 + \phi}}}{2\kappa^{ca}} \) if that renders him a higher profit than sponsoring at \( \beta^{br} \) and not sponsoring, which comes as 

\[
\frac{\alpha}{2} \left[ \phi \left( \frac{1}{\kappa^{br}} - 1 \right) - \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1 + \phi}}}{2} \left[ \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right] \right] + \frac{q \alpha}{2} \left[ 1 - \frac{\phi}{\kappa^{br}} - \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1 + \phi}}}{2} \left[ \frac{1}{2} + l + \left( \frac{1}{2} - l \right) \phi \right] \right] > \frac{\alpha - q}{2} \times \left[ \frac{\phi}{\kappa^{br}} - \kappa^{br} \left( \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right) \right]
\]

This is equivalent to 

\[
q \left[ 1 - \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1 + \phi}}}{2} \left[ \frac{1}{2} + l + \left( \frac{1}{2} - l \right) \phi \right] - \kappa^{br} \left[ \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right] \right] > \phi + \left( \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1 + \phi}}}{2} - \kappa^{br} \right) \left[ \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right]
\]

i.e., \( q \geq q_3 \), where 

\[
q_3 = \frac{\phi + \left( \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1 + \phi}}}{2} - \kappa^{br} \right) \left[ \frac{1}{2} - l + \left( \frac{1}{2} + l \right) \phi \right]}{1 - \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1 + \phi}}}{2} \left[ \frac{1}{2} + l + \left( \frac{1}{2} - l \phi \right] - \kappa^{br} \left[ \frac{1}{2} - l + \left( \frac{1}{2} + l \phi \right] \right)}
\]

In summary, if \( q \geq q_3 \), \( M_i \) sponsors at \( \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1 + \phi}}}{2\kappa^{ca}} \); if \( q < q_3 \) and \( \frac{\phi}{2} - l + (\frac{1}{2} + l) \phi < \)
Consider the case where $\frac{\phi}{1+\phi} < \frac{1+\phi^2}{1+\phi}$, and $\kappa^{br} > \sqrt{\frac{\phi}{2-\ell+(\frac{1}{2}+l)\phi}}$. In this case sponsoring at $\beta^{br}$ wouldn’t be optimal for $M_i$. We need to compare $M_i$’s profit when sponsoring at $\frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1+\phi}}}{2\kappa^{ca}}$ versus not sponsoring. If sponsoring at $\frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1+\phi}}}{2\kappa^{ca}}$, compared with not sponsoring, this will lead $M_i$’s profit to change by

\[
\alpha \left[ \frac{1}{\kappa^{br} - 1} - \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1+\phi}}}{2} \left[ 1 - \frac{l + \left( \frac{1}{2} - l \right) \phi}{1 + \left( \frac{1}{2} - l \right) \phi} \right] \right] + q \left[ 1 - \frac{\phi}{\kappa^{br} - 1} - \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1+\phi}}}{2} \left[ 1 + \frac{l + \left( \frac{1}{2} - l \right) \phi}{1 - \left( \frac{1}{2} - l \right) \phi} \right] \right].
\]

The above amount is positive when $q > q_2$, where $q_2 \equiv \frac{\phi (1 - \frac{l + \left( \frac{1}{2} - l \right) \phi}{1 - \left( \frac{1}{2} - l \right) \phi} - 1)}{1 + \left( \frac{1}{2} - l \right) \phi}$.

Finally, consider the case when $\kappa^{ca} > \frac{1+\phi^2}{1+\phi}$. In this case, $M_i$ will not sponsor the retailer at $\frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1+\phi}}}{2\kappa^{ca}}$. However, she will sponsor at $\beta^{br}$ if that leads to a higher profit for $M_i$ across both $M_i$ and $M_j$’s brand keywords. This requires $\frac{\phi}{2 - \ell + (\frac{1}{2} + l)\phi} < \kappa^{br} < \sqrt{\frac{\phi}{2 - \ell + (\frac{1}{2} + l)\phi}}$.

Therefore, in this region, if $\frac{\phi}{2 - \ell + (\frac{1}{2} + l)\phi} < \kappa^{br} < \sqrt{\frac{\phi}{2 - \ell + (\frac{1}{2} + l)\phi}}$, then $\beta_i^* = \beta^{br}$. Otherwise, $\beta_i^* = 0$.

By intersecting the two response functions $\beta_i^*(\beta_j)$ and $\beta_j^*(\beta_i)$, we have the equilibrium sponsorship rates as,

\[
(\beta_i^*, \beta_j^*) \in \begin{cases} 
(0, 0), \\
((\beta_1, \beta_2) | \beta_1 + \beta_2 = 1 - \frac{1}{\kappa} \frac{2\phi}{1+\phi} \text{ and } \beta_1 > 0, \beta_2 > 0) \cup \{(0, 0)\}, \\
((\beta_1, \beta_2) | \beta_1 + \beta_2 = 1 - \frac{1}{\kappa} \frac{2\phi}{1+\phi} \text{ and } \beta_1 > 0, \beta_2 > 0) 
\end{cases}
\]

where $q_1 = \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1+\phi}}}{1 - \frac{\phi}{\kappa^{br} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1+\phi}}}} \left[ \frac{1}{2} - \ell + (\frac{1}{2} + l)\phi \right]$, $q_2 = \frac{\phi (1 - \frac{l + \left( \frac{1}{2} - l \right) \phi}{1 - \left( \frac{1}{2} - l \right) \phi} - 1)}{1 + \left( \frac{1}{2} - l \right) \phi} - \frac{\kappa^{ca} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1+\phi}}}{1 - \frac{\phi}{\kappa^{br} - 1 + \sqrt{(\kappa^{ca} + 1)^2 - 4 \frac{2\phi}{1+\phi}}}} \left[ \frac{1}{2} - \ell + (\frac{1}{2} + l)\phi \right]$.