Time-dependent angularly averaged inverse transport

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May 7, 2009

Abstract

This paper concerns the reconstruction of the absorption and scattering parameters in a time-dependent linear transport equation from knowledge of angularly averaged measurements performed at the boundary of a domain of interest. Such measurement settings find applications in medical and geophysical imaging. We show that the absorption coefficient and the spatial component of the scattering coefficient are uniquely determined by such measurements. We obtain stability results on the reconstruction of the absorption and scattering parameters with respect to the measured albedo operator. The stability results are obtained by a precise decomposition of the measurements into components with different singular behavior in the time domain.

1 Introduction

Inverse transport theory has many applications in e.g. medical and geophysical imaging. It consists of reconstructing optical parameters in a domain of interest from measurements of the transport solution at the boundary of that domain. The optical parameters are the total absorption (extinction) parameter $\sigma(x)$ and the scattering parameter $k(x, v', v)$, which measures the probability of a particle at position $x \in X \subset \mathbb{R}^n$ to scatter from direction $v' \in \mathbb{S}^{n-1}$ to direction $v \in \mathbb{S}^{n-1}$, where $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$.

The domain of interest is probed as follows. A known flux of particles enters the domain and the flux of outgoing particles is measured at the domain's boundary. Several inverse theories may then be envisioned based on available data. In this paper, we assume availability of time dependent measurements that are angularly averaged. Also the source term used to probe the domain is not resolved angularly in order to e.g. save time in the acquisition of data. More precisely, the incoming density of particles $\phi(t, x, v)$ as a function of time t, at position $x \in \partial X$ at the boundary of the domain of interest, and for incoming directions v, is of the form $\phi_S(t, x, v) = \phi(t, x)S(x, v)$, where $\phi(t, x)$ is arbitrary but $S(x, v)$ is fixed. This paper is concerned with the reconstruction of the optical parameters from such measurements. We show that the attenuation coefficient is uniquely determined and that the spatial structure of the scattering coefficient can be reconstructed provided that scattering vanishes in the vicinity of the domain's boundary (except in dimension $n = 2$ and when X is a disc, where our theory does not require k to vanish in the vicinity of ∂X). For instance, when $k(x, v', v) = k_0(x)g(v', v)$ with $g(v', v)$ known a priori, then $k_0(x)$ is uniquely determined by the measurements. Similar results were announced in [1] when measurements are available in the modulation frequency variable, which is the dual (Fourier) variable to the time variable.

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Several other regimes have been considered in the literature. The uniqueness of the reconstruction of the optical parameters from knowledge of angularly resolved measurements both in the time-dependent and time-independent settings was proved in [7, 8]; see also [15] for a review. Stability in the time-independent case has been analyzed in dimension $n = 2, 3$ under smallness assumptions for the optical parameters in [12, 13] and in dimension $n = 2$ in [16]. Stability results in the presence of full, angularly resolved, measurements have been obtained in [3, 4, 17]. The intermediate case of angularly averaged measurements with angularly resolved sources was considered in [11]. The lack of stability of the reconstruction in the time independent setting with angularly averaged measurements and isotropic sources is treated in [6]. See also [2] for a recent review of results in inverse transport theory.

The rest of the paper is structured as follows. In section 2, we recall known results on the transport equation, present our measurement setting and a decomposition of the resulting measurement operator (the averaged albedo operator) in Proposition 2.1, and study the temporal behavior of the averaged albedo operator when the scattering coefficient vanishes in the vicinity of the boundary of the (convex) domain X. Our main results on uniqueness and stability are presented in section 3. We show that the absorption coefficient and the spatial structure of the scattering coefficient (the phase function describing scattering from v to v' has to be known in advance) can be reconstructed stably from angularly averaged time dependent data. The reconstruction of the scattering coefficient requires inversion of a weighted Radon transform in the general case. When X is a sphere, i.e., when measurements are performed at the boundary of a sphere, then the scattering coefficient may be obtained by inverting a *classical* Radon transform. In section 4, we show that the results are significantly modified when k does not vanish at the boundary of the domain X.

The mathematical derivation of the results is fairly technical and is based on a careful analysis of the temporal behavior of the decomposition of the albedo operator into components that are multi-linear in the scattering coefficient. We show that the ballistic and single scattering components can be separated from the rest of the data. These two components are then used to obtain the uniqueness and stability results. It turns out that the structure of single scattering is very different depending on whether k vanishes on ∂X or not. When k does not vanish on ∂X, the main singularities of the single scattering component do not allow us to "see inside" the domain as they only depend on values of k at the domain's boundary in dimension $n \geq 3$. The singular structure of single scattering and the resulting stability estimates are presented in detail both when k vanishes on ∂X and when it does not. Since the case of non-vanishing k on ∂X is practically interesting mostly as a negative result (for then we are not able to reconstruct k inside the domain from such singularities in dimension $n \geq 3$), we have presented the results without proofs in section 4 and refer the reader to [5] for the mathematical details. The proof of the results when k vanishes in the vicinity of ∂X are presented in detail in sections 5 and 6. In Appendix A, we give the proof of elementary lemmas, which appear in section 5. In Appendix B, we complete the proof of the technical but central Proposition 2.1.

2 Forward problem and albedo operator

2.1 The linear Boltzmann transport equation

We now introduce notation and recall some known results on the linear transport equation. Let X be a bounded convex open subset of $\mathbb{R}^n$, $n \geq 2$, with a C^1 boundary ∂X. Let $\nu(x)$ denote the outward normal unit vector to ∂X at $x \in \partial X$. Let $\Gamma_\pm = \{(x, v) \in \partial X \times \mathbb{S}^{n-1} \mid \pm \nu(x) \cdot v > 0\}$ be the sets of incoming and outgoing conditions. For $(x, v) \in \bar{X} \times \mathbb{S}^{n-1}$, we define $\tau_\pm(x, v)$ and

$\tau(x,v)$ by $\tau_\pm(x,v) := \inf\{s \in (0,+\infty) \mid x \pm sv \notin X\}$ and $\tau(x,v) := \tau_-(x,v) + \tau_+(x,v)$. For $x \in \partial X$, we define $\mathbb{S}^{n-1}_{x,\pm} := \{v \in \mathbb{S}^{n-1} \mid \pm\nu(x)\cdot v > 0\}$.

We consider two nonnegative (measurable) functions $\sigma : X \times \mathbb{S}^{n-1} \to \mathbb{R}$ and $k : X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{R}$ and two convex open subsets Y and Z of $\mathbb{R}^n$ with C^1 boundary such that:

$$\begin{aligned}&Z \subseteq Y \subseteq X; \sigma \text{ is bounded and continuous on } Y \times \mathbb{S}^{n-1} \text{and supported on } \bar{Y} \times \mathbb{S}^{n-1};\\ &k \text{ is bounded and continuous on } Z \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \text{ and supported on } \bar{Z} \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}.\end{aligned} \tag{2.1}$$

We consider also the real δ defined by $\delta := \inf_{(x,z)\in\partial X\times Z} |x - z|$ and we assume throughout this paper except in section 4 that

$$\delta > 0. \tag{2.2}$$

In other words, except in section 4, k vanishes in the $\delta-$vicinity of the boundary ∂X. In practice, this simply means that the array of detectors has to be located some distance away from the scattering region, which is not too restrictive an assumption.

Let $T > \eta > 0$. We consider the following linear Boltzmann transport equation

$$\begin{aligned}&\frac{\partial u}{\partial t}(t,x,v) + v\cdot\nabla_x u(t,x,v) + \sigma(x,v)u(t,x,v)\\ &\qquad = \int_{\mathbb{S}^{n-1}} k(x,v',v)u(t,x,v')dv',\ (t,x,v) \in (0,T)\times X \times \mathbb{S}^{n-1},\\ &u_{|(0,T)\times\Gamma_-}(t,x,v) = \phi(t,x,v),\\ &u(0,x,v) = 0,\ (x,v) \in X \times \mathbb{S}^{n-1},\end{aligned} \tag{2.3}$$

where $\phi \in L^1((0,T), L^1(\Gamma_-, d\xi))$ and $\mathrm{supp}\phi \subseteq [0,\eta]$. Signals are then emitted for a maximal duration η and are recorded at the domain's boundary for a duration T that will be chosen sufficiently large so that information can travel through the domain X and be measured. Here, $d\xi(x,v) = |v\cdot\nu(x)|dvd\mu(x)$, where $d\mu$ is the surface measure on ∂X and dv is the surface measure on $\mathbb{S}^{n-1}$.

The theory for (2.3) is well-developed; we refer the reader to [4, 7, 9]. For our purpose, it is sufficient to recall that the solution may be decomposed as

$$u(t) = G_-(t)\phi + \int_0^t \sum_{m=1}^{\infty} H_{m-1}(t-s)A_2G_-(s)\phi ds, \tag{2.4}$$

where we have defined the following operators:

$$\begin{aligned}&A_2 f = \int_{\mathbb{S}^{n-1}} k(x,v',v)f(x,v')dv', \qquad H_m(t) = \int_0^t H_{m-1}(t-s)A_2U_1(s)ds, \quad m \geq 1\\ &U_1(t) = H_0(t)f = e^{-\int_0^t \sigma(x-sv,v)ds} f(x-tv,v)\chi_{[0,\tau_-(x,v))}(t),\ (x,v) \in X\times\mathbb{S}^{n-1},\\ &G_-(t)\phi(x,v) = e^{-\int_0^{\tau_-(x,v)}\sigma(x-sv,v)ds}\phi_-(t-\tau_-(x,v), x-\tau_-(x,v)v, v),\ (t,x,v) \in (0,T)\times X\times\mathbb{S}^{n-1},\end{aligned}$$

with $\chi_{[0,\tau_-(x,v))}$ defined on $\mathbb{R}$ such that $\chi_{[0,\tau_-(x,v))}(t) = 1$ when $0 \leq t < \tau_-(x,v)$ and $\chi_{[0,\tau_-(x,v))}(t) = 0$ otherwise. Functions $\phi \in L^1((0,\eta), L^1(\Gamma_-, d\xi))$ are extended to $t \in \mathbb{R}$ by 0 outside of the interval $(0,\eta)$. The first term $G_-(t)\phi$ in the above series is the ballistic part of $u(t)$ while the term corresponding to $m \geq 1$ is m-linear in the scattering kernel k. The term corresponding to $m = 1$ is the single scattering term.

The albedo operator A given by the formula

$$A\phi = u_{|(0,T)\times\Gamma_+},\ \text{ for } \phi \in L^1((0,\eta), L^1(\Gamma_-, d\xi)) \text{ where } u \text{ solves (2.3)}, \tag{2.5}$$

is also well-defined and bounded; see [4, 7] for a derivation of the albedo operator and for the reconstruction of the optical parameters when the full albedo operator is known. We assume here that only partial knowledge of the albedo operator is available from measurements.

2.2 The operator $A_{S,W}$ and its distributional kernel

We now define more precisely the type of measurements we consider in this paper. The directional behavior of the source term is determined by a fixed function $S(x,v)$, which is bounded and continuous on Γ_-. We assume that the incoming conditions have the following structure

$$\phi_S(t',x',v') = S(x',v')\phi(t',x'),\ t'\in(0,\eta),\ (x',v')\in\Gamma_-, \tag{2.6}$$

where $\phi(t,x)$ is an arbitrary function in $L^1((0,\eta)\times\partial X)$. We model the detectors by the kernel $W(x,v)$, which we assume is a continuous and bounded function on Γ_+. The available measurements are therefore modeled by the availability of the averaged albedo operator $A_{S,W}$ from $L^1((0,\eta)\times\partial X, dtd\mu(x))$ to $L^1((0,T)\times\partial X, dtd\mu(x))$ and defined by

$$A_{S,W}\phi(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}} A(\phi_S)(t,x,v)W(x,v)(\nu(x)\cdot v)dv,\ \text{for a.e. } (t,x)\in(0,T)\times\partial X. \tag{2.7}$$

The functions S and W are fixed throughout the paper. The case $W\equiv 1$ corresponds to measurements of the current of exiting particles at the domain's boundary.

The decomposition of the transport solution (2.4) translates into a similar decomposition of the albedo operator of the form

$$A_{S,W}\phi(t,x) = \sum_{m=0}^{+\infty} A_{m,S,W}\phi(t,x), \tag{2.8}$$

for $(t,x)\in(0,T)\times\partial X$, where we have defined

$$A_{0,S,W}\phi(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x)\cdot v)W(x,v)\left(G_-(.)\phi_S\right)_{|(0,T)\times\Gamma_+}(t,x,v)dv, \tag{2.9}$$

$$A_{m,S,W}\phi(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x)\cdot v)W(x,v)\left(\int_{-\infty}^{t} H_{m-1}(t-s)A_2G_-(s)\phi_S ds\right)_{|(0,T)\times\Gamma_+}(t,x,v)dv, \tag{2.10}$$

for a.e. $(t,x)\in(0,T)\times\partial X$ where ϕ_S is defined by (2.6). The kernels of the operators $A_{m,S,W}$ can be written explicitly. Consider the nonnegative (measurable) function E from $\partial X\times\partial X\to\mathbb{R}$ defined by

$$E(x_1,x_2) = \exp\Big(-\int_0^{|x_1-x_2|}\sigma(x_1 - s\frac{x_1-x_2}{|x_1-x_2|}, \frac{x_1-x_2}{|x_1-x_2|})ds\Big) \tag{2.11}$$

for $(x_1,x_2)\in\partial X\times\partial X$. We use the same notation $E(x_1,x_2)$ when x_1 and x_2 are either in X or on ∂X. For $m\geq 3$, we also define the nonnegative (measurable) function $E(x_1,\ldots,x_m)$ by induction:

$$E(x_1,\ldots,x_m) = E(x_1,\ldots,x_{m-1})E(x_{m-1},x_m), \qquad m\geq 3,$$

which measures the total attenuation along the broken path $(x_1,\ldots,x_m)\in\partial X\times X^{m-2}\times\partial X$.

For $m\in\mathbb{N}$, $m\geq 1$ and for any subset U of $\mathbb{R}^m$ we denote by χ_U the characteristic function from $\mathbb{R}^m$ to $\mathbb{R}$ defined by $\chi_U(y)=1$ when $y\in U$ and $\chi_U(y)=0$ otherwise. We then obtain the following result on the structure of the kernels of the albedo operator.

Proposition 2.1. *We have*

$$A_{m,S,W}(\phi)(t,x) = \int_{(0,\eta)\times\partial X} \gamma_m(t-t',x,x')\phi(t',x')dt'd\mu(x'), \tag{2.12}$$

for $m \geq 0$ and for a.e. $(t,x) \in (0,T) \times \partial X$, where

$$\gamma_0(\tau,x,x') := \frac{E(x,x')}{|x-x'|^{n-1}} \left[W(x,v)S(x',v)(\nu(x)\cdot v)|\nu(x')\cdot v|\right]_{v=\frac{x-x'}{|x-x'|}} \delta(\tau-|x-x'|), \quad (2.13)$$

$$\gamma_1(\tau,x,x') := \chi_{(0,+\infty)}(\tau-|x'-x|) \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x)\cdot v)W(x,v)\left[E(x,x-sv,x')k(x-sv,v',v)\right.$$

$$\left.\times\, \chi_{(0,\tau_-(x,v))}(s)S(x',v')|\nu(x')\cdot v'|\right]\Big|_{v'=\frac{x-x'-sv}{|x-x'-sv|}\,;\, s=\frac{\tau^2-|x-x'|^2}{2(\tau-v\cdot(x-x'))}} \frac{2^{n-2}(\tau-(x-x')\cdot v)^{n-3}}{|x-x'-\tau v|^{2n-4}} dv, \quad (2.14)$$

for $(\tau,x,x') \in \mathbb{R}\times\partial X\times\partial X$ and where γ_m for $m \geq 2$ admits a similar, more complex, expression given in section 5 (see (5.21)–(5.22)*).*

Proposition 2.1 is proved in detail in Appendix B. As in (2.4), γ_0 is the kernel of the ballistic contribution to $A_{S,W}$, γ_1 that of the single scattering contribution and γ_m that of the contribution that is $m-$linear in k.

2.3 Regularity of the albedo kernels

The reconstruction of the optical parameters is based on an analysis of the behavior in time of the kernels of the albedo operator. We define the scattering kernels

$$\Gamma_0 = \sum_{m=0}^{+\infty} \gamma_m, \qquad \Gamma_1 = \Gamma_0 - \gamma_1, \qquad \Gamma_2 = \Gamma_1 - \gamma_2. \quad (2.15)$$

Thus, Γ_k accounts for scattering of order at least k in the albedo operator. Our first result is the following.

Theorem 2.2. *Under the assumption $k \in L^\infty(X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1})$ and under assumption* (2.2)*, which implies that the scattering coefficient vanishes in the vicinity of ∂X, we have*

$$(\tau-|x-x'|)^{\frac{3-n}{2}}\gamma_1(\tau,x,x') \in L^\infty((0,T)\times\partial X\times\partial X) \qquad \text{when } n \geq 2; \quad (2.16)$$

$$\Gamma_2(\tau,x,x') \in L^\infty((0,T)\times\partial X\times\partial X), \qquad \text{when } n = 2; \quad (2.17)$$

$$\left(1+\ln\left(\frac{\tau+|x-x'|}{\tau-|x-x'|}\right)\right)^{-1} \frac{\Gamma_2(\tau,x,x')}{(\tau-|x-x'|)} \in L^\infty((0,T)\times\partial X\times\partial X), \quad \text{when } n = 3; \quad (2.18)$$

$$(\tau-|x-x'|)^{\frac{1-n}{2}}\Gamma_2(\tau,x,x') \in L^\infty((0,T)\times\partial X\times\partial X), \quad \text{when } n \geq 4. \quad (2.19)$$

Theorem 2.2 is proved in section 5. The results in (2.17)–(2.19) quantify how "smoother" multiple scattering is compared to the single scattering contribution considered in (2.16).

2.4 Single scattering contribution and weighted X-ray transform

We want to analyze the behavior of the function $\gamma_1(\tau,x,x')$ given by the right hand side of (2.14) for all $(\tau,x,x') \in \mathbb{R}\times\partial X\times\partial X$. Under hypothesis (2.2), i.e., when the scattering coefficient vanishes in the vicinity of where measurements are collected, Theorem 2.3 below describes the behavior in time of the single scattering term as a function of $E(x,x')$ for $(x,x') \in \partial X^2$, (which is uniquely determined by the ballistic term; see (2.13)) and the weighted X-ray transform P_{ϑ_0} of $x \mapsto k(x,v_0,v_0)$ (for a fixed $v_0 \in \mathbb{S}^{n-1}$), where P_{ϑ_0} is defined by

$$P_{\vartheta_0}f(v,x) = \int_{\tau_-(x,v)}^{\tau_+(x,v)} \vartheta_0(v,tv+x)f(tv+x)dt, \quad (2.20)$$

for a.e. $(v,x)\in\mathbb{S}^{n-1}\times\partial X$ and $f\in L^2(X,\sup_{v\in\mathbb{S}^{n-1}}\vartheta_0(v,x)dx)$, and where the weight $\vartheta_0:\mathbb{S}^{n-1}\times X\to\mathbb{R}$ is the function defined by

$$\vartheta_0(v,x)=\big(\tau_-(x,v)\tau_+(x,v)\big)^{-\frac{n-1}{2}},\ (v,x)\in\mathbb{S}^{n-1}\times X. \tag{2.21}$$

Theorem 2.3. *Let $(x,x_0')\in\partial X^2$ be such that $x_0'+s(x-x_0')\in Z$ for some $s\in(0,1)$. Define $v_0=\frac{x-x_0'}{|x-x_0'|}$ and $t_0=|x-x_0'|$ and let $k_{v_0}(y):=k(y,v_0,v_0)$ for $y\in X$. Then we have the following.*

$$\gamma_1(\tau,x,x_0')=\frac{1}{\sqrt{\tau-t_0}}\frac{\sqrt{2}W(x,v_0)S(x_0',v_0)(\nu(x)\cdot v_0)|\nu(x_0')\cdot v_0|E(x,x_0')}{\sqrt{t_0}} \tag{2.22}$$

$$\times P_{\vartheta_0}k_{v_0}(v_0,x)+o\Big(\frac{1}{\sqrt{\tau-t_0}}\Big),\quad as\ \tau\to t_0^+,\qquad when\ n=2$$

$$\gamma_1(\tau,x,x_0')=(\tau-t_0)^{\frac{n-3}{2}}(2t_0)^{\frac{1-n}{2}}\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})S(x_0',v_0)W(x,v_0)|\nu(x_0')\cdot v_0|(\nu(x)\cdot v_0) \tag{2.23}$$

$$\times E(x,x_0')P_{\vartheta_0}k_{v_0}(v_0,x)+o((\tau-t_0)^{\frac{n-3}{2}}),\ as\ \tau\to t_0^+\qquad when\ n\geq 3.$$

Theorem 2.3 is proved in section 6. Theorem 2.3 may remain valid under more general assumptions. For instance, when σ is bounded and continuous on X, k is continuous on $X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ and $k(x,.,.)$ decays sufficiently rapidly as x approaches the boundary ∂X for any $x\in X$, then the same asymptotic expansion holds for γ_1.

3 Uniqueness and stability results under condition (2.2)

We recall that $\Gamma_0=\sum_{m=0}^{+\infty}\gamma_m$ the distributional kernel of $A_{S,W}$ and that $\Gamma_0-\gamma_0=\Gamma_1$ denotes the distributional kernel of the multiple scattering of $A_{S,W}$. For the rest of the paper, we assume that the duration of measurement $T>\mathrm{diam}(X):=\sup_{(x,y)\in X^2}|x-y|$ so that the singularities of the ballistic and single scattering contributions are indeed captured by the available measurements.

Let $(\tilde\sigma,\tilde k)$ be a pair of absorption and scattering coefficients that also satisfy (2.1) and (2.2) for the same (Y,Z) related to (σ,k). We denote by a superscript $\tilde{}$ any object (such as the albedo operator $\tilde A$ or the distributional kernels $\tilde\Gamma_0$ and $\tilde\gamma_0$) associated to $(\tilde\sigma,\tilde k)$. Let $\|.\|_{\eta,T}:=\|\cdot\|_{\mathcal{L}(L^1((0,\eta)\times\partial X)),L^1((0,T)\times\partial X))}$.

3.1 Stability estimates in integral form

The following theorem presents stability results for the reconstruction of the attenuation coefficient and of the scattering coefficient when the latter vanishes in the vicinity of the boundary ∂X.

Theorem 3.1. *Assume $m_{S,W}=\min\big(\inf_{\Gamma_-}S,\inf_{\Gamma_+}W\big)>0$. Let (σ,k) and $(\tilde\sigma,\tilde k)$ satisfy conditions (2.1)–(2.2). Let $x_0'\in\partial X$. Then there exist constants $C_1=C_1(m_{S,W},X,Y)$ and $C_2=C_2(m_{S,W},X,Z)$ such that*

$$\int_{\mathbb{S}^{n-1}_{x_0',-}}|E-\tilde E|(x_0'+\tau_+(x_0',v_0)v_0,x_0')|\nu(x_0')\cdot v_0|dv_0\leq C_1\|A_{S,W}-\tilde A_{S,W}\|_{\eta,T}, \tag{3.1}$$

$$\Big|E(x,x_0')P_{\vartheta_0}k_{v_0'}(v_0',x_0')-\tilde E(x,x_0')P_{\vartheta_0}\tilde k_{v_0'}(v_0',x_0')\Big|\leq C_2\Big\|(\tau-|z-z'|)^{\frac{3-n}{2}}(\Gamma_1-\tilde\Gamma_1)(\tau,z,z')\Big\|_{L^\infty}, \tag{3.2}$$

for $x\in\partial X$ such that $px_0'+(1-p)x\in Z$ for some $p\in(0,1)$ where $v_0'=\frac{x-x_0'}{|x-x_0'|}$, P_{ϑ_0} is defined in (2.20), and $k_{v_0'}(y):=k(y,v_0',v_0')$ for $y\in X$ ($\tilde k_{v_0'}$ is defined similarly).

Theorem 3.1 is the main result of the paper. The first estimate (3.1) shows that integrals of the attenuation coefficients are stably determined by the measurements $A_{S,W}$. The attenuation coefficient may then stably be reconstructed by inverse Radon transform as we will see in (3.7) below. The second inequality shows that a weighted integral of the scattering coefficient is stably determined by an appropriate measure of the multiple scattering coefficient Γ_1. What may then be stably reconstructed in the scattering kernel will be made explicit in Theorems 3.3 and 3.4 below. Theorem 3.1 is proved in section 6.

3.2 Uniqueness and stability when X is a ball of $\mathbb{R}^n$

When X is an open Euclidean ball of $\mathbb{R}^n$, which is important from the practical point of view in medical imaging as it is relatively straightforward to place sources and detectors on a sphere, we are able to invert the weighted X-ray transform $P_{\vartheta_0}f$ for functions of the form $f(x) \in L^2(X, \sup_{v\in\mathbb{S}^{n-1}} \vartheta_0(v,x)dx)$ using the classical inverse X-ray transform (inverse Radon transform in dimension $n = 2$). More general convex domains X, which require one to solve more complex weighted X-ray transforms, are considered in the next subsection.

Up to rescaling, we assume $X = B_n(0,1)$, the ball in $\mathbb{R}^n$ centered at 0 of radius 1. Consider the X-ray transform P defined by

$$Pf(v,x) = \int_{\tau_-(x,v)}^{\tau_+(x,v)} f(sv+x)ds \text{ for a.e. } (v,x) \in \mathbb{S}^{n-1}\times\partial X, \tag{3.3}$$

for $f \in L^2(X)$ (we extend f by 0 outside X). We have the following result:

Proposition 3.2. *When $X = B_n(0,1)$ we have*

$$P_{\vartheta_0}f(v,x) = P(\varrho f)(v,x), \text{ for a.e. } (v,x) \in \mathbb{S}^{n-1}\times\partial X, \tag{3.4}$$

for $f \in L^2(X, \sup_{v\in\mathbb{S}^{n-1}} \vartheta_0(v,x)dx)$ where $\varrho(y) := (1-|y|^2)^{-\frac{n-1}{2}}$, $y \in X$.

Proof of Proposition 3.2. It is easy to see that

$$\tau_\pm(tv+qv^\perp, v) = \sqrt{1-q^2} \mp t, \tag{3.5}$$

$$\vartheta_0(v,x) = (1-q^2-t^2)^{-\frac{n-1}{2}} = (1-|x|^2)^{-\frac{n-1}{2}}, \tag{3.6}$$

for $(t,q) \in \mathbb{R}^2$, $t^2+q^2 \le 1$ and for $(v,v^\perp) \in \mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$, $v\cdot v^\perp = 0$, where $x = tv+qv^\perp$ (we recall that ϑ_0 is defined by (2.21)). Then Proposition 3.2 follows from the definition (2.20). □

Assume that (σ,k) satisfies (2.1) and (2.2). Assume also that $k(x,v,v') = k_0(x)g(v,v')$ for a.e. $(x,v,v') \in X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ where g is a given continuous function on $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$, $\inf_{v\in\mathbb{S}^{n-1}} g(v,v) > 0$, and where $k_0 \in L^\infty(X)$. Then from the decomposition of the angularly averaged albedo operator $A_{S,W}$ (Proposition 2.1) and from Theorems 2.2, 2.3, and from Proposition 3.2 and methods of reconstruction of a function from its X-ray transform, it follows that (σ,k_0) can be reconstructed from the asymptotic expansion in time of $A_{S,W}$ provided that $\sigma = \sigma(x)$ and $\min(\inf_{\Gamma_-} S, \inf_{\Gamma_+} W) > 0$. In addition we have the following stability estimates.

Theorem 3.3. *Assume $X = B_n(0,1)$ and $m_{S,W} = \min\big(\inf_{\Gamma_-} S, \inf_{\Gamma_+} W\big) > 0$. Let (σ,k) and $(\tilde\sigma,\tilde k)$ satisfy conditions (2.1) and (2.2). Assume that σ, $\tilde\sigma$ do not depend on the velocity variable ($\sigma(x,v) = \sigma(x)$) and let $M = \max(\|\sigma\|_{L^\infty(Y)}, \|\tilde\sigma\|_{L^\infty(Y)})$. Assume $k(x,v,v') = k_0(x)g(v,v')$ and $\tilde k(x,v,v') = \tilde k_0(x)g(v,v')$, $g(v,v) > 0$, for $(x,v,v') \in X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ where g is an a priori known continuous function on $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$.*

Then there exists $C_3 = C_3(m_{S,W}, X, Y, M)$ *and* $C_4 = C_4(m_{S,W}, X, Y, Z, M, g)$ *such that*

$$\|\sigma - \tilde{\sigma}\|_{H^{-\frac{1}{2}}(Y)} \leq C_3 \|\sigma - \tilde{\sigma}\|^{\frac{1}{2}}_{L^\infty(Y)} \|A_{S,W} - \tilde{A}_{S,W}\|^{\frac{1}{2}}_{\eta,T}; \tag{3.7}$$

$$\begin{aligned} \|k_0 - \tilde{k}_0\|_{H^{-\frac{1}{2}}(Z)} \leq & \; C_4 \|k_0 - \tilde{k}_0\|^{\frac{1}{2}}_{L^\infty(Z)} \Big(\|\tilde{k}_0\|_{L^\infty(Z)} \|A_{S,W} - \tilde{A}_{S,W}\|_{\eta,T} \\ & + \Big\| (\tau - |z - z'|)^{\frac{3-n}{2}} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, z, z') \Big\|_{L^\infty} \Big)^{\frac{1}{2}}. \end{aligned} \tag{3.8}$$

Theorem 3.3 is proved as Theorem 3.4 given below for a larger class of domains X. Under the assumptions of Theorem 3.3 and additional regularity assumptions on (σ, k), we obtain stability estimates similar to those given in Corollary 3.5 below for a larger class of domains X. Note that $\|\sigma - \tilde{\sigma}\|_{L^\infty(Y)}$ and $\|k_0 - \tilde{k}_0\|_{L^\infty(Z)}$ are bounded a priori by positive constants. These and similar estimates below show how measurement errors translate into reconstruction errors. These are Hölder-type estimates in the sense that measurement errors of size δ generate reconstruction errors of size δ^α for some $\alpha > 0$. Such estimates should be compared to those obtained for other measurement settings in inverse transport theory; see e.g. [2].

3.3 Uniqueness and stability estimates for more general domains X

Theorem 3.4. *Assume that the open subset* X *of* $\mathbb{R}^n$ *is convex with a real analytic boundary and that* $\min\big(\inf_{\Gamma_-} S, \inf_{\Gamma_+} W\big) > 0$. *Let* (σ, k) *and* $(\tilde{\sigma}, \tilde{k})$ *satisfy conditions* (2.1) *and* (2.2). *Assume also that* σ, $\tilde{\sigma}$ *do not depend on the velocity variable* ($\sigma(x, v) = \sigma(x)$) *and* $k(x, v, v') = k_0(x) g(x, v, v')$ *and* $\tilde{k}(x, v, v') = \tilde{k}_0(x) g(x, v, v')$, $g(x, v, v') > 0$, *for* $(x, v, v') \in X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ *where* g *is an a priori known real analytic function on* $X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ *and where* $\mathrm{supp} k_0 \cup \mathrm{supp} \tilde{k}_0 \subseteq \bar{Z}$, $(k_0, \tilde{k}_0) \in L^\infty(Z)$. *Then estimates* (3.7)–(3.8) *still hold.*

Theorem 3.4 is proved in section 6. Now we give stability estimates under additional regularity assumptions on the optical parameters and on X. Assume that X is convex with a real analytic boundary and that $\min\big(\inf_{\Gamma_-} S, \inf_{\Gamma_+} W\big) > 0$. Let Y and Z be open convex subsets of X, $\bar{Z} \subset X$, $Z \subseteq Y \subseteq X$, with a C^1 boundary. Let g be an a priori known real analytic function on $X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$, $g(x, v, v') > 0$ for $(x, v, v') \in X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$. Let $r_1 > 0$, $r_2 > 0$. Consider the class

$$\begin{aligned} N \; := \; & \Big\{ (\sigma, k) \in H^{\frac{n}{2} + r_1}(Y) \times L^\infty(Z \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}) \mid \|\sigma\|_{H^{\frac{n}{2}+r_1}(Y)} \leq M_1, \\ & k = k_0 g, \; \mathrm{supp} k_0 \subseteq \bar{Z}, \; \|k_0\|_{H^{\frac{n}{2}+r_2}(Z)} \leq M_2 \Big\}. \end{aligned} \tag{3.9}$$

Note that there exists a function $D_1 : \mathbb{N} \times (0, +\infty) \to (0, +\infty)$ such that

$$\begin{aligned} & \|\sigma\|_{L^\infty(Y)} \leq D_1(n, r_1) \|\sigma\|_{H^{\frac{n}{2}+r_1}(Y)} \leq D_1(n, r_1) M_1, \\ & \|k_0\|_{L^\infty(Z)} \leq D_1(n, r_2) \|k_0\|_{H^{\frac{n}{2}+r_2}(Z)} \leq D_1(n, r_2) M_2, \\ & \|k\|_{L^\infty(Z)} \leq \|g\|_{L^\infty(Z)} \|k_0\|_{L^\infty(Z)} \leq D_1(n, r_2) M_2 \|g\|_{L^\infty(Z)}, \end{aligned} \tag{3.10}$$

for $(\sigma, k) \in N$. We also use the classical interpolation formula

$$\|f\|_{H^s(O)} \leq \|f\|^{\frac{s_2 - s}{s_2 - s_1}}_{H^{s_1}(O)} \|f\|^{\frac{s - s_1}{s_2 - s_1}}_{H^{s_2}(O)}, \tag{3.11}$$

for $s_1 < s < s_2$ and for $(O, s_1, s_2) \in \{(Y, -\frac{1}{2}, \frac{n}{2} + r_1), \; (Z, -\frac{1}{2}, \frac{n}{2} + r_2)\}$. Using Theorem 3.4 and (3.10), and applying (3.11) on $f = \sigma - \tilde{\sigma}$ and $f = k_0 - \tilde{k}_0$ we obtain the following result.

Corollary 3.5. *Let (σ,k), $(\tilde\sigma,\tilde k)\in N$. Then, for $-\frac{1}{2}\le s\le \frac{n}{2}+r_1$ and for $0<r<r_1$, there exists $C_5=C_5(m_{S,W},X,Y,M_1,r_1,s)$ such that*

$$\|\sigma-\tilde\sigma\|_{H^s(Y)} \le C_5\|\sigma-\tilde\sigma\|_{L^\infty(Y)}^{\frac{\kappa}{2}}\|A_{S,W}-\tilde A_{S,W}\|_{\eta,T}^{\frac{\kappa}{2}}, \tag{3.12}$$

$$\|\sigma-\tilde\sigma\|_{H^{\frac{n}{2}+r}(Y)} \le C_6\|A_{S,W}-\tilde A_{S,W}\|_{\eta,T}^{\frac{\kappa'}{2-\kappa'}}, \tag{3.13}$$

where $(\kappa,\kappa')=\left(\frac{n+2(r_1-s)}{n+1+2r_1},\frac{2(r_1-r)}{n+1+2r_1}\right)$ and $C_6=C_5^{\frac{2}{2-\kappa'}}D_1(n,r)^{\frac{\kappa'}{2-\kappa'}}$ ($D_1(n,r)$ is defined by (3.10)). In addition, there exists $C_7=C_7(m_{S,W},X,Y,Z,g,M_1,r_1,M_2,r_2,s)$ such that

$$\|k_0-\tilde k_0\|_{H^s(Z)} \le C_7\|k_0-\tilde k_0\|_{L^\infty(Z)}^{\frac{\kappa}{2}}\Big(\|A_{S,W}-\tilde A_{S,W}\|_{\eta,T} + \left\|(\tau-|z-z'|)^{\frac{3-n}{2}}(\Gamma_1-\tilde\Gamma_1)(\tau,z,z')\right\|_{L^\infty}\Big)^{\frac{\kappa}{2}}, \tag{3.14}$$

$$\|k_0-\tilde k_0\|_{H^{\frac{n}{2}+r}(Z)}\le C_8\Big(\|A_{S,W}-\tilde A_{S,W}\|_{\eta,T}+\left\|(\tau-|z-z'|)^{\frac{3-n}{2}}(\Gamma_1-\tilde\Gamma_1)(\tau,z,z')\right\|_{L^\infty}\Big)^{\frac{\kappa'}{2-\kappa'}}, \tag{3.15}$$

for $-\frac{1}{2}\le s\le\frac{n}{2}+r_2$ and $0<r<r_2$, where $(\kappa,\kappa')=\left(\frac{n+2(r_2-s)}{n+1+2r_2},\frac{2(r_2-r)}{n+1+2r_2}\right)$ and $C_8=C_7^{\frac{2}{2-\kappa'}}D_1(n,r)^{\frac{\kappa'}{2-\kappa'}}$ ($D_1(n,r)$ is defined by (3.10)).

Remark 3.6. (i.) Theorem 3.4 and Corollary 3.5 remain valid when: X is only assumed to be convex with C^2 boundary; the weight ϑ_o defined by (2.21) (resp. the function g which appears in the assumptions of Theorem 3.4 and Corollary 3.5) is sufficiently close (in the C^2 norm) to an analytic weight $\theta_{0,a}$ on the vicinity of $\bar Z\times\mathbb{S}^{n-1}$ (resp. an analytic function g_a on the vicinity of $\bar Z\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$); see proof of Theorem 3.4 and [10, Theorem 2.3].

(ii.) When $n=3$ then under hypothesis (2.2), we have

$$\left\|(\tau-|z-z'|)^{\frac{3-n}{2}}(\Gamma_1-\tilde\Gamma_1)(\tau,z,z')\right\|_{L^\infty}=\left\|\sum_{m=1}^{\infty}(A_{m,S,W}-\tilde A_{m,S,W})\right\|_{\mathcal{L}(L^1((0,\eta)\times\partial X),L^\infty((0,T)\times\partial X))}.$$

where the distributional kernel of the bounded operator $\sum_{m=1}^{+\infty}(A_{m,S,W}-\tilde A_{m,S,W})$ from $L^1((0,\eta)\times\partial X)$ to $L^1((0,T)\times\partial X)$ is given by $\Gamma_1-\tilde\Gamma_1$. Therefore when $n=3$ and under condition (2.2), the right-hand side of the stability estimates (3.8) and (3.14) can be expressed with operator norms only (instead of using a norm on the distributional kernel of the multiple scattering).

4 Non-vanishing scattering on ∂X

Throughout this section, we consider the class of optical parameters (σ,k) such that (σ,k) satisfies (2.1) with

$$X=Y=Z \text{ (thus } \delta=0); \quad \text{the function } k \text{ is continuous on } \bar X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}. \tag{4.1}$$

Under assumptions (2.1) and (4.1), the albedo operator $A_{S,W}$ is defined as in section 2 and the decomposition given in Proposition 2.1 still holds. The behavior in time of the measurement operator $A_{S,W}$ is however significantly modified when k does not vanish on ∂X. Because they appear to be less interesting practically, the results of this section are given without proofs. We refer the reader to [5] for the details.

4.1 Behavior in time of the averaged albedo operator

The analog of Theorem 2.2 is given by the following Theorems 4.1 and 4.2.

Theorem 4.1. *Under the conditions $k \in L^\infty(X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$, the following holds:*

$$\sqrt{\tau^2 - |x-x'|^2}\gamma_1(\tau, x, x') \in L^\infty((0,T) \times \partial X \times \partial X) \qquad \text{when } n = 2; \tag{4.2}$$

$$\frac{\tau|x-x'|}{\ln\left(\frac{\tau+|x-x'|}{\tau-|x-x'|}\right)}\gamma_1(\tau, x, x') \in L^\infty((0,T) \times \partial X \times \partial X) \qquad \text{when } n = 3; \tag{4.3}$$

$$\tau|x-x'|^{n-2}\gamma_1(\tau, x, x') \in L^\infty((0,T) \times \partial X \times \partial X) \qquad \text{when } n \geq 4; \tag{4.4}$$

The results in the following Theorem 4.2 quantify how "smoother" multiple scattering is compared to single scattering contribution considered in (4.2)–(4.4).

Theorem 4.2. *Assume that $k \in L^\infty(X \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$. Then we have:*

$$\text{statement (2.17) still holds}, \qquad \text{when } n = 2; \tag{4.5}$$

$$\frac{\tau|x-x'|\Gamma_2(\tau, x, x')}{(\tau - |x-x'|)\left(1 + \ln\left(\frac{\tau+|x-x'|}{\tau-|x-x'|}\right)\right)^2} \in L^\infty((0,T) \times \partial X \times \partial X), \qquad \text{when } n = 3; \tag{4.6}$$

$$\frac{\tau|x-x'|^{n-2}}{\tau - |x-x'|}\Gamma_2(\tau, x, x') \in L^\infty((0,T) \times \partial X \times \partial X), \qquad \text{when } n \geq 4. \tag{4.7}$$

The analog of the single scattering asymptotic expansion in time given in Theorem 2.3 is:

Theorem 4.3. *Let $(x, x_0') \in \partial X^2$ be such that $x + s(x - x_0') \in X$ for some $s \in (0,1)$. Set $v_0 = \frac{x-x_0'}{|x-x_0'|}$ and $t_0 = |x - x_0'|$. Then under condition (4.1), we have the following results.*

$$\text{(2.22) still holds} \quad \text{when } n = 2,$$

$$\begin{aligned}
\gamma_1(\tau, x, x_0') &= \ln\left(\frac{1}{\tau - t_0}\right)\frac{\pi}{t_0^2}W(x, v_0)S(x_0', v_0)(\nu(x) \cdot v_0)|\nu(x_0') \cdot v_0|E(x, x_0') \qquad (4.8)\\
&\quad \times (k(x, v_0, v_0) + k(x_0', v_0, v_0)) + o\Big(\ln\left(\frac{1}{\tau - t_0}\right)\Big), \text{ as } \tau \to t_0^+, \qquad \text{when } n = 3,\\
\gamma_1(\tau, x, x_0') &= t_0^{1-n}E(x, x_0')\Big[S(x_0', v_0)|\nu(x_0') \cdot v_0| \int_{\mathbb{S}^{n-1}_{x,+}} \frac{W(x,v)(\nu(x) \cdot v)k(x, v_0, v)}{1 - v \cdot v_0}dv\\
&\quad + W(x, v_0)(\nu(x) \cdot v_0) \int_{\mathbb{S}^{n-1}_{x_0',-}} \frac{k(x_0', v', v_0)S(x_0', v')|\nu(x_0') \cdot v'|}{1 - v' \cdot v_0}dv'\Big] \qquad (4.9)\\
&\quad + o(1), \text{ as } \tau \to t_0^+, \qquad \text{when } n \geq 4.
\end{aligned}$$

Note that the asymptotic expansion in time of γ_1 depends on the values of k on ∂X in dimension $n \geq 3$, and no longer on k inside X. Such singularities are thus useless in the reconstruction of the scattering coefficient inside X. The proof of these results can be found in [5].

4.2 Stability results

The singularities exhibited in the preceding results still allow us to perform stable reconstructions when k does not vanish on ∂X. The analog of Theorem 3.1 is as follows.

Theorem 4.4. *Let (σ,k) and $(\tilde\sigma,\tilde k)$ satisfy* (2.1) *and* (4.1). *Let $x_0'\in\partial X$. Then we have:*

$$\int_{\partial X}\left[\frac{|E-\tilde E|(x,x_0')}{|x-x_0'|^{n-1}}W(x,v_0)S(x',v_0)(\nu(x)\cdot v_0)|\nu(x_0')\cdot v_0|\right]_{\substack{t_0=|x-x_0'|\\ v_0=\frac{x-x_0'}{|x-x_0'|}}} d\mu(x)\le \|A_{S,W}-\tilde A_{S,W}\|_{\eta,T}. \tag{4.10}$$

Let $x\in\partial X$ be such that $px_0'+(1-p)x\in X$ for some $p\in(0,1)$. Set $v_0=\frac{x-x_0'}{|x-x_0'|}$ and $t_0=|x-x_0'|$. When $n=2$, we have

$$\begin{aligned}&W(x,v_0)S(x_0',v_0)(\nu(x)\cdot v_0)|\nu(x_0')\cdot v_0|\left|E(x,x_0')P_{\vartheta_0}k_{v_0}(v_0,x)-\tilde E(x,x_0')P_{\vartheta_0}\tilde k_{v_0}(v_0,x)\right|\\ &\le \frac12\left\|\sqrt{\tau^2-|z-z'|^2}(\Gamma_1-\tilde\Gamma_1)(\tau,z,z')\right\|_{L^\infty},\end{aligned} \tag{4.11}$$

where $\|\cdot\|_{L^\infty}:=\|\cdot\|_{L^\infty((0,T)\times\partial X\times\partial X)}$. When $n=3$, then

$$\begin{aligned}&\left|E(x,x_0')(k(x,v_0,v_0)+k(x_0',v_0,v_0))-\tilde E(x,x_0')(\tilde k(x,v_0,v_0)+\tilde k(x_0',v_0,v_0))\right| \\ &\times W(x,v_0)S(x_0',v_0)(\nu(x)\cdot v_0)|\nu(x_0')\cdot v_0|\le\frac{1}{\pi}\left\|\frac{\tau|z-z'|}{\ln\left(\frac{\tau+|z-z'|}{\tau-|z-z'|}\right)}(\Gamma_1-\tilde\Gamma_1)(\tau,z,z')\right\|_{L^\infty}.\end{aligned} \tag{4.12}$$

When $n\ge 4$, then

$$\begin{aligned}&S(x_0',v_0)|\nu(x_0')\cdot v_0|\left|\int_{\mathbb{S}^{n-1}_{x,+}}\frac{W(x,v)(\nu(x)\cdot v)}{1-v\cdot v_0}\Big(E(x,x_0')k(x,v_0,v)-\tilde E(x,x_0')\tilde k(x,v_0,v)\Big)dv\right.\\ &\left.+W(x,v_0)(\nu(x)\cdot v_0)\int_{\mathbb{S}^{n-1}_{x_0',-}}\frac{S(x_0',v')|\nu(x_0')\cdot v'|}{1-v'\cdot v_0}\Big(E(x,x_0')k(x_0',v',v_0)-\tilde E(x,x_0')\tilde k(x_0',v',v_0)\Big)dv'\right|\\ &\le\left\|\tau|z-z'|^{n-2}(\Gamma_1-\tilde\Gamma_1)(\tau,z,z')\right\|_{L^\infty}.\end{aligned} \tag{4.13}$$

Note that if we assume that $\inf_{\Gamma_-}S>0$ and $\inf_{\Gamma_+}W>0$, then (4.10) and (3.1) are equivalent by performing the change of variables "$v_0=\frac{x-x_0'}{|x-x_0'|}$" in (4.10). Note also that (4.11) is similar to but different from (3.2) in dimension $n=2$. Results in (4.12) and (4.13) show that the spatial structure of k may be stably reconstructed at the domain's boundary in $n\ge 3$. They do not imply stable reconstruction of k inside the domain.

4.3 Improved results when X is a ball

First consider the case $n=2$ and $X=B_2(0,1)$. Let (σ,k) satisfy (2.1) and (4.1). Assume that $k(x,v,v')=k_0(x)g(v,v')$ for a.e. $(x,v,v')\in X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ where g is a given continuous function on $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$, $\inf_{v\in\mathbb{S}^{n-1}}g(v,v)>0$, and where $k_0\in L^\infty(X)$. Then from the decomposition of the angularly averaged albedo operator $A_{S,W}$ (Proposition 2.1) and from Theorems 4.1, 4.2, 4.3, and from Proposition 3.2 and methods of reconstruction of a function from its X-ray transform, it follows that (σ,k_0) can be reconstructed from the asymptotic expansion in time of $A_{S,W}$ provided that $\sigma=\sigma(x)$ and $\inf_{(x',v')\in\Gamma_-}S(x',v')>0$ and $\inf_{(x,v)\in\Gamma_+}W(x,v)>0$. In addition we have the following stability estimates.

Theorem 4.5. *Assume $X=B_n(0,1)$, and* $\min\big(\inf_{\Gamma_-}S,\inf_{\Gamma_+}W\big)>0$. *Let (σ,k) and $(\tilde\sigma,\tilde k)$ satisfy conditions* (2.1) *and* (4.1). *Assume that σ, $\tilde\sigma$ do not depend on the velocity variable*

$(\sigma(x,v)=\sigma(x))$ and let $M=\max(\|\sigma\|_{L^\infty(Y)},\|\tilde\sigma\|_{L^\infty(Y)})$. Assume $k(x,v,v')=k_0(x)g(v,v')$ and $\tilde k(x,v,v')=\tilde k_0(x)g(v,v')$, $g(v,v)>0$, for $(x,v,v')\in X\times\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$ where g is an a priori known continuous function on $\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}$. Then we have :

(i) (3.7) *still holds;*

(ii) when $n=2$, there exists $C_8=C_8(S,W,X,Y,Z,M,g)$ such that

$$\begin{aligned}\|\rho(k_0-\tilde k_0)\|_{H^{-\frac12}(X)} \le\ & C_8\|k_0-\tilde k_0\|_\infty^{\frac34}\Big(\|\tilde k_0\|_\infty\|A_{S,W}-\tilde A_{S,W}\|_{\eta,T}\\ &+\Big\|\sqrt{\tau^2-|z-z'|^2}(\Gamma_1-\tilde\Gamma_1)(\tau,z,z')\Big\|_{L^\infty}\Big)^{\frac14}.\end{aligned} \tag{4.14}$$

Theorem 4.5 can be proved as Theorem 3.4. Under the assumptions of Theorem 4.5 and additional regularity assumptions on (σ,k) one obtains stability estimates similar to those given in Corollary 3.5.

5 Proof of Theorem 2.2

For $0<b<a$, we recall that

$$\int_0^{2\pi}\frac{1}{a-b\sin(\Omega)}d\Omega=\frac{2\pi}{\sqrt{a^2-b^2}}. \tag{5.1}$$

5.1 Proof of (2.16)

First, we give an estimate on the single scattering term. From (2.14), it follows that

$$|\gamma_1(\tau,x,x')|\le 2^{n-2}\|W\|_\infty\|S\|_\infty\|k\|_\infty G_1(\tau,x,x') \tag{5.2}$$

for a.e. $(\tau,x,x')\in\mathbb{R}\times\partial X\times\partial X$, where

$$G_1(\tau,x,x')=\chi_{(0,+\infty)}(\tau-|x-x'|)\int_{\mathbb{S}^{n-1}}\chi_{\mathrm{supp}k}(x-sv)_{|s=\frac{\tau^2-|x-x'|^2}{2(\tau-v\cdot(x-x'))}}\frac{(\tau-(x-x')\cdot v)^{n-3}}{|x-x'-\tau v|^{2n-4}}dv. \tag{5.3}$$

Let $(\tau,x,x')\in(0,T)\times\partial X\times\partial X$ be such that $x\neq x'$ and $\tau>|x-x'|$. Assume without loss of generality $x'-x=|x'-x|(1,0\dots0)$. Let $v\in\mathbb{S}^{n-1}$ and $s:=\frac{\tau^2-|x-x'|^2}{2(\tau-(x-x')\cdot v)}$. Straightforward computations give $s+|x-x'-sv|=\tau$. Using (2.2) we obtain that

$$\text{if } \tau<\delta \text{ or } s>\tau-\delta, \text{ then } x-sv\notin\mathrm{supp}k. \tag{5.4}$$

Let $n=2$. From (5.3) and (5.1), it follows that

$$G_1(\tau,x,x')\le\int_0^{2\pi}\frac{1}{\tau-|x-x'|\sin(\Omega)}d\Omega=\frac{2\pi}{\sqrt{\tau^2-|x-x'|^2}}\le\frac{2\pi}{\sqrt{\delta}\sqrt{\tau-|x-x'|}}, \tag{5.5}$$

for $\tau\ge\delta$. Using (5.3) and (5.4), we obtain

$$\text{if } \tau<\delta \text{ then } G_1(\tau,x,x')=0. \tag{5.6}$$

Combining (5.2) with (5.5)–(5.6), we obtain (2.16) for $n=2$.

Let $n \geq 3$ and $\tau \geq \delta$ (the case $\tau < \delta$ is already considered in (5.6)). Performing the change of variables "$r = \frac{\tau^2-|x-x'|^2}{2(\tau-|x-x'|\sin(\Omega))} - \frac{\tau-|x-x'|}{2}$" with "$v = \Phi(\Omega,\omega) := (\sin(\Omega), \cos(\Omega)\omega)$, $\Omega \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\omega \in \mathbb{S}^{n-2}$" on the right-hand side of (5.3), we obtain

$$\begin{aligned} G_1(\tau,x,x') &= 2^{2-n}\frac{(\tau^2-|x-x'|^2)^{\frac{n-3}{2}}}{|x-x'|^{n-2}}\int_0^{|x-x'|}\frac{\sqrt{r(|x-x'|-r)}^{n-3}}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}(\frac{\tau+|x-x'|}{2}-r)^{n-2}} \\ &\int_{\mathbb{S}^{n-2}}\left[\chi_{\mathrm{supp}k}(x-sv)\right]_{\substack{\Omega=\arcsin(|x-x'|^{-1}(\tau-\frac{(\tau^2-|x-x'|^2)}{2(r+\frac{\tau-|x-x'|}{2})})) \\ s=r+\frac{\tau-|x-x'|}{2},\quad v=\Phi(\Omega,\omega)}} d\omega dr. \end{aligned} \tag{5.7}$$

Now assume $\tau > \frac{\delta}{2}+|x-x'|$. Then

$$\begin{aligned} &|x-x'|^{2-n}\int_0^{|x-x'|}\frac{\sqrt{r(|x-x'|-r)}^{n-3}}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}(\frac{\tau+|x-x'|}{2}-r)^{n-2}}dr \\ &\leq \left(\frac{\delta}{4}\right)^{4-2n}|x-x'|^{2-n}\int_0^{|x-x'|}\sqrt{r(|x-x'|-r)}^{n-3}dr = \left(\frac{\delta}{4}\right)^{4-2n}\int_0^1\sqrt{r(1-r)}dr \leq \left(\frac{\delta}{4}\right)^{4-2n}. \end{aligned}$$

Therefore using (5.7) we obtain

$$(\tau-|x-x'|)^{-\frac{n-3}{2}}G_1(\tau,x,x') \leq 2^{n-2}\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})(T+\mathrm{diam}(X))^{\frac{n-3}{2}}\left(\frac{\delta}{2}\right)^{4-2n}. \tag{5.8}$$

Finally assume $\delta \leq \tau \leq \frac{\delta}{2}+|x-x'|$ and $|x-x'| < \tau \leq T$. From (5.7) and (5.4), it follows that

$$(\tau-|x-x'|)^{-\frac{n-3}{2}}G_1(\tau,x,x') \leq \frac{\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})(T+\mathrm{diam}(X))^{\frac{n-3}{2}}}{2^{n-2}|x-x'|^{n-2}}\int_{r_-(\tau,x,x')}^{r_+(\tau,x,x')}\frac{\sqrt{r(|x-x'|-r)}^{n-3}}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}(\frac{\tau+|x-x'|}{2}-r)^{n-2}}dr, \tag{5.9}$$

$$r_-(\tau,x,x') := \frac{|x-x'|+\delta-\tau}{2},\ r_+(\tau,x,x') := \frac{\tau-\delta+|x-x'|}{2}. \tag{5.10}$$

Note that

$$\begin{aligned} &\int_{r_-(\tau,x,x')}^{r_+(\tau,x,x')}\frac{\sqrt{r(|x-x'|-r)}^{n-3}}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}(\frac{\tau+|x-x'|}{2}-r)^{n-2}}dr = 2\int_{r_-(\tau,x,x')}^{\frac{|x-x'|}{2}}\frac{\sqrt{r(|x-x'|-r)}^{n-3}}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}(\frac{\tau+|x-x'|}{2}-r)^{n-2}}dr \\ &\leq 2\left(\frac{\tau}{2}\right)^{2-n}|x-x'|^{n-3}\int_{r_-(\tau,x,x')}^{\frac{|x-x'|}{2}}\frac{1}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}}dr = 2^{n-1}\tau^{2-n}|x-x'|^{n-3}\int_{r_-(\tau,x,x')}^{\frac{|x-x'|}{2}}\frac{1}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}}dr. \end{aligned} \tag{5.11}$$

Using (5.10) we obtain

$$\int_{r_-(\tau,x,x')}^{\frac{|x-x'|}{2}}\frac{1}{(\frac{\tau-|x-x'|}{2}+r)^{n-2}}dr = C(n,\tau) := \begin{cases} \ln\left(\frac{\tau}{\delta}\right), & \text{if } n=3, \\ \frac{1}{n-3}\left(\left(\frac{\delta}{2}\right)^{3-n}-\left(\frac{\tau}{2}\right)^{3-n}\right) & \text{otherwise.} \end{cases} \tag{5.12}$$

From (5.9), (5.11), (5.12) and the estimates $\delta \leq \tau < \frac{\delta}{2}+|x-x'|$, it follows that

$$(\tau-|x-x'|)^{-\frac{n-3}{2}}G_1(\tau,x,x') \leq 2^n\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\delta^{-(n-1)}(T+\mathrm{diam}(X))^{\frac{n-3}{2}}C(n,T), \tag{5.13}$$

where the constant $C(n,T)$ is defined in (5.12). Combining (5.2) with (5.6), (5.8) and (5.13), we obtain (2.16) for $n \geq 3$. □

5.2 Preliminary results for the proof of (2.17), (2.18) and (2.19)

To prove Theorem 2.2 (2.17), (2.18) and (2.19), we need the explicit expressions for γ_m, $m \geq 2$ given below and the following Lemmas 5.1–5.2, whose proof is given in Appendix A.

Let $m \geq 1$ and $z, z' \in \mathbb{R}^n$ such that $z \neq z'$. Let $\mu \geq 0$. We denote by $\mathcal{E}_{m,n}(\mu, z, z')$ the subset of $(\mathbb{R}^n)^m$ defined by

$$\mathcal{E}_{m,n}(\mu, z, z') = \{(y_1, \ldots, y_m) \in (\mathbb{R}^n)^m \mid |y_1| + \ldots + |y_m| + |z - z' - y_1 - \ldots - y_m| < \mu\}. \quad (5.14)$$

When $\mu \leq |z - z'|$, then $\mathcal{E}_{m,n}(\mu, z, z') = \emptyset$.

Lemma 5.1. *Let $n \geq 2$. Let $(\mu, z, z') \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ be such that $\mu > |z - z'| > 0$. Then*

$$\mathrm{Vol}_n(\mathcal{E}_{1,n}(\mu, z, z')) \leq \frac{\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\pi(\mu + |z - z'|)}{4} \left(\frac{\sqrt{\mu^2 - |z - z'|^2}}{2}\right)^{n-1}. \quad (5.15)$$

Lemma 5.2. *Let $n \geq 2$ and $\delta > 0$. Let N denote the nonnegative measurable function from $(0, T) \times \partial X \times \mathbb{R}^n$ to $[0, +\infty[$ defined by*

$$N(\mu, z, z') = \chi_{(0,+\infty)}(\mu - |z - z'|) \int_{\mathbb{S}^{n-1}} \frac{(\mu - (z - z') \cdot v)^{n-3}}{|z - z' - \mu v|^{2n-4}} dv, \quad (5.16)$$

for $(\mu, z, z') \in (0, T) \times \partial X \times \mathbb{R}^n$. When $n = 2$, then

$$C(N, 2) := \sup_{\substack{(\mu,z,z') \in (0,T) \times \partial X \times \mathbb{R}^n \\ \mu > |z - z'|}} \int_{y \in \mathcal{E}_{1,n}(\mu,z,z')} \frac{N(\mu - |y|, z, z' + y)}{|y|} dy < \infty. \quad (5.17)$$

When $n = 3$, then

$$\begin{aligned} C_1(N, 3) &:= \sup_{\substack{(\mu,z,z') \in (0,T) \times \partial X \times \mathbb{R}^n \\ \mu > |z - z'|}} (\mu - |z - z'|)^{-1} \left(1 + \ln\left(\frac{\mu + |z - z'|}{\mu - |z - z'|}\right)\right)^{-1} \\ &\quad \times \int_{\substack{y \in \mathcal{E}_{1,n}(\mu,z,z') \\ \mu - |y| \geq \delta,\ |z - z' - y| \geq \delta,\ |y| \geq \delta}} N(\mu - |y|, z, z' + y) dy < \infty, \quad (5.18) \\ C_2(N, 3) &:= \sup_{\substack{(\mu,z,z') \in (0,T) \times \partial X \times \mathbb{R}^n \\ \mu > |z - z'|}} (\mu - |z - z'|)^{-1} \mu |z - z'| \left(1 + \ln\left(\frac{\mu + |z - z'|}{\mu - |z - z'|}\right)\right)^{-2} \\ &\quad \times \int_{y \in \mathcal{E}_{1,n}(\mu,z,z')} \frac{N(\mu - |y|, z, z' + y)}{|y|^2} dy < \infty, \quad (5.19) \end{aligned}$$

When $n \geq 4$, then

$$C(N, n) := \sup_{(\mu,z,z') \in (0,T) \times \partial X \times \mathbb{R}^n} \mu |z - z'|^{n-2} N(\mu, z, z') < \infty. \quad (5.20)$$

The explicit expression for γ_m, $m \geq 2$, is given by

$$\begin{aligned} \gamma_2(\tau, x, x') &:= \int_{\substack{y \in \mathcal{E}_{1,n}(\tau,x,x') \\ x' + y \in X}} \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x) \cdot v) W(x, v) \big[E(x, x - (\tau - |y| - s_1)v, x' + y, x') \\ &\chi_{(0,\tau_-(x,v))}(\tau - |y| - s_1) k(x - (\tau - s_1 - |y|)v, v_1, v) k(x' + y, v', v_1) S(x', v') \\ &|\nu(x') \cdot v'|\big]_{\substack{s_1 = \frac{|x - x' - y - (\tau - |y|)v|^2}{2(\tau - |y| - (x - x' - y) \cdot v)} \\ v_1 = \frac{x - x' - y - (\tau - s_1 - |y|)v}{s_1},\quad v' = \frac{y}{|y|}}} \frac{2^{n-2}(\tau - |y| - (x - x' - y) \cdot v)^{n-3}}{|x - x' - y - (\tau - |y|)v|^{2n-4} |y|^{n-1}} dy dv, \quad (5.21) \end{aligned}$$

and for $\tau \in \mathbb{R}$ and a.e. $(x, x') \in \partial X \times \partial X$ and for $m \geq 3$:

$$
\begin{aligned}
\gamma_m(\tau, x, x') &:= \int_{\substack{(y_2,\ldots,y_m)\in\mathcal{E}_{m-1,n}(\tau,x,x') \\ x'+\sum_{j=i}^m y_i \in X \text{ for } i=2,\ldots,m}} \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x)\cdot v) W(x,v) \\
&\times \frac{2^{n-2}\left(\tau - |y_2| - \ldots - |y_m| - (x - x' - y_2 - \ldots - y_m)\cdot v\right)^{n-3}}{|y_2|^{n-1}\ldots|y_m|^{n-1}|x - x' - y_2 - \ldots - y_m - (\tau - |y_2| - \ldots - |y_m|)v|^{2n-4}} \\
&\times \left[\chi_{(0,\tau_-(x,v))}(\tau - s_1 - |y_2| - \ldots - |y_m|) E(x, x - (\tau - s_1 - |y_2| - \ldots - |y_m|)v,\right. \\
&x' + y_m \ldots + y_2, \ldots, x' + y_m, x') k(x - (\tau - s_1 - |y_2| - \ldots - |y_m|)v, v_1, v) \\
&\times \prod_{i=1}^{m-2} k(x' + y_m + \ldots + y_{i+1}, v_{i+1}, v_i) k(x' + y_m, v', v_{m-1}) S(x', v') \\
&|\nu(x')\cdot v'|\Big]_{\substack{v_1 = \frac{x - x' - y_2 - \ldots - y_m - (\tau - s_1 - |y_2| - \ldots - |y_m|)v}{s_1} \\ s_1 = \frac{|x - x' - y_2 - \ldots - y_m - (\tau - |y_2| - \ldots - |y_m|)v|^2}{2(t - |y_2| - \ldots - |y_{m-1}| - (x - x' - y_2 - \ldots y_{m-1})\cdot v)},\ v' = \frac{y_m}{|y_m|},\ v_i = \frac{y_i}{|y_i|},\ i=2\ldots m-1}} dy_2 \ldots dy_m dv. \quad (5.22)
\end{aligned}
$$

5.3 Proof of (2.17), (2.18) and (2.19)

Let $\tau > 0$ and let $x, x' \in \partial X$ such that $|x - x'| < \tau$ and $x \neq x'$. We set $t_0 = |x - x'|$. Using spherical coordinates ("$y_i = s_i\omega_i$", $(s_i, \omega_i) \in (0, +\infty) \times \mathbb{S}^{n-1}$, $i = 2 \ldots m$) and (5.14), we obtain

$$
\int_{\mathcal{E}_{m-1,n}(\tau,x,x')} \frac{dy_2 \ldots dy_m}{|y_2|^{n-1}\ldots|y_m|^{n-1}} \leq \mathrm{Vol}(\mathbb{S}^{n-1})^{m-1} \int_{\substack{s_2+\ldots+s_m<\tau \\ s_i \geq 0 \text{ for } i=2\ldots m}} ds_2 \ldots ds_m = \mathrm{Vol}(\mathbb{S}^{n-1})^{m-1} \frac{\tau^{m-1}}{(m-1)!}, \tag{5.23}
$$

for all $m \in \mathbb{N}$, $m \geq 2$. Note that using (2.2) and (5.14), we obtain

$$
x' + \sum_{i=2}^m y_i \in \mathrm{supp}k \Rightarrow \delta \leq |x - x' - \sum_{i=2}^m y_i| < \tau - \sum_{i=2}^m |y_i|, \tag{5.24}
$$

$$
x' + y_m \in \mathrm{supp}k \Rightarrow \delta \leq |y_m|, \tag{5.25}
$$

for $(y_2, \ldots, y_m) \in \mathcal{E}_{m-1,n}(\tau, x, x')$ (recall that $(x', x) \in \partial X^2$).

We first look for an upper bound on $|\gamma_m(\tau, x, x')|$, $m \geq 2$. Using the explicit expression for the multiple scattering kernels $\gamma_m(\tau, x, x')$ (see (5.21)–(5.22)) and the fact that σ is a nonnegative function, we obtain that

$$
|\gamma_2(\tau, x, x')| \leq 2^{n-2}\|W\|_\infty \|S\|_\infty \|k\|_\infty^2 G_2(\tau, x, x'), \tag{5.26}
$$

where N, $\mathcal{E}_{1,n}(\tau, x, x')$ and G_2 are defined by (5.16), (5.14) and

$$
G_2(\mu, z, z') := \int_{\substack{y\in\mathcal{E}_{1,n}(\mu,z,z') \\ z'+y\in \mathrm{supp}k}} \frac{N(\mu - |y|, z, z' + y)}{|y|^{n-1}} dy, \tag{5.27}
$$

for $(\mu, z, z') \in (0, +\infty) \times \partial X \times \mathbb{R}^n$. We also obtain

$$
|\gamma_m(\tau, x, x')| \leq 2^{n-2}\|W\|_\infty \|S\|_\infty \|k\|_\infty^m G_m(\tau, x, x'), \tag{5.28}
$$

where

$$
G_m(\tau, x, x') = \int_{\substack{(y_3,\ldots,y_m)\in\mathcal{E}_{m-2,n}(\tau,x,x') \\ x'+\sum_{i=j}^m y_i \in \mathrm{supp}k,\ j=3\ldots m}} \frac{G_2(\tau - \sum_{i=3}^m |y_i|, x, x' + \sum_{i=3}^m y_i)}{|y_3|^{n-1}\ldots|y_m|^{n-1}} dy_3 \ldots dy_m, \tag{5.29}
$$

and G_2 (resp. $\mathcal{E}_{m-2,n}(\tau,x,x')$) is defined by (5.27) (resp. (5.14)).

We now prove (2.17). Assume $n=2$. From (5.26) and (5.17) we obtain

$$|\gamma_2(\tau,x,x')| \le 2^{n-2}\|W\|_\infty\|S\|_\infty\|k\|_\infty^2 C(N,2). \tag{5.30}$$

Then using (5.29), (5.17) and (5.23), we obtain

$$G_m(\tau,x,x') \le C(N,2)\int_{\mathcal{E}_{m-2,n}(\tau,x,x')} \frac{dy_3\ldots dy_m}{|y_3|\ldots|y_m|} = C(N,2)(2\pi)^{m-2}\frac{\tau^{m-2}}{(m-2)!},\ \text{for } m\ge 3. \tag{5.31}$$

Finally combining (5.31) and (5.28), we obtain

$$|\gamma_m(\tau,x,x')| \le 2^{n-2}C(N,2)(2\pi)^{m-2}\|W\|_\infty\|S\|_\infty\|k\|_\infty^m\frac{\tau^{m-2}}{(m-2)!},\ \text{for } m\ge 3. \tag{5.32}$$

Statement (2.17) follows from (5.30) and (5.32).

We prove (2.18). Assume $n=3$. Using (5.24)–(5.25) (with "$m=2$") and (5.27), we obtain

$$G_2(\tau,x,x') \le \delta^{-2}\int_{\substack{y\in\mathcal{E}_{1,3}(\tau,x,x')\\ \tau-|y|\ge\delta,\ |x-x'-y|\ge\delta,\ |y|\ge\delta}} N(\tau-|y|,x,x'+y)dy. \tag{5.33}$$

Therefore using (5.18) we obtain

$$\sup_{\substack{(s,z,z')\in(0,T)\times\partial X\times\partial X\\ s>|z-z'|>0}} (s-|z-z'|)^{-1}\left(1+\ln\left(\frac{s+|z-z'|}{s-|z-z'|}\right)\right)^{-1} G_2(s,z,z') \le \delta^{-2}C_1(N,3)<\infty. \tag{5.34}$$

Now assume $n=3$ and $m\ge 3$. Using (5.29) and using (5.19) and the estimate $\sup_{r\in(0,1)} r(1-\ln(r))^2<\infty$ we obtain

$$G_m(\tau,x,x') \le D\int_{\substack{(y_3,\ldots,y_m)\in\mathcal{E}_{m-2,n}(\tau,x,x'_0)\\ (x'+\sum_{i=3}^m y_i,x'+y_m)\in(\mathrm{supp}k)^2}} \frac{(|x-x'-y_3-\ldots-y_m|+\tau-\sum_{i=3}^m|y_i|)dy_m\ldots dy_3}{|y_3|^{n-1}\ldots|y_m|^{n-1}(\tau-\sum_{i=3}^m|y_i|)|x-x'-y_3-\ldots-y_m|}, \tag{5.35}$$

where $D:=\sup_{r\in(0,1)} r(1-\ln(r))^2C_2(N,3)$. If $m=3$, then using (5.24)–(5.25) with "$(y_2,\ldots,y_m)$" replaced by "$(y_3,\ldots,y_m)$", we obtain

$$G_3(\tau,x,x') \le 2\tau\delta^{-4}D\mathrm{Vol}(\mathcal{E}_{1,3}(\tau,x,x')) \tag{5.36}$$

(we also used the estimate $|x-x'-y_3|+\tau-|y_3|\le 2\tau$ for $y_3\in\mathcal{E}_{1,3}(\tau,x,x')$). If $m\ge 4$, then using (5.24)–(5.25) with "$(y_2,\ldots,y_m)$" replaced by "$(y_3,\ldots,y_m)$", we obtain

$$\begin{aligned} G_m(\tau,x,x') &\le 2\tau\delta^{-4}D\mathrm{Vol}(\mathcal{E}_{1,3}(\tau,x,x'))\int_{(y_3,\ldots,y_{m-1})\in\mathcal{E}_{m-3,3}(\tau,x,x')}\frac{dy_3\ldots dy_{m-1}}{|y_3|^{n-1}\ldots|y_{m-1}|^{n-1}}\\ &= 2\tau\delta^{-4}D\mathrm{Vol}(\mathbb{S}^{n-1})^{m-3}\frac{\tau^{m-3}}{(m-3)!}\mathrm{Vol}(\mathcal{E}_{1,3}(\tau,x,x')) \end{aligned} \tag{5.37}$$

(we also used the estimate $|x-x'-y_3-\ldots-y_m|+\tau-|y_3|-\ldots-|y_m|\le 2\tau$ for $(y_3,\ldots,y_m)\in\mathcal{E}_{m-2,3}(\tau,x,x')$ and we used (5.23)). Statement (2.18) follows from (5.26), (5.28), (5.34) and (5.36)–(5.37) and (5.15).

We prove (2.19). Let $n \geq 4$ and $m \geq 2$. Using (5.27), (5.29) and (5.20), we obtain

$$G_m(\tau, x, x') \leq C(N,n) \int\limits_{\substack{(y_2,\ldots,y_m)\in\mathcal{E}_{m-1,n}(\tau,x,x_0')\\ x'+y_m\in\mathrm{supp}k\\ x'+\sum_{i=2}^m y_i\in\mathrm{supp}k}} \frac{dy_2\ldots dy_m}{|y_2|^{n-1}\ldots|y_m|^{n-1}|x-x'-\sum_{i=2}^m y_i|^{n-2}(\tau-\sum_{i=2}^m|y_i|)}. \quad (5.38)$$

From (5.38) and (5.24)–(5.25), it follows that

$$G_m(\tau, x, x') \leq \delta^{-2n+2} C(N,n)\mathrm{Vol}(\mathcal{E}_{1,n}(\tau, x, x')) \quad (5.39)$$

for $m = 2$, and

$$\begin{aligned} G_m(\tau, x, x') &\leq \delta^{-2n+2} C(N,n) \int\limits_{(y_2,\ldots,y_m)\in\mathcal{E}_{m-1,n}(\tau,x,x')} \frac{dy_2\ldots dy_m}{|y_2|^{n-1}\ldots|y_{m-1}|^{n-1}} \\ &= \delta^{-2n+2} C(N,n)\mathrm{Vol}(\mathbb{S}^{n-1})^{m-2}\mathrm{Vol}(\mathcal{E}_{1,n}(\tau, x, x'))\frac{\tau^{m-2}}{(m-2)!} \quad (5.40)\end{aligned}$$

for $m \geq 3$ (we also use (5.23) to prove (5.40)). From (5.39), (5.40) and (5.15) it follows that

$$G_m(\tau, x, x') \leq \delta^{-2n+2} C(N,n)\mathrm{Vol}(\mathbb{S}^{n-1})^{m-2}\frac{\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\pi(\mu+|z-z'|)}{4}\left(\frac{\sqrt{\mu^2-|z-z'|^2}}{2}\right)^{n-1}\frac{\tau^{m-2}}{(m-2)!} \quad (5.41)$$

for $m \geq 2$. Statement (2.19) follows from (5.26), (5.28) and (5.41).

6 Proof of Theorems 2.3, 3.1 and 3.4

Proof of Theorem 2.3. For the sake of simplicity and without loss of generality we assume $v_0 = (1, 0, \ldots, 0)$. Assume that conditions (2.1)–(2.2) are satisfied. For $n \geq 2$ consider the following open subset of $(0, +\infty) \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$

$$\mathcal{D} := \{(s, v, v') \in (0, +\infty) \times \mathbb{S}^{n-1}_{x,+} \times \mathbb{S}^{n-1}_{x_0',-} \mid s \in (0, \tau_-(x, v))\}. \quad (6.1)$$

Then we introduce the bounded function Ψ_n on $\mathcal{D}$ defined by

$$\Psi_n(s, v, v') = 2^{n-2} W(x, v)(\nu(x)\cdot v)E(x, x-sv, x_0')k(x-sv, v', v)S(x_0', v')|\nu(x_0')\cdot v'|, \quad (6.2)$$

for $(s, v, v') \in \mathcal{D}$. Note that from convexity of X it follows that $\tau_\pm$ is continuous on $\Gamma_\mp$ and $E(x, x-sv, x_0') = e^{-\int_0^s \sigma(x-pv,v)dp - \int_0^{|x-x_0'-sv|}\sigma(x-sv-p\frac{x-x_0'-sv}{|x-x_0'-sv|}, \frac{x-x_0'-sv}{|x-x_0'-sv|})dp}$ for $v \in \mathbb{S}^{n-1}_{x,+}$ and $0 < s < \tau_-(x, v)$. Under (2.1)–(2.2) we obtain that

$$\begin{aligned}&\Psi_n(s, v, v') = 0 \text{ for } (s, v, v') \in (0, +\infty) \times \mathbb{S}^{n-1}_{x,+} \times \mathbb{S}^{n-1}_{x_0',-} \text{ such that } x - sv \notin \bar{Z}, \\ &\text{and the function } \Psi_n \text{ is continuous at any point } (s, v, v') \in \mathcal{D} \text{ such that } x - sv \in Z.\end{aligned} \quad (6.3)$$

We first prove (2.22) for $n = 2$. Let $\tau > t_0$. From (6.2), (2.14), it follows that

$$\begin{aligned}\gamma_1(\tau, x, x_0') &= \int_{-\alpha_0}^{\pi-\alpha_0} \frac{1}{\tau - t_0\cos(\Omega)}\left[\chi_{(0,\tau_-(x,v))}(s)\Psi_2(s, v, v')\right]_{\substack{v=(\cos\Omega,\sin\Omega)\\ v'=\frac{t_0(1,0)-sv}{\tau-s}\\ s=\frac{\tau^2-t_0^2}{2(\tau-t_0\cos(\Omega))}}} d\Omega \\ &= \gamma_{1,1}(\tau, x, x_0') + \gamma_{1,2}(\tau, x, x_0'), \quad (6.4)\end{aligned}$$

where $\mathbb{S}^{n-1}_{x,+} = \{(\cos\Omega, \sin\Omega) \mid -\alpha_0 < \Omega < \pi - \alpha_0\}$ $(0 < \alpha_0 < \pi)$ and

$$\gamma_{1,1}(\tau, x, x'_0) = \int_0^{\pi} \frac{\chi_{(0,\pi-\alpha_0)}(\Omega)}{\tau - t_0\cos(\Omega)} \left[\chi_{(0,\tau_-(x,v))}(s)\Psi_2(s, v, v')\right]_{\substack{v=(\cos\Omega,\sin\Omega)\\ v'=\frac{t_0(1,0)-sv}{\tau-s},\ s=\frac{\tau^2-t_0^2}{2(\tau-t_0\cos(\Omega))}}} d\Omega, \quad (6.5)$$

$$\gamma_{1,2}(\tau, x, x'_0) = \int_{-\pi}^{0} \frac{\chi_{(-\alpha_0,0)}(\Omega)}{\tau - t_0\cos(\Omega)} \left[\chi_{(0,\tau_-(x,v))}(s)\Psi_2(s, v, v')\right]_{\substack{v=(\cos\Omega,\sin\Omega)\\ v'=\frac{t_0(1,0)-sv}{\tau-s},\ s=\frac{\tau^2-t_0^2}{2(\tau-t_0\cos(\Omega))}}} d\Omega. \quad (6.6)$$

We now prove that

$$\sqrt{\tau - t_0}\gamma_{1,i}(\tau, x, x'_0) \to \frac{W(x, v_0)S(x'_0, v_0)(\nu(x)\cdot v_0)|\nu(x'_0)\cdot v_0|E(x'_0, x)}{\sqrt{2t_0}} \int_0^{t_0} \frac{k(x - sv_0, v_0, v_0)}{\sqrt{s(t_0 - s)}}ds,$$
as $\tau \to t_0^+$, (6.7)

for $i = 1, 2$. Then adding (6.7) for $i = 1$ and $i = 2$, we obtain (2.22). We only prove (6.7) for $i = 1$ since the proof for $i = 2$ is similar. Let $\tau > t_0$. Using the change of variables $s = \frac{\tau^2-t_0^2}{2(\tau-t_0\cos(\Omega))} - \frac{\tau-t_0}{2}$, we obtain

$$\gamma_{1,1}(\tau, x, x'_0) = \frac{1}{\sqrt{\tau^2 - t_0^2}} \int_0^{t_0} \chi_{(0,\pi-\alpha_0)}(\Omega(s,\tau)) \frac{\chi_{(0,\tau_-(x,v(s,\tau)))}(s + \frac{\tau-t_0}{2})\Psi_2(s, v(s,\tau), v'(s,\tau))}{\sqrt{s(t_0 - s)}}d\tau, \quad (6.8)$$

where

$$\begin{aligned} &v(s,\tau) = (\cos\Omega(s,\tau), \sin\Omega(s,\tau)),\ \Omega(s,\tau) = \arccos\Big(\frac{\tau - \frac{\tau^2-t_0^2}{2s+\tau-t_0}}{t_0}\Big),\\ &v'(s,\tau) = \frac{t_0(1,0) - \left(s + \frac{\tau-t_0}{2}\right)v(s,\tau)}{\frac{\tau+t_0}{2} - s}. \end{aligned} \quad (6.9)$$

Let $s \in (0, t_0)$. From (6.9), it follows that

$$v(s,\tau) \to (1,0) \text{ as } \tau \to t_0^+,\ v'(s,\tau) \to (1,0) \text{ as } \tau \to t_0^+. \quad (6.10)$$

Note that using the definition of v_0 and using the assumption $x'_0 + \varepsilon(x - x'_0) \in X$ for some $\varepsilon \in (0,1)$ we obtain $t_0 = \tau_-(x, v_0)$. Note also that the function $s \mapsto \frac{1}{\sqrt{s(t_0-s)}}$, $s \in (0, t_0)$, is integrable in $(0, t_0)$. Therefore, using (6.3), the boundedness of Ψ_2 on $\mathcal{D}$ and the Lebesgue dominated convergence theorem, we obtain (6.7). This proves (2.22) when $n = 2$.

Let $n \geq 3$ and prove (2.23). From (6.2) and (2.14), it follows that

$$\gamma_1(\tau, x, x'_0) = \int_{\mathbb{S}^{n-1}} \frac{(\tau - t_0 v_0\cdot v)^{n-3}}{|t_0 v_0 - \tau v|^{2n-4}} \chi_{(0,+\infty)}(\nu(x)\cdot v)\Psi_n(s, v, v')_{\substack{v'=\frac{t_0v_0-sv}{\tau-s}\\ s=\frac{\tau^2-t_0^2}{2(\tau-t_0v\cdot v_0)}}} dv, \quad (6.11)$$

for $\tau > |x - x'_0|$.

Let $\Phi(\Omega, \omega) = (\sin\Omega, \cos(\Omega)\omega_1, \ldots, \cos(\Omega)\omega_{n-1})$ for $\Omega \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\omega = (\omega_1, \ldots, \omega_{n-1}) \in \mathbb{S}^{n-2}$. Using spherical coordinates we obtain

$$\begin{aligned} \gamma_1(\tau, x, x'_0) = &\int_{-\pi/2}^{\pi/2} \cos(\Omega)^{n-2} \frac{(\tau - t_0\sin(\Omega))^{n-3}}{(t_0^2 + \tau^2 - 2t_0\tau\sin(\Omega))^{n-2}} \\ &\int_{\mathbb{S}^{n-2}} \chi_{(0,+\infty)}(\nu(x)\cdot\Phi(\Omega,\omega))\Psi_n(s, \Phi(\Omega,\omega), v')_{\substack{v'=\frac{t_0v_0-s\Phi(\Omega,\omega)}{\tau-s}\\ s=\frac{\tau^2-t_0^2}{2(\tau-t_0\sin(\Omega))}}} d\omega d\Omega, \end{aligned} \quad (6.12)$$

for $\tau > t_0$. Performing the change of variables "$r = \frac{\tau^2 - t_0^2}{2(\tau - t_0 \sin(\Omega))} - \frac{\tau - t_0}{2}$" on the first integral on the right-hand side of (6.12), we obtain

$$\gamma_1(\tau, x, x_0') = 2^{2-n} t_0^{2-n} (\tau^2 - t_0^2)^{\frac{n-3}{2}} \int_0^{t_0} \frac{\sqrt{r(t_0 - r)}^{n-3}}{(\frac{\tau - t_0}{2} + r)^{n-2} (\frac{\tau + t_0}{2} - r)^{n-2}} \int_{\mathbb{S}^{n-2}} \left[\chi(\Phi(\Omega, \omega)) \Psi_n(r + \frac{\tau - t_0}{2}, \Phi(\Omega, \omega), v') \right]_{\substack{\Omega = \arcsin\left(t_0^{-1}\left(\tau - \frac{(\tau^2 - t_0^2)}{2(r + \frac{\tau - t_0}{2})}\right)\right) \\ s = r + \frac{\tau - t_0}{2},\ v' = \frac{t_0 v_0 - s\Phi(\Omega, \omega)}{\tau - s}}} d\omega dr. \tag{6.13}$$

Therefore using (6.13), (6.3) and (6.2) and using Lebesgue dominated convergence theorem, we obtain (2.23). This concludes the proof of Theorem 2.3. □

Proof of Theorem 3.1. We now prove (3.1). Let $x_0' \in \partial X$. For $\varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2$ and $\varepsilon_3 \in (0, +\infty)$, let $(f_{\varepsilon_1}, g_{\varepsilon_2}) \in C^1(\partial X) \times C^1(\mathbb{R})$ satisfy

$$g_{\varepsilon_2} \geq 0,\ f_{\varepsilon_1} \geq 0,\ \mathrm{supp} g_{\varepsilon_2} \subseteq (0, \min(\varepsilon_2, \eta)), \tag{6.14}$$

$$\mathrm{supp} f_{\varepsilon_1} \subseteq \{x' \in \partial X \mid |x' - x_0'| < \varepsilon_1\}, \tag{6.15}$$

$$\int_0^\eta g_{\varepsilon_2}(t') dt' = 1,\ \int_{\partial X} f_{\varepsilon_1}(x') d\mu(x') = 1, \tag{6.16}$$

for $\varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2$. Therefore, $\phi_\varepsilon := g_{\varepsilon_2} f_{\varepsilon_1}$ is an approximation of the delta function at $(0, x_0') \in \mathbb{R} \times \partial X$ for $\varepsilon := (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2$. Let $\psi_{\varepsilon_3} \in L^\infty((0, T) \times \partial X)$ be defined by

$$\psi_{\varepsilon_3}(t, x) = \chi_{(-\varepsilon_3, \varepsilon_3)}(t - |x - x_0'|)(2\chi_{(0, +\infty)}((E - \tilde{E})(x, x_0')) - 1),\ (t, x) \in (0, T) \times \partial X, \tag{6.17}$$

for $\varepsilon_3 > 0$. From (2.12) and (2.15) it follows that

$$\int_{(0,T) \times \partial X} \psi_{\varepsilon_3}(t, x)(A_{S,W} - \tilde{A}_{S,W}) \phi_\varepsilon(t, x) dt d\mu(x) = I_0(\psi_{\varepsilon_3}, \phi_\varepsilon) + \int_{(0,T) \times \partial X \times (0,\eta) \times \partial X} \psi_{\varepsilon_3}(t, x) \phi_\varepsilon(t', x')(\Gamma_1 - \tilde{\Gamma}_1)(t - t', x, x') dt d\mu(x) dt' d\mu(x'), \tag{6.18}$$

for $\varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty)$ and $\varepsilon_3 \in (0, +\infty)$, where

$$I_0(\psi_{\varepsilon_3}, \phi_\varepsilon) = \int_{\substack{(0,T)_t \times \partial X_x \times \partial X_{x'} \\ |x - x'| < t}} \psi_{\varepsilon_3}(t, x) \phi_\varepsilon(t - |x - x'|, x') \frac{E(x, x') - \tilde{E}(x, x')}{|x - x'|^{n-1}} \times [W(x, v) S(x', v)(\nu(x) \cdot v)|\nu(x') \cdot v|]_{v = \frac{x - x'}{|x - x'|}} dt d\mu(x) d\mu(x'). \tag{6.19}$$

From (2.16), (2.17), (2.18) and (2.19) it follows that

$$(\tau - |x - x'|)^{\frac{3-n}{2}} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, x, x') \in L^\infty((0, T) \times \partial X \times \partial X). \tag{6.20}$$

Combining (6.18) and the equality $\|\phi_\varepsilon\|_{L^1((0,\eta) \times \partial X)} = 1$ and the estimate $\|\psi_{\varepsilon_3}\|_{L^\infty((0,T) \times \partial X)} \leq 1$ and (6.20) we obtain

$$I_0(\psi_{\varepsilon_3}, \phi_\varepsilon) \leq \|A_{S,W} - \tilde{A}_{S,W}\|_{\eta, T} + C \Delta_1(\psi_{\varepsilon_3}, \phi_\varepsilon), \tag{6.21}$$

for $\varepsilon = (\varepsilon_1, \varepsilon_2) \in (0, +\infty)$ and $\varepsilon_3 \in (0, +\infty)$, where $C = \|(\tau - |x - x'|)^{\frac{3-n}{2}} (\Gamma_1 - \tilde{\Gamma}_1)(\tau, x, x') \|_{L^\infty((0,T) \times \partial X_x \times \partial X_{x'})}$ and

$$\Delta_1(\psi_{\varepsilon_3}, \phi_\varepsilon) = \int_{\substack{(0,T)_t \times \partial X_x \times (0,\eta)_{t'} \times \partial X_{x'} \\ |x - x'| < t - t'}} \psi_{\varepsilon_3}(t, x) \phi_\varepsilon(t', x')(t - t' - |x - x'|)^{\frac{n-3}{2}} dt d\mu(x) dt' d\mu(x'). \tag{6.22}$$

Note that the function $\Phi_{1,\varepsilon_3} : [0,\eta) \times \partial X \to \mathbb{R}$ defined by

$$\Phi_{1,\varepsilon_3}(t',x') := \int_{\substack{(0,T)_t\times\partial X_x\\ |x-x'|<t-t'}} \psi_{\varepsilon_3}(t,x)(t-t'-|x-x'|)^{\frac{n-3}{2}} dtd\mu(x),\ (t',x')\in[0,\eta)\times\partial X, \quad (6.23)$$

is continuous on $[0,\eta)\times\partial X$ for $\varepsilon_3\in(0,+\infty)$. Therefore, from (6.14)–(6.16) and the equality $\Delta_1(\psi_{\varepsilon_3},\phi_\varepsilon) = \int_{(0,\eta)\times\partial X}\phi_\varepsilon(t',x')\Phi_{1,\varepsilon_3}(t',x')dt'd\mu(x')$, it follows that

$$\lim_{\varepsilon_3\to0^+}\lim_{\varepsilon_2\to0^+}\lim_{\varepsilon_1\to0^+}\Delta_1(\psi_{\varepsilon_3},\phi_\varepsilon) = \lim_{\varepsilon_3\to0^+}\Phi_{1,\varepsilon_3}(0,x_0') = 0 \quad (6.24)$$

(we also used (6.23), (6.17) and the Lebesgue dominated convergence theorem to prove that $\lim_{\varepsilon_3\to0^+}\Phi_{1,\varepsilon_3}(0,x_0') = 0$). Note that under condition (2.2) the function $\Phi_{0,\varepsilon_2,\varepsilon_3} : \partial X \to \mathbb{R}$ defined by

$$\begin{aligned}\Phi_{0,\varepsilon_2,\varepsilon_3}(x') \;=\; &\int_{\substack{(0,T)_t\times\partial X_x\\ |x-x'|<t}} \psi_{\varepsilon_3}(t,x)g_{\varepsilon_2}(t-|x-x'|)\frac{E(x,x')-\tilde E(x,x')}{|x-x'|^{n-1}}\\ &\times[W(x,v)S(x',v)(\nu(x)\cdot v)|\nu(x')\cdot v|]_{v=\frac{x-x'}{|x-x'|}}\,dtd\mu(x),\end{aligned} \quad (6.25)$$

is continuous on ∂X for $(\varepsilon_2,\varepsilon_3)\in(0,+\infty)^2$. Therefore, from the equality $I_0(\psi_{\varepsilon_3},\phi_\varepsilon) = \int_{\partial X}\Phi_{0,\varepsilon_2,\varepsilon_3}(x')$ $\times f_{\varepsilon_1}(x')d\mu(x')$ (see (6.19)) it follows that

$$\lim_{\varepsilon_1\to0^+} I_0(\psi_{\varepsilon_3},\phi_\varepsilon) = \Phi_{0,\varepsilon_2,\varepsilon_3}(x_0'),\ \text{for } (\varepsilon_2,\varepsilon_3)\in(0,+\infty)^2. \quad (6.26)$$

Thus, using the Lebesgue dominated convergence theorem and (6.25) we obtain

$$\lim_{\varepsilon_3\to0^+}\lim_{\varepsilon_2\to0^+}\lim_{\varepsilon_1\to0^+} I_0(\psi_{\varepsilon_3},\phi_\varepsilon) = \int_{\partial X_x}\frac{\left|E(x,x_0')-\tilde E(x,x_0')\right|}{|x-x_0'|^{n-1}}[W(x,v)S(x_0',v)(\nu(x)\cdot v)|\nu(x_0')\cdot v|]_{v=\frac{x-x_0'}{|x-x_0'|}}\,d\mu(x). \quad (6.27)$$

Combining (6.27), (6.24) and (6.21) we obtain the formula (4.10). Using (4.10) and the estimates $\inf_{\Gamma_-}S>0$ and $\inf_{\Gamma_+}W>0$ and the change of variables $x = x_0'+\tau_+(x_0',v_0)v_0$ ($\frac{\nu(x)\cdot v_0}{|x-x_0'|^{n-1}}d\mu(x) = dv_0$) we obtain (3.1) where the constant C_1, which appears on the right-hand side of (3.1), is given by $C_1 = \left(\inf_{\Gamma_-}S\inf_{\Gamma_+}W\right)^{-1}$.

We now prove (3.2). Let $x\in\partial X$ be such that $px_0'+(1-p)x\in Z$ for some $p\in(0,1)$. We set $t_0 = |x-x_0'|$ and $v_0 = \frac{x-x_0'}{|x-x_0'|}$. From (2.16), (2.17), (2.18) and (2.19) it follows that

$$\begin{aligned}(\tau-|x-x_0'|)^{\frac{3-n}{2}}|\gamma_1-\tilde\gamma_1|(\tau,x,x_0') &\le (\tau-|x-x_0'|)^{\frac{3-n}{2}}|\Gamma_2-\tilde\Gamma_2|(\tau,z,z')\\ &+\|(s-|z-z'|)^{\frac{3-n}{2}}(\Gamma_1-\tilde\Gamma_1)(s,z,z')\|_{L^\infty((0,T)_s\times\partial X_z\times\partial X_{z'})},\end{aligned} \quad (6.28)$$

for $\tau>|x-x_0'|$. From (2.17), (2.18)–(2.19) it turns out that $\lim_{\tau\to|x-x_0'|^+}(\tau-|x-x_0'|)^{\frac{3-n}{2}}|\Gamma_2-\tilde\Gamma_2|(\tau,z,z') = 0$. Therefore applying (2.22) and (2.23) on the left-hand side of (6.28) we obtain

$$\begin{aligned}&2^{\frac{1-n}{2}}|x-x_0'|^{-\frac{n-1}{2}}C_nS(x_0',v_0)W(x,v_0)|\nu(x_0')\cdot v_0|(\nu(x)\cdot v_0)\\ &\times\left|\int_0^{t_0}\frac{e^{-\int_0^{t_0}\sigma(x_0'+sv_0,v_0)ds}k(x-pv_0,v_0,v_0)-e^{-\int_0^{t_0}\tilde\sigma(x_0'+sv_0,v_0)ds}\tilde k(x-pv_0,v_0,v_0)}{p^{\frac{n-1}{2}}(t_0-p)^{\frac{n-1}{2}}}dp\right|\\ &\le\|(s-|z-z'|)^{\frac{3-n}{2}}(\Gamma_1-\tilde\Gamma_1)(s,z,z'))\|_{L^\infty((0,T)_s\times\partial X_z\times\partial X_{z'})},\end{aligned} \quad (6.29)$$

where $C_n = 2$ if $n=2$ and $C_n = \mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})$ if $n\ge3$. Then note that $C_X := \inf_{x_1\in\partial X,\ z\in\bar Z}\nu(x_1)\cdot\frac{x_1-z}{|x_1-z|}>0$ since X is a bounded convex subset of $\mathbb{R}^n$ with C^1 boundary and $\bar Z\subset X$. Therefore (3.2) follows from (6.29) where the constant C_2 which appears on the right-hand side of (3.2) is given by $C_2 = \frac{2^{\frac{n-1}{2}}\mathrm{diam}(X)^{\frac{n-1}{2}}}{C_nC_X^2\inf_{\Gamma_-}S\inf_{\Gamma_+}W}$. Theorem 3.1 is proved. □

Proof of Theorem 3.4. We first prove (3.7). We extend σ and $\tilde\sigma$ by 0 outside Y. For a bounded and continuous function f on Y consider the X-ray transform $Pf:\mathbb{S}^{n-1}\times\mathbb{R}^n\to\mathbb{R}$ defined by (3.3) (we extend f by 0 outside Y). We recall the following estimate

$$\|f\|_{H^{-\frac{1}{2}}(Y)}\le\left(\int_{\mathbb{S}^{n-1}}\int_{\Pi_v}|Pf(v,x)|^2dxdv\right)^{\frac{1}{2}},\tag{6.30}$$

where $\Pi_v:=\{x\in\mathbb{R}^n\mid v\cdot x=0\}$ for $v\in\mathbb{S}^{n-1}$. Note that using the estimate $\|\sigma\|_\infty\le M$, we obtain

$$\int_0^{\tau_+(x_0',v)}\sigma(x_0'+sv,v)ds\le M\tau_+(x_0',v)\le M\mathrm{diam}(X),\ \text{for }(x_0',v)\in\Gamma_-.\tag{6.31}$$

Replacing σ by $\tilde\sigma$ on the left-hand side of (6.31) we obtain an estimate similar to (6.31) for $\tilde\sigma$. Therefore using the estimate $|e^{t_1}-e^{t_2}|\ge e^{-M\mathrm{diam}(X)}|t_1-t_2|$ for $(t_1,t_2)\in[0,+\infty)^2$, $\max(t_1,t_2)\le M\mathrm{diam}(X)$, we obtain

$$\left|e^{-\int_0^{\tau_+(x_0',v)}\sigma(x_0'+sv,v)ds}-e^{-\int_0^{\tau_+(x_0',v)}\tilde\sigma(x_0'+sv,v)ds}\right|\ge e^{-M\mathrm{diam}(X)}\left|P(\sigma-\tilde\sigma)(v,x_0')\right|,\tag{6.32}$$

for $(x_0',v)\in\Gamma_-$. Integrating the left-hand side of (3.1) over ∂X and using (6.32), we obtain

$$\int_{\Gamma_-}|P(\sigma-\tilde\sigma)(v,x_0')|\,d\xi(v,x_0')\le e^{M\mathrm{diam}(X)}\mathrm{Vol}(\partial X)C_1\|A_{S,W}-\tilde A_{S,W}\|_{\eta,T},\tag{6.33}$$

where C_1 is the constant that appears on the right-hand side of (3.1). Note that using that X is a convex open subset of $\mathbb{R}^n$ with C^1 boundary we obtain $\int_{\Gamma_-}|P(\sigma-\tilde\sigma)(v,x_0')|\,d\xi(v,x_0')=\int_{\mathbb{S}^{n-1}}\int_{\Pi_v}|P(\sigma-\tilde\sigma)(v,x)|dxdv$. Therefore using (6.33) and the estimate $|P(\sigma-\tilde\sigma)(v,x)|^2\le\|\sigma-\tilde\sigma\|_{L^\infty(Y)}\mathrm{diam}(X)|P(\sigma-\tilde\sigma)(v,x)|$ for $(v,x)\in T\mathbb{S}^{n-1}$ (see (6.31) and the estimates $\sigma\ge0$, $\tilde\sigma\ge0$) we obtain

$$\left(\int_{\mathbb{S}^{n-1}}\int_{\Pi_v}|P(\sigma-\tilde\sigma)(v,x)|^2dxdv\right)^{\frac{1}{2}}\le C_3\|\sigma-\tilde\sigma\|_\infty^{\frac{1}{2}}\|A_{S,W}-\tilde A_{S,W}\|_{\eta,T}^{\frac{1}{2}}.\tag{6.34}$$

where $C_3=\big(\mathrm{diam}(X)e^{M\mathrm{diam}(X)}\mathrm{Vol}(\partial X)C_1\big)^{\frac{1}{2}}$. Combining (6.34) and (6.30), we obtain (3.7).

We now prove (3.8). Let $f\in L^2(X)$, $\mathrm{supp}f\subseteq\bar Z$. We consider the weighted X-ray transform of f, $P_\vartheta f$, defined by

$$P_\vartheta f(x,v)=\int_0^{\tau_+(v,x)}f(pv+x)\vartheta(pv+x,v)dp,\ \text{for a.e. }(x,v)\in\Gamma_-,\tag{6.35}$$

where $\vartheta:X\times\mathbb{S}^{n-1}\to(0,+\infty)$ is the analytic function given by

$$\vartheta(x,v)=(\tau_-(x,v)\tau_+(x,v))^{-\frac{n-1}{2}}g(x,v,v),\ \text{for }(x,v)\in X\times\mathbb{S}^{n-1}.\tag{6.36}$$

From [10, theorem 2.2] and from [14, theorem 4] we obtain

$$\|f\|_{H^{-\frac{1}{2}}(Z)}\le C\|P_\vartheta f\|_{L^2(\Gamma_-,d\xi)},\tag{6.37}$$

where $C=C(X,Z,g)$ is a constant that does not depend on f. Let $x_0'\in\partial X$ and let $x\in\partial X$ such that $px_0'+(1-p)x\in Z$ for some $p\in(0,1)$ where $v_0=\frac{x-x_0'}{|x-x_0'|}$ and $t_0=|x-x_0'|$. Note that using (2.2) (since $\tilde k\in L^\infty(Z)$ and $\mathrm{supp}\tilde k\subseteq\bar Z\subseteq\{x\in X\mid\inf_{x'\in\partial X}|x-x'|\ge\delta\}$), we obtain

$$\begin{aligned}\int_0^{\tau_+(x_0,v_0')}\frac{\tilde k(x_0'+pv_0',v_0',v_0')}{p^{\frac{n-1}{2}}(\tau_+(x_0',v_0')-p)^{\frac{n-1}{2}}}dp &\le \|\tilde k\|_{L^\infty(Z)}\int_\delta^{\tau_+(x_0,v_0')-\delta}\frac{1}{p^{\frac{n-1}{2}}(\tau_+(x_0',v_0')-p)^{\frac{n-1}{2}}}dp\\ &\le \|\tilde k\|_{L^\infty(Z)}\delta^{-(n-1)}\tau_+(x_0',v_0') \le \|\tilde k\|_{L^\infty(Z)}\delta^{-(n-1)}\mathrm{diam}(X).\end{aligned}\tag{6.38}$$

We use the estimate

$$|P_\vartheta(k_0-\tilde k_0)(x_0',v_0')| \le e^{P\sigma(v_0',x_0')}|P_\vartheta\tilde k_0(x_0',v_0')|\left|e^{-P\sigma(v_0',x_0')}-e^{-P\tilde\sigma(v_0',x_0')}\right| + e^{P\sigma(v_0',x_0')}\left|e^{-P\sigma(v_0',x_0')}P_\vartheta k_0(x_0',v_0')-e^{-P\tilde\sigma(v_0',x_0')}P_\vartheta\tilde k_0(x_0',v_0')\right|. \tag{6.39}$$

Integrating both sides of inequality (6.39) over $v_0'\in\mathbb{S}^{n-1}_{x_0',-}$ and using the estimate $e^{P\sigma(v_0',x_0')}\le e^{M\mathrm{diam}(X)}$, and using (6.38), (3.1)–(3.2), we obtain

$$\int_{\mathbb{S}^{n-1}_{x_0',-}}|P_\vartheta(k_0-\tilde k_0)|(x_0',v_0')|\nu(x_0')\cdot v|dv \le \delta^{-(n-1)}\mathrm{diam}(X)e^{M\mathrm{diam}(X)}C_1\|\tilde k\|_\infty\|A_{S,W}-\tilde A_{S,W}\|_{\eta,T} + \frac{\mathrm{Vol}(\mathbb{S}^{n-1})e^{M\mathrm{diam}(X)}C_2}{2}\left\|(\tau-|z-z'|)^{\frac{n-3}{2}}(\Gamma_1-\tilde\Gamma_1)(\tau,z,z')\right\|_{L^\infty((0,T)\times\partial X\times\partial X)}, \tag{6.40}$$

where C_1 and C_2 are the constants that appear on the right-hand side of (3.1) and (3.2).

From the estimate $|P_\vartheta(k_0-\tilde k_0)(v_0',x_0')|\le\|k-\tilde k\|_{L^\infty(Z)}\delta^{-(n-1)}\mathrm{diam}(X)$ for a.e. $(x_0',v_0')\in\Gamma_-$ (see (6.38)), it follows that

$$\|P_\vartheta(k_0-\tilde k_0)\|^2_{L^2(\Gamma_-,d\xi)}\le\|k-\tilde k\|_{L^\infty(Z)}\delta^{-(n-1)}\mathrm{diam}(X)\int_{\partial X}\int_{\mathbb{S}^{n-1}_{x_0',-}}|P_\vartheta(k_0-\tilde k_0)(x_0',v_0')||\nu(x_0')\cdot v|dvd\mu(x_0'). \tag{6.41}$$

Combining (6.40)–(6.41) and (6.37), we obtain (3.8). □

A Proof of some lemmas

We recall the following change of variables for the proof of Lemmas 5.1, 5.2.

$$\int_{\mathcal{E}_{1,n}(\tau,t_0v,0))}f(y)dy = \begin{cases}\int_{(0,2\pi)\times(t_0,\tau)} f\Big(\frac{t_0+s\cos(\varphi)}{2},\frac{\sqrt{s^2-t_0^2}}{2}\sin\varphi\Big)\\ \times\frac{(s^2-t_0^2\cos^2(\varphi))}{4\sqrt{s^2-t_0^2}}dsd\varphi,\ \text{if } n=2,\\ \int_{\mathbb{S}^{n-2}\times(0,\pi)\times(t_0,\tau)} f\Big(\frac{t_0+s\cos(\varphi)}{2},\frac{\sqrt{s^2-t_0^2}}{2}\sin\varphi\omega\Big)\\ \times\Big(\frac{\sin(\varphi)\sqrt{s^2-t_0^2}}{2}\Big)^{n-2}\frac{s^2-t_0^2\cos^2(\varphi)}{4\sqrt{s^2-t_0^2}}d\omega dsd\varphi,\ \text{if } n\ge3,\end{cases} \tag{A.1}$$

for $f\in L^1(\mathbb{R}^n)$ and $(\tau,t_0,v)\in(0,+\infty)\times(0,+\infty)\times\mathbb{S}^{n-1}$ such that $\tau>t_0$.

Proof of Lemma 5.1. Let $n\ge2$. Using a rotation and (5.14), we have

$$\mathrm{Vol}_n(\mathcal{E}_{1,n}(\tau,x,x'))=\mathrm{Vol}_n(\mathcal{E}_{1,n}(\tau,t_0e_1,0)), \tag{A.2}$$

where $t_0=|x-x'|$ and $e_1=(0,\ldots,0)\in\mathbb{R}^n$. From (A.1), it follows that

$$\mathrm{Vol}_n(\mathcal{E}_{1,n}(\tau,t_0e_1,0))=\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\int_{t_0}^{\tau}\int_0^{\pi}\Big(\frac{\sin(\varphi)\sqrt{s^2-t_0^2}}{2}\Big)^{n-2}\frac{s^2-t_0^2\cos^2(\varphi)}{4\sqrt{s^2-t_0^2}}dsd\varphi. \tag{A.3}$$

From (A.3) and the estimate $\sin(\varphi)\sqrt{s^2-t_0^2} \le \sqrt{\tau^2-t_0^2}$ for $s \in (t_0,\tau)$, we obtain

$$\begin{aligned} \mathrm{Vol}_n(\mathcal{E}_{1,n}(\tau,t_0e_1,0)) &\le \mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\Big(\frac{\sqrt{\tau^2-t_0^2}}{2}\Big)^{n-2}\int_{t_0}^{\tau}\int_0^{\pi}\frac{s^2-t_0^2\cos^2(\phi)}{4\sqrt{s^2-t_0^2}}dsd\varphi \\ &\le \frac{1}{2}\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\Big(\frac{\sqrt{\tau^2-t_0^2}}{2}\Big)^{n-2}\mathrm{Vol}(\mathcal{E}_{1,2}(\tau,t_0e_1,0)). \end{aligned} \tag{A.4}$$

We recall that $\mathrm{Vol}(\mathcal{E}_{1,2}(\tau,t_0e_1,0)) = \frac{\pi(t_0+\tau)\sqrt{\tau^2-t_0^2}}{4}$. Therefore (5.15) follows from (A.4). Lemma 5.1 is proved. ☐

Proof of Lemma 5.2. We first prove (5.17). Let $n=2$. Note that

$$N(\mu,z,z') = \int_0^{2\pi}\frac{1}{\mu-|z-z'|\sin(\Omega)}d\Omega.$$

for $(\mu,z,z') \in (0,T)\times\partial X\times\mathbb{R}^n$, $\mu > |z-z'|$. Therefore, using (5.1) we obtain

$$N(\mu,z,z') = \frac{2\pi}{\sqrt{\mu^2-|z-z'|^2}}, \tag{A.5}$$

for $(\mu,z,z') \in (0,T)\times\partial X\times\mathbb{R}^n$, $\mu > |z-z'|$. Now let $(\mu,z,z') \in (0,T)\times\partial X\times\mathbb{R}^n$ be fixed with $\mu > |z-z'|$. Set $t_0 = |z-z'|$. Using (A.1) ("$\tau=\mu$", "$t_0v = z-z'$"), we obtain

$$\begin{aligned} \int_{\mathcal{E}_{1,2}(\mu,z,z')}\frac{2\pi}{|y|\sqrt{(\mu-|y|)^2-|z-z'-y|^2}}dy &= \int_{\mathcal{E}_{1,2}(\mu,t_0(1,0),0)}\frac{2\pi}{|y|\sqrt{(\mu-|y|)^2-|t_0(1,0)-y|^2}}dy \\ &= 4\pi\int_{t_0}^{\mu}\int_0^{2\pi}G_{2,2}(\mu,s,\varphi)d\varphi d\varepsilon, \end{aligned} \tag{A.6}$$

$$\text{where}\qquad G_{2,2}(\mu,s,\varphi) := \frac{(s-t_0\cos(\varphi))}{\sqrt{s^2-t_0^2}\sqrt{\mu-s}\sqrt{\mu-t_0\cos(\varphi)}}, \tag{A.7}$$

for $\varphi \in (0,2\pi)$ and $s \in (t_0,\mu)$. From (A.7) and the estimates $\mu - t_0\cos(\varphi) \ge s - t_0\cos(\varphi)$, $s+t_0 \ge s-t_0\cos(\varphi)$, it follows that

$$G_{2,2}(\mu,s,\varphi) \le \frac{1}{\sqrt{s-t_0}\sqrt{\mu-s}}, \tag{A.8}$$

for $\varphi \in (0,2\pi)$ and $s \in (t_0,\mu)$. Performing the change of variables $s = t_0+\varepsilon(\mu-t_0)$ we have

$$\int_{t_0}^{\mu}\frac{1}{\sqrt{s-t_0}\sqrt{\mu-s}}ds = \int_0^1\frac{1}{\sqrt{\varepsilon(1-\varepsilon)}}d\varepsilon < +\infty, \tag{A.9}$$

for $s \in (t_0,\mu)$. Combining (A.5), (A.6), (A.8), (A.9), we obtain

$$\sup_{\substack{(\mu,z,z')\in(0,T)\times\partial X\times\mathbb{R}^2\\ \mu>|z-z'|}}\int_{\mathcal{E}_{1,2}(\mu,z,z')}\frac{N(\mu-|y|,z,z'+y)}{|y|}dy \le 8\pi^2\int_0^1\frac{1}{\sqrt{\varepsilon(1-\varepsilon)}}d\varepsilon < \infty. \tag{A.10}$$

Statement (5.17) follows from (A.10).

We prove (5.18). Let $n=3$. Note that

$$N(\mu,z,z') = 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d}{d\Omega}\ln\left(\mu^2+|z-z'|^2-2\mu|z-z'|\sin(\Omega)\right)d\Omega,$$

for $(\mu, z, z') \in (0,T) \times \partial X \times \mathbb{R}^n$, $\mu > |z - z'|$. Therefore

$$N(\mu, z, z') = \frac{2\pi}{\mu|z-z'|} \ln\left(\frac{\mu + |z-z'|}{\mu - |z-z'|}\right), \quad \text{(A.11)}$$

for $(\mu, z, z') \in (0,T) \times \partial X \times \mathbb{R}^n$, $\mu > |z - z'|$. Now let $(\mu, z, z') \in (0,T) \times \partial X \times \mathbb{R}^n$ be fixed with $\mu > |z - z'|$. Set $t_0 = |z - z'|$. Using (A.11) and (A.1), we obtain

$$\begin{aligned}\int_{\substack{y\in\mathcal{E}_{1,3}(\mu,z,z')\\ \mu-|y|\geq\delta,\ |z-z'-y|\geq\delta,\ |y|\geq\delta}} N(\mu-|y|, z, z'+y)dy &\leq 2\pi\delta^{-2}\int_{y\in\mathcal{E}_{1,3}(\mu,z,z')} \ln\left(\frac{\mu-|y|+|z-z'-y|}{\mu-|y|-|z-z'-y|}\right)dy \\ &= \frac{\pi^2}{2\delta^2}\int_{t_0}^{\mu}\int_0^{\pi} G_{2,3,a}(\mu,s,\varphi)d\varphi ds, \quad \text{(A.12)}\end{aligned}$$

where

$$G_{2,3,a}(\mu, s, \varphi) := (s^2 - t_0^2\cos^2(\varphi))\sin(\varphi)\ln\left(\frac{\mu - t_0\cos(\varphi)}{\mu - s}\right), \quad \text{(A.13)}$$

for $\varphi \in (0,\pi)$ and $s \in (t_0, \mu)$. Using (A.13) and the estimates $\ln\left(\frac{\mu-t_0\cos(\varphi)}{\mu-s}\right) \leq \ln\left(\frac{\mu+t_0}{\mu-s}\right)$, $s^2 - t_0^2\cos^2(\varphi) \leq \mu^2$, we obtain

$$\int_0^{\pi} G_{2,3,a}(\mu,s,\varphi)d\varphi \leq \mu^2\int_0^{\pi}\sin(\varphi)d\varphi \ln\left(\frac{\mu+t_0}{\mu-s}\right). \quad \text{(A.14)}$$

We recall the following integral value

$$\int_{t_0}^{\mu}\ln\left(\frac{\mu+t_0}{\mu-s}\right)ds = (\mu - t_0)\ln\left(\frac{\mu+t_0}{\mu-t_0}\right) + \mu - t_0. \quad \text{(A.15)}$$

Combining (A.12), (A.14) and (A.15) we obtain

$$\int_{\substack{y\in\mathcal{E}_{1,3}(\mu,z,z')\\ \mu-|y|\geq\delta,\ |z-z'-y|\geq\delta,\ |y|\geq\delta}} N(\mu-|y|, z, z'+y)dy \leq \frac{\pi^2\mu^2}{\delta^2}(\mu-t_0)\left(\ln\left(\frac{\mu+t_0}{\mu-t_0}\right)+1\right), \quad \text{(A.16)}$$

which proves (5.18).

We next prove (5.19). Let $n = 3$. Let $(\mu, z, z') \in (0,T) \times \partial X \times \mathbb{R}^n$ be fixed, $\mu > |z - z'|$. Set $t_0 = |z - z'|$. Using (A.11) and (A.1), we obtain

$$\begin{aligned}\int_{y\in\mathcal{E}_{1,3}(\mu,z,z')}\frac{N(\mu-|y|,z,z'+y)}{|y|^2}dy &= \int_{y\in\mathcal{E}_{1,3}(\mu,z,z')}\frac{2\pi\ln\left(\frac{\mu-|y|+|z-z'-y|}{\mu-|y|-|z-z'-y|}\right)}{|y|^2(\mu-|y|)|z-z'-y|}dy \\ = \int_{\mathcal{E}_{1,3}(\mu,t_0(1,0,0),0)}\frac{2\pi\ln\left(\frac{\mu-|y|+|t_0(1,0,0)-y|}{\mu-|y|-|t_0(1,0,0)-y|}\right)}{|y|^2(\mu-|y|)|t_0(1,0,0)-y|}dy &= 8\pi^2\int_{t_0}^{\mu}\int_0^{\pi}G_{2,3,b}(\mu,s,\varphi)d\varphi ds, \quad \text{(A.17)}\end{aligned}$$

where

$$\begin{aligned}G_{2,3,b}(\mu,s,\varphi) &:= \frac{\sin(\varphi)\ln\left(\frac{\mu-t_0\cos(\varphi)}{\mu-s}\right)}{(s+t_0\cos(\varphi))(2\mu-s-t_0\cos(\varphi))} \\ &= \ln\left(\frac{\mu-t_0\cos(\varphi)}{\mu-s}\right)\left(\frac{\sin(\varphi)}{2\mu(s+t_0\cos(\varphi))}+\frac{\sin(\varphi)}{2\mu(2\mu-s-t_0\cos(\varphi))}\right) \quad \text{(A.18)}\end{aligned}$$

for $\varphi \in (0,\pi)$ and $s \in (t_0,\mu)$. From (A.18) and the estimates $2\mu - s - t_0\cos(\varphi) \geq \mu - t_0\cos(\varphi)$, $0 \leq \ln\left(\frac{\mu - t_0\cos(\varphi)}{\mu - s}\right) \leq \ln\left(\frac{\mu+t_0}{\mu-s}\right)$, it follows that

$$G_{2,3,b}(\mu,s,\varphi) \leq \ln\left(\frac{\mu+t_0}{\mu-s}\right)\left(\frac{\sin(\varphi)}{2\mu(s-t_0\cos(\varphi))} + \frac{\sin(\varphi)}{2\mu(\mu-t_0\cos(\varphi))}\right),$$

for $\varphi \in (0,\pi)$ and $s \in (t_0,\mu)$. Therefore

$$\begin{aligned}\int_0^\pi G_{2,3,b}(\mu,s,\varphi)d\varphi &\leq \frac{\ln\left(\frac{\mu+t_0}{\mu-s}\right)}{2\mu t_0}\left(\ln\left(\frac{s+t_0}{s-t_0}\right) + \ln\left(\frac{\mu+t_0}{\mu-t_0}\right)\right)\\ &\leq \frac{\ln\left(\frac{\mu+t_0}{\mu-s}\right)}{2\mu t_0}\left(\ln\left(\frac{\mu+t_0}{s-t_0}\right) + \ln\left(\frac{\mu+t_0}{\mu-t_0}\right)\right). \qquad (A.19)\end{aligned}$$

Using the estimate $\ln\left(\frac{\mu+t_0}{\mu-s}\right) \leq \ln\left(\frac{2(\mu+t_0)}{\mu-t_0}\right)$ for $s \in (t_0, \frac{t_0+\mu}{2})$, we obtain

$$\begin{aligned}&\int_{t_0}^{\mu} \ln\left(\frac{\mu+t_0}{s-t_0}\right)\ln\left(\frac{\mu+t_0}{\mu-s}\right)ds = 2\int_{t_0}^{\frac{t_0+\mu}{2}} \ln\left(\frac{\mu+t_0}{s-t_0}\right)\ln\left(\frac{\mu+t_0}{\mu-s}\right)ds \qquad (A.20)\\ &\leq 2\ln\left(\frac{2(\mu+t_0)}{\mu-t_0}\right)\int_{t_0}^{\frac{t_0+\mu}{2}} \ln\left(\frac{\mu+t_0}{s-t_0}\right)ds \leq 2(\mu-t_0)\left(\ln\left(\frac{\mu+t_0}{\mu-t_0}\right)+\ln 2\right)\left(\ln\left(\frac{\mu+t_0}{\mu-t_0}\right)+1\right).\end{aligned}$$

Combining (A.17)–(A.20) and (A.15), we obtain

$$\int_{y\in\mathcal{E}_{1,3}(\mu,z,z')} \frac{N(\mu-|y|,z,z'+y)}{|y|^2}dy \leq 4\pi^2\frac{\mu-t_0}{\mu t_0}\left(3\ln\left(\frac{\mu+t_0}{\mu-t_0}\right)+2\ln 2\right)\left(\ln\left(\frac{\mu+t_0}{\mu-t_0}\right)+1\right). \qquad (A.21)$$

Statement (5.19) follows from (A.21).

We now prove (5.20). Let $n \geq 4$ and let $(\mu,z,z') \in (0,T)\times\partial X\times\mathbb{R}^n$ be such that $\mu > |z-z'|$ (we recall that $N(\mu,z,z') = 0$ if $\mu \leq |z-z'|$). Using spherical coordinates, we obtain

$$N(\mu,z,z') = \mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(\mu-|z-z'|\sin(\Omega))^{n-3}}{(|z-z'|^2+\mu^2-2\mu|z-z'|\sin(\Omega))^{n-2}}\cos(\Omega)^{n-2}d\Omega. \qquad (A.22)$$

Performing the change of variables "$r = \frac{\mu^2-|z-z'|^2}{2(\mu-|z-z'|\sin(\Omega))} - \frac{\mu-|z-z'|}{2}$", we obtain

$$\begin{aligned}N(\mu,z,z') &= \frac{\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})(\mu^2-|z-z'|^2)^{\frac{n-3}{2}}}{|z-z'|^{n-2}}\int_0^{|z-z'|}\frac{\sqrt{r(|z-z'|-r)}^{n-3}}{(\frac{\mu-|z-z'|}{2}+r)^{n-2}(\frac{\mu+|z-z'|}{2}-r)^{n-2}}dr\\ &= \frac{2\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})(\mu^2-|z-z'|^2)^{\frac{n-3}{2}}}{|z-z'|^{n-2}}\int_0^{\frac{|z-z'|}{2}}\frac{\sqrt{r(|z-z'|-r)}^{n-3}}{(\frac{\mu-|z-z'|}{2}+r)^{n-2}(\frac{\mu+|z-z'|}{2}-r)^{n-2}}dr\\ &\leq \frac{2\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})|(\mu^2-|z-z'|^2)^{\frac{n-3}{2}}}{|z-z'|^{n-2}}\int_0^{\frac{|z-z'|}{2}}\frac{1}{(\frac{\mu-|z-z'|}{2}+r)^{\frac{n-1}{2}}(\frac{\mu+|z-z'|}{2}-r)^{\frac{n-1}{2}}}dr\end{aligned}$$

$$\begin{aligned}&\leq \frac{2\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})(\mu^2-|z-z'|^2)^{\frac{n-3}{2}}}{|z-z'|^{n-2}}\frac{\mu}{2}^{\frac{1-n}{2}}\int_0^{\frac{|z-z'|}{2}}\frac{1}{(\frac{\mu-|z-z'|}{2}+r)^{\frac{n-1}{2}}}dr \qquad (A.23)\\ &\leq \frac{2^{n-1}}{n-3}\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})|z-z'|^{2-n}\left(\frac{\mu+|z-z'|}{2\mu}\right)^{\frac{n-3}{2}}\mu^{-1} \leq \frac{2^{n-1}}{n-3}\mathrm{Vol}_{n-2}(\mathbb{S}^{n-2})|z-z'|^{2-n}\mu^{-1}.\end{aligned}$$

□

B Proof of Proposition 2.1

We start with the derivation of (2.13) and (2.14). From (2.9) and the definition of G_-, we obtain $A_{0,S,W}\phi(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}}(\nu(x)\cdot v)W(x,v)E(x,x-\tau_-(x,v)v)S(x-\tau_-(x,v)v,v)\phi(t-\tau_-(x,v),x-\tau_-(x,v)v)dv$, $(t,x)\in(0,T)\times\partial X$ and for $\phi\in L^1((0,\eta)\times\partial X)$. Therefore, performing the change of variables "x'"$= x-\tau(x,v)v$ ($dv = \frac{|\nu(x')\cdot v|}{|x-x'|^{n-1}}d\mu(x')$ and $\tau(x,v)=|x-x'|$), we obtain (2.13).

From the definition of A_2 and G_- we note that $A_2G_-(s)\phi_S(z,w) := \int_{\mathbb{S}^{n-1}}k(z,v',w)E(z,z-\tau_-(z,v')v')S(z-\tau_-(z,v')v',v')\phi(s-\tau_-(z,v'),z-\tau_-(z,v')v')dv'$, for a.e. $(z,w)\in X\times\mathbb{S}^{n-1}$ and for $\phi\in L^1((0,\eta)\times\partial X)$. Performing the change of variables "$x' = z-\tau_-(z,v')v'$", we obtain the equality $\left(A_2G_-(s)\phi_S\right)(z,w) = \int_{\partial X}\left[k(z,v',w)S(x',v')|\nu(x')\cdot v'|\right]_{v'=\frac{z-x'}{|z-x'|}}\frac{E(z,x')}{|z-x'|^{n-1}}\phi(s-|z-x'|,x')d\mu(x')$, for a.e. $(z,w)\in X\times\mathbb{S}^{n-1}$ and $\phi\in L^1((0,\eta)\times\partial X)$. Using also the definition of $A_{1,S,W}$ (see (2.10) for $m=1$) we obtain the following equality for any $\phi\in L^1((0,\eta)\times\partial X)$ and for a.e. $(t,x)\in(0,T)\times\partial X$

$$A_{1,S,W}(\phi)(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}}\int_{-\infty}^{t}\int_{\partial X}\left[k(x-(t-s)v,v',v)S(x',v')|\nu(x')\cdot v'|\right]_{v'=\frac{x-(t-s)v-x'}{|x-(t-s)v-x'|}}(\nu(x)\cdot v)$$
$$\times\frac{E(x,x-(t-s)v,x')}{|x-(t-s)v-x'|^{n-1}}\chi_{(0,\tau_-(x,v))}(t-s)\phi(s-|x-(t-s)v-x'|,x')W(x,v)d\mu(x')dsdv. \quad \text{(B.1)}$$

Then, performing the changes of variables "s"$= t-s$ and "t'"$= t-s-|x-sv-x'|$ ($s = \frac{(t-t')^2-|x-x'|^2}{2(t-t'-v\cdot(x-x'))}$, $\frac{dt'}{ds} = \frac{2((t-t')-(x-x')\cdot v)^2}{|x-x'-(t-t')v|^2}$), we obtain (2.14).

In order to prove (5.21)–(5.22), we introduce and prove Proposition B.1 below, which gives the distributional kernel of the operators H_m defined in section 2.1. Let $\bar{E}$ denotes the nonnegative measurable function from $\mathbb{R}^n\times\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$\bar{E}(x_1,x_2) = e^{-\int_0^{|x_1-x_2|}\sigma(x_1-s\frac{x_1-x_2}{|x_1-x_2|},\frac{x_1-x_2}{|x_1-x_2|})ds}\Theta(x_1,x_2),\ \text{for a.e. } (x_1,x_2)\in\mathbb{R}^n\times\mathbb{R}^n, \quad \text{(B.2)}$$

where $\Theta(x_1,x_2)=1$ if $\{px_1+(1-p)x_2 \mid p\in[0,1]\}\subseteq X$ and $\Theta(x_1,x_2)=0$ otherwise. For $m\geq 3$, we define recursively the nonnegative measurable real function $\bar{E}(x_1,\ldots,x_m)$ by the formula

$$\bar{E}(x_1,\ldots,x_m) = \bar{E}(x_1,\ldots,x_{m-1})\bar{E}(x_{m-1},x_m), \quad \text{(B.3)}$$

for $(x_1,\ldots,x_m)\in(\mathbb{R}^n)^m$.

Proposition B.1. *We have*

$$H_m(t)\phi(x,v) = \int_{X\times\mathbb{S}^{n-1}}\beta_m(t,x,v,x',v')\phi(x',v')dx'dv', \quad \text{(B.4)}$$

for $t\in(0,T)$ and a.e. $(x,v)\in X\times\mathbb{S}^{n-1}$ and for $m\geq 2$, where

$$\beta_2(t,x,v,x',v') = \int_0^t\chi_{(0,t-s_2)}(|x'-(x-s_2v')|)\frac{2^{n-2}\left(t-s_2-(x-s_2v'-x')\cdot v\right)^{n-3}}{|x-s_2v'-x'-(t-s_2)v|^{2n-4}}$$
$$\times\left[\bar{E}(x,x-(t-s_1-s_2)v,x'+s_2v',x')k(x-(t-s_1-s_2)v,v_1,v)\right.$$
$$\left.\times k(x'+s_2v',v',v_1)\right]_{v_1=\frac{x-s_2v'-x'-(t-s_1-s_2)v}{s_1},\ s_1=\frac{|x-s_2v'-x'-(t-s_2)v|^2}{2(t-s_2-(x-x'-s_2v')\cdot v)}}ds_2, \quad \text{(B.5)}$$

$$\beta_m(t,x,v,x',v') = \int_{(\mathbb{S}^{n-1})^{m-2}} \int_{\substack{\sum_{j=2}^m s_j \le t \\ s_j\ge 0,\ j=2\dots m}} \chi_{(0,t-\sum_{j=2}^m s_j)}(|x'+s_m v' + \sum_{j=2}^{m-1} s_j v_j - x|)$$

$$\times \frac{2^{n-2}\left(t-\sum_{j=2}^m s_j - (x-x'-\sum_{j=2}^{m-1} s_j v_j - s_m v')\cdot v\right)^{n-3}}{|x-x'-\sum_{j=2}^{m-1} s_j v_j - s_m v' - (t-\sum_{j=2}^m s_j)v|^{2n-4}}$$

$$\times \Big[\bar E(x, x-(t-\sum_{j=1}^m s_j)v, x'+s_m v' + \sum_{j=2}^{m-1} s_j v_j, x'+s_m v' + \sum_{j=3}^{m-1} s_j v_j, \dots, x'+s_m v', x')$$

$$\times k(x-(t-\sum_{j=1}^m s_j)v, v_1, v) k(x'+s_m v' + \sum_{j=2}^{m-1} s_j v_j, v_2, v_1)\dots k(x'+s_m v' + \sum_{j=i+1}^{m-1} s_j v_j, v_{i+1}, v_i)\dots$$

$$k(x'+s_m v', v', v_{m-1})\Big]_{\substack{v_1 = \frac{x-x'-\sum_{j=2}^{m-1} s_j v_j - s_m v' - (t-\sum_{j=1}^m s_j)v}{s_1} \\ s_1 = \frac{|x-x'-\sum_{j=2}^{m-1} s_j v_j - s_m v' - (t-\sum_{j=2}^m s_j)v|^2}{2(t-\sum_{j=2}^m s_j - (x-x'-\sum_{j=2}^{m-1} s_j v_j - s_m v')\cdot v)}}} ds_2\dots ds_m dv_2 \dots dv_{m-1},\ m\ge 3. \quad \text{(B.6)}$$

Proof of Proposition B.1. Note that

$$\begin{aligned}
H_2(t)\phi(x,v) &= \left(\int_0^t \int_0^{t-s_1} U_1(t-s_1-s_2)A_2U_1(s_1)A_2U_1(s_2)\phi ds_2 ds_1\right)(x,v)\\
&= \left(\int_0^t \left(\int_0^{t-s_2} U_1(t-s_1-s_2)A_2U_1(s_1)A_2 ds_1\right) U_1(s_2)\phi ds_2\right)(x,v)\\
&= \int_0^t \int_0^{t-s_2} \bar E(x, x-(t-s_1-s_2)v) \int_{\mathbb{S}^{n-1}} k(x-(t-s_1-s_2)v, v_1, v)\\
&\quad \times \bar E(x-(t-s_1-s_2)v, x-(t-s_1-s_2)v - s_1 v_1)\\
&\quad \times \int_{\mathbb{S}^{n-1}} k(x-(t-s_2-s_1)v - s_1v_1, v_2, v_1)\\
&\quad \times \bar E(x-(t-s_1-s_2)v - s_1 v_1, x-(t-s_1-s_2)v - s_1v_1 - s_2v_2)\\
&\quad \times \phi(x-(t-s_1-s_2)v - s_1v_1 - s_2v_2, v_2) dv_2 dv_1 ds_1 ds_2,
\end{aligned}$$

for $t\in(0,T)$ and $(x,v)\in X\times\mathbb{S}^{n-1}$, where the functions $\bar E$ are defined by (B.2)–(B.3).

Using the change of variables "$y(s_1,v_1) = (t-s_2-s_1)v + s_1 v_1$" we obtain

$$H_2(t)\phi(x,v) = \int_0^t \int_{\mathbb{S}^{n-1}} \big[\bar E(x, x-(t-s_1-s_2)v, x-y, x-y-s_2v_2) k(x-(t-s_1-s_2)v, v_1, v)$$

$$\times\, k(x-y, v_2, v_1)\big]_{\substack{v_1 = \frac{y-(t-s_1-s_2)v}{s_1} \\ s_1 = \frac{|y-(t-s_2)v|^2}{2(t-s_2-y\cdot v)}}} \frac{2^{n-2}\left((t-s_2)-y\cdot v\right)^{n-3}}{|y-(t-s_2)v|^{2n-4}} \phi(x-y-s_2v_2, v_2) dy dv_2 ds_2.$$

Hence we obtain (B.4) for $m=2$. Note that

$$\begin{aligned}
(H_3(t)\phi)(x,v) &= \int_0^t H_2(t-s_3)A_2U_1(s_3)\phi ds_3\\
&= \int_0^t \int_{X\times\mathbb{S}^{n-1}} \beta_2(t-s_3, x, v, x_2, v_2)(A_2U_1(s_3))\phi(x_2,v_2) dx_2 dv_2 ds_3\\
&= \int_{X\times\mathbb{S}^{n-1}} \int_0^t \beta_2(t-s_3, x, v, x_2, v_2) \int_{\mathbb{S}^{n-1}} k(x_2, v', v_2)\bar E(x_2, x_2 - s_3 v')\\
&\quad \times \phi(x_2 - s_3 v', v') dv' ds_3 dx_2 dv_2.
\end{aligned}$$

Hence performing the change of variables "$x' = x_2 - s_3 v'$" ($dx' = dx_2$) and using definitions (B.5) and (B.6) (for $m = 3$) we obtain

$$(H_3(t)\phi)(x,v) = \int_{X\times\mathbb{S}^{n-1}} \beta_3(t,x,v,x',v')\phi(x',v')dx'dv'. \tag{B.7}$$

The proof of (B.6) follows by induction from (B.7) and the recurrence formula $H_m(t) = \int_0^t H_{m-1}(t-s)A_2U_1(s)ds$ for $t \geq 0$. $\square$

Proof of (5.21)–(5.22). We recall that

$$(A_2G_-(s)\phi_S)(z,w) = \int_{\partial X} [k(z,v',w)S(x',v')|\nu(x')\cdot v'|]_{v'=\frac{z-x'}{|z-x'|}} \frac{E(z,x')}{|z-x'|^{n-1}}\phi(s-|z-x'|,x')d\mu(x'), \tag{B.8}$$

for a.e. $(z,w) \in X \times \mathbb{S}^{n-1}$ and $\phi \in L^1((0,\eta)\times\partial X)$.

Let $m = 2$. Then from (2.10) and the definition of the operator H_1, it follows that

$$\begin{aligned}
&A_{2,S,W}(\phi)(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x)\cdot v)W(x,v)\int_{-\infty}^t\int_0^{t-s}\int_{\mathbb{S}^{n-1}}\int_{\partial X}[k(x-(t-s-s_1)v,v_1,v)\\
&\times k(x-(t-s-s_1)v - s_1v_1,v',v_1)S(x',v')|\nu(x')\cdot v'|]_{v'=\frac{x-(t-s-s_1)v-s_1v_1-x'}{|x-(t-s-s_1)v-s_1v_1-x'|}} E(x,x-(t-s-s_1)v,\\
&x-(t-s-s_1)v-s_1v_1,x')\frac{\phi(s-|x-(t-s-s_1)v-s_1v_1-x'|,x')}{|x-(t-s-s_1)v-s_1v_1-x'|^{n-1}}\\
&\chi_{X^2}(x-(t-s-s_1)v,x-(t-s-s_1)v-s_1v_1)d\mu(x')dv_1ds_1dsdv.
\end{aligned}$$

Performing the change of variables $y(s_1,v_1) = (t-s-s_1)v + s_1v_1$, we obtain

$$\begin{aligned}
&A_{2,S,W}(\phi)(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}\times\partial X\times\mathbb{R}^n} (\nu(x)\cdot v)W(x,v)\int_{-\infty}^t \chi_{(0,t-s)}(|y|)\\
&\times[E(x,x-(t-s-s_1)v,x-y,x')\chi_{X^2}(x-(t-s-s_1)v,x-y)\\
&\times k(x-(t-s-s_1)v,v_1,v)k(x-y,v',v_1)S(x',v')|\nu(x')\cdot v'|]_{s_1=\frac{|(t-s)v-y|^2}{2(t-s-y\cdot v)},\ v_1=\frac{y-(t-s-s_1)v}{s_1},\ v'=\frac{x-y-x'}{|x-y-x'|}}\\
&\times\frac{2^{n-2}(t-s-y\cdot v)^{n-3}\phi(s-|x-y-x'|,x')}{|(t-s)v-y|^{2n-4}|x-y-x'|^{n-1}}dsdyd\mu(x')dv.
\end{aligned} \tag{B.9}$$

Performing the change of variables "y"$= x - x' - y$ and $t' = s - |y|$ we obtain (5.21).

Let $m = 3$. Then from (2.10), (B.4) (for "$m = 2$") and (B.8) it follows that

$$\begin{aligned}
&A_{3,S,W}(\phi)(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x)\cdot v)W(x,v)\int_{-\infty}^t\int_{X\times\mathbb{S}^{n-1}}\beta_2(t-s,x,v,x_2,v_2) \qquad \text{(B.10)}\\
&\int_{\partial X}[k(x_2,v',v_2)S(x',v')|\nu(x')\cdot v'|]_{v'=\frac{x_2-x'}{|x_2-x'|}}\frac{E(x_2,x')}{|x_2-x'|^{n-1}}\phi(s-|x_2-x'|,x')d\mu(x')dx_2dv_2dsdv,
\end{aligned}$$

for $t \in (0,T)$ and $x \in \partial X$. From (B.10) and (B.5) we obtain

$$\begin{aligned}
&A_{3,S,W}(\phi)(t,x) = \int_{\mathbb{S}^{n-1}_{x,+}} (\nu(x)\cdot v)W(x,v)\int_{X\times\mathbb{S}^{n-1}\times\partial X}\int_{-\infty}^t\int_0^{t-s}\chi_{(0,t-s-s_2)}(|x_2-(x-s_2v_2)|)\\
&\frac{2^{n-2}\left(t-s-s_2-(x-s_2v_2-x_2)\cdot v\right)^{n-3}}{|x_2-x'|^{n-1}\left|x-s_2v_2-x_2-(t-s-s_2)v\right|^{2n-4}}[E(x,x-(t-s_1-s_2)v,x_2+s_2v_2,x_2,x')
\end{aligned}$$

$$
\times\chi_{X^2}(x-(t-s_1-s_2)v, x_2+s_2v_2)k(x-(t-s_1-s_2)v, v_1, v)k(x_2+s_2v_2, v_2, v_1)k(x_2, v', v_2)
S(x', v')|\nu(x')\cdot v'|]_{\substack{v_1=\frac{x-s_2v_2-x_2-(t-s-s_1-s_2)v}{s_1}\\ s_1=\frac{|x-s_2v_2-x_2-(t-s-s_2)v|^2}{2(t-s-s_2-(x-x_2-s_2v_2)\cdot v)}\\ v'=\frac{x_2-x'}{|x_2-x'|}}}\ \phi(s-|x_2-x'|, x')ds_2dsdx_2dv_2d\mu(x')dv.
$$

Performing the change of variables $y_2 = s_2v_2$ and $y_3 = x_2 - x'$ we obtain (5.22) for "$m = 3$".

Let $m \geq 3$. From (2.10), (B.4) and (B.8) it follows that

$$
\begin{aligned}
A_{m+1,S,W}(\phi)(t,x) &= \int_{\mathbb{S}^{n-1}_{x,+}}\int_{\partial X}(\nu(x)\cdot v)W(x,v)\int_X\int_{(-\infty,t-|x_m-x'|)\times\mathbb{S}^{n-1}}\beta_m(t-t'-|x_m-x'|, x, v, x_m, v_m)\\
&\quad [k(x_m, v', v_m)S(x', v')|\nu(x')\cdot v'|]_{v'=\frac{x_m-x'}{|x_m-x'|}}\frac{E(x_m,x')}{|x_m-x'|^{n-1}}\phi(t',x')d\mu(x')dt'dx_mdv_mdv\\
&= \int_{(0,\eta)\times\partial X}\gamma_{m+1}(t-t', x, x')\phi(t',x')dt'd\mu(x'),
\end{aligned}
\tag{B.11}
$$

where

$$
\begin{aligned}
&\gamma_{m+1}(\tau, x, x') := \int_{\mathbb{S}^{n-1}_{x,+}}(\nu(x)\cdot v)W(x,v)\int_{X\times\mathbb{S}^{n-1}}\chi_{(0,+\infty)}(\tau-|x_m-x'|)\\
&\int_{(\mathbb{S}^{n-1})^{m-2}}\int_{\substack{\sum_{j=2}^m s_j\leq\tau-|x_m-x'|\\ s_j\geq 0,\ j=2\ldots m}}\chi_{(0,\tau-|x_m-x'|-\sum_{j=2}^m s_j)}(|x_m+\sum_{j=2}^m s_jv_j-x|)\\
&\times\frac{2^{n-2}\Big(\tau-|x_m-x'|-\sum_{j=2}^m s_j-(x-x_m-\sum_{j=2}^m s_jv_j)\cdot v\Big)^{n-3}}{|x_m-x'|^{n-1}|x-x_m-\sum_{j=2}^m s_jv_j-(\tau-|x_m-x'|-\sum_{j=2}^m s_j)v|^{2n-4}}\\
&\times\Big[E(x, x-(\tau-|x_m-x'|-\sum_{j=1}^m s_j)v, x_m+\sum_{j=2}^m s_jv_j, x_m+\sum_{j=3}^m s_jv_j, \ldots, x_m+s_mv_m, x_m, x')\\
&\times\chi_{X^m}(x-(\tau-|x_m-x'|-\sum_{j=1}^m s_j)v, x_m+s_mv_m, \ldots, x_m+\sum_{j=2}^m s_jv_j)\\
&\times k(x-(\tau-|x_m-x'|-\sum_{j=1}^m s_j)v, v_1, v)k(x_m+\sum_{j=2}^m s_jv_j, v_2, v_1)\ldots k(x_m+\sum_{j=i+1}^m s_jv_j, v_{i+1}, v_i)\ldots\\
&k(x_m+s_mv_m, v_m, v_{m-1})k(x_m, v', v_m)S(x', v')|\nu(x')\cdot v'|]_{\substack{v_1=\frac{x-x_m-\sum_{j=2}^m s_jv_j-(\tau-|x_m-x'|-\sum_{j=1}^m s_j)v}{s_1}\\ s_1=\frac{|x-x_m-\sum_{j=2}^m s_jv_j-(\tau-|x_m-x'|-\sum_{j=2}^m s_j)v|^2}{2(t-\sum_{j=2}^m s_j-(x-x'-\sum_{j=2}^m s_jv_j)\cdot v)}\\ v'=\frac{x_m-x'}{|x_m-x'|}}}\\
&ds_2\ldots ds_mdv_2\ldots dv_{m-1}dx_mdv_mdv.
\end{aligned}
\tag{B.12}
$$

Performing the change of variables $y_i = s_iv_i$, $i = 2\ldots m$, and $y_{m+1} = x_{m+1} - x'$, we obtain (5.22) for "$m \geq 4$". □

Acknowledgment

This paper was funded in part by grant NSF DMS-0554097. The authors would like to thank Ian Langmore and François Monard for stimulating discussions on the inverse transport problem.

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