Inverse Transport Theory and Applications

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Abstract. Inverse transport consists of reconstructing the optical properties of a domain from measurements performed at the domain's boundary. This paper concerns several types of measurements: time dependent, time independent, angularly resolved and angularly averaged measurements. We review recent results on the reconstruction of the optical parameters from such measurements and the stability of such reconstructions. Inverse transport finds applications e.g. in medical imaging (optical tomography, optical molecular imaging) and in geophysical imaging (remote sensing in the Earth atmosphere).

Keywords. Transport equation, radiative transfer, inverse problems, stability estimates, optical tomography, remote sensing, optical molecular imaging

1. Introduction

This paper reviews recent results on the inverse problem for the linear transport equation. The transport equation typically models the density of particles $u(t,x,v)$ in the space of positions x and velocities v as a function of time t or the energy density of high frequency waves in the space of positions and wavenumbers as a function of time. We shall use the terminology of "particles" to represent both particles and the energy density of wave packets. The most general transport equation we consider here has the form:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + \nabla_v H(x,v)\cdot\nabla_x u - \nabla_x H(x,v)\cdot\nabla_v u + \sigma(x,v)u \\
&= \int_V k(x,v',v)u(t,x,v')dv' + S(t,x,v), \qquad (t,x,v)\in(0,\mathrm{T})\times X\times V \\
&u_{|(0,\mathrm{T})\times\Gamma_-}(t,x,v) = g(t,x,v) \qquad (t,x,v)\in(0,\mathrm{T})\times\Gamma_- \\
&u(0,x,v) = 0, \qquad (x,v)\in X\times V.
\end{aligned}
\tag{1}
$$

The spatial domain X is a convex, bounded, open subset of $\mathbb{R}^d$ for dimension $d\geq 2$, with a C^1 boundary ∂X. The space of velocities V is a subset of $\mathbb{R}^d$. The theories presented in this paper apply when V is a bounded, open subset of $\mathbb{R}^d$ or the unit sphere $\mathbb{S}^{d-1}$. Most results will be presented in the latter setting to simplify. We are interested here in two types of problems: the first one in which the probing particles enter the domain through its boundary ∂X and the second one in which particles are created inside the domain. In the former problem, $g(t,x,v)$ models the density of particles entering the domain as a function of time, space, and incoming velocities. The sets of incoming conditions Γ_- and outgoing conditions Γ_+ are defined by

$$\Gamma_\pm = \{(x,v) \in \partial X \times V, \text{ s.t. } \pm v \cdot \nu(x) > 0\}, \tag{2}$$

where $\nu(x)$ is the outgoing normal vector to X at $x \in \partial X$. In the latter problem, $S(t,x,v)$ models the density of creation of particles inside the domain X.

In the absence of interaction of the particles with the underlying medium, the dynamics are governed by the Hamiltonian $H \equiv H(x,v)$ and (1) is the Liouville equation of classical mechanics. In most of the paper, we consider the propagation of particles or energy densities along straight lines so that

$$\begin{aligned} \nabla_v H(x,v) \cdot \nabla_x u - \nabla_x H(x,v) \cdot \nabla_v u &= v \cdot \nabla_x u, \qquad H(x,v) := \frac{|v|^2}{2} \\ \nabla_v H(x,v) \cdot \nabla_x u - \nabla_x H(x,v) \cdot \nabla_v u &= c\hat{v} \cdot \nabla_x u, \qquad H(x,v) := c|v|, \end{aligned} \tag{3}$$

where we define $\hat{v} = \frac{v}{|v|}$. The first Hamiltonian in (3) models the propagation of particles with kinetic energy $\frac{1}{2}|v|^2$ in the absence of external force field. The second Hamiltonian models the propagation of classical waves, such as e.g. acoustic, electromagnetic, elastic waves in a medium with constant speed c. This paper mostly focuses on the latter setting, which finds several applications in medical and geophysical imaging. Generalizations to spatially varying Hamiltonians such as e.g. $H(x,v) = c(x)|v|$, which corresponds to a varying index of refraction, will also be considered in later sections.

The optical parameters $\sigma(x,v)$ and $k(x,v',v)$ model the interaction of the propagating particles with the underlying structure. We assume here that such an interaction is linear. The parameter $\sigma(x,v)$ models the total absorption of particles caused by *true*, intrinsic absorption, and by scattering of particles into other directions. The scattering coefficient $k(x,v',v)$ indicates the amount of particles scattering from a direction v' into a direction v at position x. In most inverse problems considered in this paper, σ and k are the unknown parameters that need to be reconstructed from available measurements. The measurements considered here are measurements collected at the boundary ∂X of the domain of interest. We assume some control over the density of particles entering the domain through ∂X. The measurements are therefore functionals of the incoming density of particles $u_{|\Gamma_-}(t,x,v)$ and outgoing density of particles $u_{|\Gamma_+}(t,x,v)$. The source term S is then set to 0. While $u_{|\Gamma_-}(t,x,v) = g(t,x,v)$ is prescribed, $u_{|\Gamma_+}(t,x,v)$ is obtained by solving (1). The relationship between both quantities is the so-called *albedo* operator defined by

$$\mathcal{A} : u_{|\Gamma_-}(t,x,v) \mapsto \mathcal{A}u_{|\Gamma_-}(t,x,v) = u_{|\Gamma_+}(t,x,v). \tag{4}$$

Most of the paper will be concerned with answering the questions of what may be reconstructed in $\sigma(x,v)$ and $k(x,v',v)$ from full or partial knowledge of the albedo operator $\mathcal{A}$ and with which stability estimate. This is the inverse transport problem.

For general relevant references on theoretical and computational inverse problems, we refer the reader to [53, 59, 68, 101, 114, 140]. For general references and earlier review papers on inverse transport, we refer the reader to e.g. [2, 78, 89, 90, 129].

The inverse transport problem finds applications in many areas including medical imaging and optical tomography [6, 38, 65, 113], radiative transfer in the atmosphere and the ocean [83, 88] (where the inverse transport problem is referred to as remote sensing), neutron transport [46, 127], as well as the propagation of seismic waves in the earth crust [122]. Independent of the physical context, we refer to the parameters σ and k as the "optical" parameters throughout the paper.

A second class of inverse problems consists of reconstructing parts of the source term $S(t, x, v)$ from available boundary measurements $u_{|\Gamma_+}(t, x, v)$. Such an inverse source problem typically requires that the optical parameters $\sigma(x, v)$ and $k(x, v', v)$ be known. The inverse source problem finds applications e.g. in medical imaging techniques such as optical molecular imaging [67, 72, 142] and in ocean optics [135]. We consider inverse transport source problems in section 7.

The transport equation may be derived either phenomenologically [39, 60, 83, 110] or from first principles as a high frequency limit of (classical or quantum) waves propagating in highly heterogeneous media; see e.g. [13, 55, 85, 120, 125, 128, 137] and their references. The direct problem (1) has been analyzed extensively in the literature [35, 45, 52, 96]. We present the theory needed for the inverse transport analysis in section 2. This will allow us to properly define the albedo operator (4). A large number of theoretical results on the reconstruction of the optical parameters are based on the *singular decomposition* of the albedo operator $\mathcal{A}$; see e.g. [31, 32, 43, 44, 79]. In the absence of scattering, a localized beam of particles entering X at $(x_0, v_0) \in \Gamma_-$ exits the domain at $(x_0 + \tau v_0, v_0) \in \Gamma_+$, where τ is the time spent by the particles in the domain. In the presence of scattering, a positive fraction of the entering beam called the ballistic part still exits the domain $(x_0 + \tau v_0, v_0) \in \Gamma_+$. This ballistic part is more singular than the contributions caused by scattering. In dimensions $d \geq 2$ for the time-dependent problem and $d \geq 3$ for the time-independent problem, the single scattering contribution to $u_{|\Gamma_+}$ also turns out to be more singular, in a sense that will be clarified later, than the contribution that has scattered at least twice before exiting the domain. These two singular contributions in the albedo operator allow us to uniquely reconstruct the optical parameters.

While the analysis of the singular decomposition of the albedo operator is one of the main focuses in this paper, its main limitations are best understood by analyzing the different regimes or particle propagation in the transport equation. The singular decomposition offers a practically useful methodology to invert for the optical parameters when two situations are met: (i) scattering should not be too large for the ballistic and single scattering components to be detectable; and (ii) scattering should be sufficiently isotropic for the ballistic front not to be too blurred. Large scattering results in the regime of diffusion whereas highly anisotropic (peaked forward) scattering results in the regime of Fokker Planck propagation. These two regimes, which may be derived as asymptotic approximations to the transport equation, are reviewed in section 2. In either regime, the decomposition of the albedo operator becomes less meaningful and the development of other inversion tools is necessary.

When scattering is not too large and not too anisotropic, then inverse transport may be analyzed by means of the singular decomposition of the albedo operator. Which part of the optical parameters may be reconstructed then depends on the available measurements. The available measurements may be categorized as follows. One may first separate between time-dependent and time-independent measurements. Time dependent measurements are richer and therefore preferable. They are available e.g. in cloud parameter retrievals [109]. They are however not feasible or too expensive in many medical imaging modalities, where time-independent measurements should be analyzed. A second separation analyzes the angular dependence of the measurements. Angularly resolved measurements are very expensive to acquire and may face the hurdle of low particle counts. Most applications of inverse transport are therefore performed from angularly averaged measurements or other types of measurements in which the density of incoming particles and the measured density of outgoing particles is a function of the spatial variable $x \in \partial X$ only. We analyze what may be reconstructed in the optical parameters and with which stability under the following four scenarios: (i) time-dependent angularly resolved measurements; (ii) time-independent angularly resolved measurements; (iii) time-dependent angularly averaged measurements; and (iv) time-independent angularly averaged measurements. Although the latter setting is by far the most common in applications, it is also the worst-case scenario as far as theoretical uniqueness and stability results are concerned. These settings are analyzed in sections 3 and 4. Section 3 presents several uniqueness (and non-uniqueness) results based on the decomposition of the albedo operator in singular components. Section 4 focuses on the stability estimates that can be obtained in these different measurement settings.

In many applications of wave energy density propagation in scattering media, the spatially independent Hamiltonians in (3) offer accurate approximations. Some applications however, require spatially varying indices of refraction, which may be modeled for scalar waves by a Hamiltonian of the form $H(x,v) = c(x)|v|$ for classical waves. Most of the results obtained in the setting of free transport along straight lines generalizes to this setting of *non-Euclidean* geometry. Such generalizations are considered in section 6 below.

Once the optical parameters $\sigma(x,v)$ and $k(x,v',v)$ are known, another important inverse transport problem is the inverse source problem, where $S(t,x,v)$ is to be reconstructed from boundary measurements on Γ_+. Typical applications of the inverse source problem in medical imaging may be found in positron-electron tomography (PET), single photon emission computerized tomography (SPECT) and optical molecular imaging (OMI). The source term may then be considered as isotropic (independent of the velocity variable v) and since the photon propagation arises at a much faster time scale than the scale of emission, the source term and measurements are well-approximated by time-independent processes. The reconstruction of spatially varying source terms $S(x)$ from steady-state measurements is considered in section 7.

Practical aspects in the numerical simulation of inverse transport problems and other theoretical results not mentioned in earlier sections are addressed in section 8. Some perspective and open problems are offered in section 9.

2. Transport equations and regimes of propagation

2.1. Theory of transport

For detailed presentations of the theory of transport equations, we refer the reader to e.g. [35, 45, 96] and their references. We now recall the theoretical results that are used in the analysis of inverse transport problems.

To simplify the theoretical presentation of the main results in this paper, we shall assume that $V = \mathbb{S}^{d-1}$, the unit sphere. All results generalize after minor modifications to the setting where V is a finite union of spheres or a bounded open subset of $\mathbb{R}^d$ such that $\inf_{v\in V}|v| > 0$. The latter constraint is necessary because particles propagating with arbitrarily small speeds take arbitrary long times to escape the domain. The presence of arbitrarily small speeds of propagation therefore requires modifications in the derivation of the theoretical results; see e.g. [44].

We say that the optical parameters (σ, k) are admissible when

$$\begin{aligned}
&0 \le \sigma \in L^\infty(X\times V)\\
&0 \le k(x,v',\cdot) \in L^1(V) \text{ a.e. in } X\times V\\
&\sigma_p(x,v') := \int_V k(x,v',v)dv \in L^\infty(X\times V).
\end{aligned} \tag{5}$$

In practice, the optical parameters are non-negative and bounded. The above hypotheses are therefore not restrictive and are assumed to hold for the rest of the paper.

We define the times of escape of free-moving particles from X as

$$\tau_\pm(x,v) = \inf\{s>0 | x \pm sv \notin X\} \tag{6}$$

and $\tau(x,v) = \tau_+(x,v) + \tau_-(x,v)$. On the boundary sets $\Gamma_\pm$, we introduce the measure $d\xi(x,v) = |v\cdot\nu(x)|d\mu(x)dv$, where $d\mu(x)$ is the surface measure on ∂X.

Let us introduce the operators

$$\begin{aligned}
Ku(x,v) &= \int_V k(x,v',v)u(x,v')dv', \qquad D(K) = L^1(X\times V)\\
Tu &= -v\cdot\nabla_x u - \sigma u + Ku\\
D(T) &= \big\{u \in L^1(X\times V) | v\cdot\nabla_x u \in L^1(X\times V) \text{ and } u_{|\Gamma_-} = 0\big\}.
\end{aligned} \tag{7}$$

We now recall results of existence for the time-dependent and time-independent transport problems.

2.1.1. Time dependent case. We first consider the evolution equation

$$\begin{aligned} \Big(\frac{\partial}{\partial t} - T\Big)u &= 0, && (t,x,v) \in (0,\mathrm{T}) \times X \times V \\ u_{|\Gamma_-}(t,x,v) &= g(t,x,v) && (t,x,v) \in (0,\mathrm{T}) \times \Gamma_- \\ u(0,x,v) &= 0 && (x,v) \in X \times V. \end{aligned} \tag{8}$$

Let $T_1 = -v \cdot \nabla_x - \sigma$ with domain $D(T_1) = D(T)$. The unbounded operators T_1 and T are generators of strongly continuous semigroups $U_1(t)$ and $U(t)$, respectively, in $L^1(X \times V)$; see e.g [45, Proposition 2 p.226]. Moreover, for $f \in L^1(X \times V)$, we have

$$U_1(t)f(x,v) = e^{-\int_0^t \sigma(x-sv,v)ds} f(x-tv,v)\chi_X(x-tv), \qquad \text{a.e. in } X \times V, \tag{9}$$

where $\chi_X(y) = 1$ when $y \in X$ and $\chi_X(y) = 0$ otherwise.

The Duhamel formula then allows us to relate U to U_1 as

$$U(t) = U_1(t) + \int_0^t U_1(t-s)KU(s)ds, \qquad t \geq 0. \tag{10}$$

A natural functional setting for the transport solution is

$$\mathcal{W} = \Big\{ u \in L^1((0,\mathrm{T}) \times X \times V) | \frac{\partial u}{\partial t} + v \cdot \nabla_x u \in L^1((0,\mathrm{T}) \times X \times V) \Big\}, \tag{11}$$

with its natural norm. It is shown e.g. in [36, 37] that the trace maps

$$\gamma_-(\psi) = \big(\psi(0,\cdot), \psi_{|(0,\mathrm{T})\times\Gamma_-}\big) \quad \text{and} \quad \gamma_+(\psi) = \big(\psi(\mathrm{T},\cdot), \psi_{|(0,\mathrm{T})\times\Gamma_+}\big), \tag{12}$$

are well defined as a map from $\mathcal{W}$ to $L^1(X \times V, \tau_+(x,v)dxdv) \times L^1((0,\mathrm{T}) \times \Gamma_-, \min(\mathrm{T}-t, \tau_+(x,v))dtd\xi)$ for γ_- and with a similar expression for γ_+. Both maps are continuous, onto, and admit continuous liftings. We then introduce the Banach space

$$W := \big\{ u \in \mathcal{W} | \gamma_-(u) \in L^1(X \times V) \times L^1((0,\mathrm{T}) \times \Gamma_-, dtd\xi) \big\}. \tag{13}$$

For $u \in W$, the trace $\gamma_+(u)$ is then well-defined as an element in $L^1(X \times V) \times L^1((0,\mathrm{T}) \times \Gamma_+, dtd\xi)$ so that the measurements $u_{(0,\mathrm{T})\times|\Gamma_+}$ are indeed well-defined for $u \in W$.

The above setting allows us to incorporate the boundary condition on Γ_-. Let us assume that $g \in L^1((0,\eta); L^1(\Gamma_-, d\xi))$ for some $\eta > 0$ and let g be extended by 0 for times outside of $(0,\eta)$. Then we consider the lifting $G_-(t)\phi$ of $(0,\phi)$ to W defined by

$$G_-(t)\phi(t,x,v) = \exp\Big(-\int_0^{\tau_-(x,v)} \sigma(x-sv,v)ds\Big)\phi\big(t-\tau_-(x,v), x-\tau_-(x,v)v, v\big), \tag{14}$$

for $(t,x,v) \in (0,\mathrm{T}) \times X \times V$. As a consequence, we have the following result [45, Theorem 3 p.229]

Theorem 2.1 *The equation* (8) *admits a unique solution u in W given by*

$$u(t) = G_-(t)g + \int_0^t U(t-s)KG_-(s)gds. \tag{15}$$

Moreover, the albedo operator is given by

$$\mathcal{A}g = u_{|(0,\mathrm{T})\times\Gamma_+}, \tag{16}$$

and is a bounded operator from $L^1((0,\eta), L^1(\Gamma_-, d\xi))$ *to* $L^1((0,\mathrm{T}), L^1(\Gamma_+, d\xi))$.

For consistency with the time-independent case, we recast the above transport equation and the transport solution as

$$(I-\mathcal{K})u = \mathcal{I}g, \qquad u = (I-\mathcal{K})^{-1}\mathcal{I}g,$$

where we have defined formally the operators

$$\begin{aligned} \mathcal{I}g(t) &= G_-(t)g & t \in (0,\mathrm{T}) \\ \mathcal{K}\phi(t) &= \int_0^t U_1(t-s)K\phi(s)ds, & t \in (0,\mathrm{T}). \end{aligned}$$

Using these operators, we may recast the transport solution as

$$u = \mathcal{I}g + \mathcal{K}\mathcal{I}g + (I-\mathcal{K})^{-1}\mathcal{K}^2\mathcal{I}g, \tag{17}$$

where $u_0 := \mathcal{I}g$ is the ballistic component, $u_1 := \mathcal{K}\mathcal{I}g$ the single scattering component and $u_2 := u - u_0 - u_1 = (I-\mathcal{K})^{-1}\mathcal{K}^2\mathcal{I}g$ is the multiple scattering component.

2.1.2. Time independent case. A similar theory may be developed for the time-independent transport problem

$$\begin{aligned} -Tu &= S, & (x,v) \in X \times V \\ u_{|\Gamma_-}(x,v) &= g(x,v) & (x,v) \in \Gamma_-, \end{aligned} \tag{18}$$

where $S(x,v) \in L^1(X \times V)$ is a volume source term. The assumptions (5) on the optical parameters are however no longer sufficient to guarantee a unique solution to (18). The physical reason is that the "creation" of particles caused by scattering Ku needs to be compensated by another mechanism in order for a steady equilibrium to take place. Two mechanisms compensate for this creation of particles: the absorption of particles inside X and the leakage of particles leaving X across the boundary ∂X. In applications in medical and geophysical imaging, total absorption is defined as the sum of scattering plus intrinsic absorption. Leakage is therefore not important. In nuclear physics however, neutrons are also created by fission so that leakage at the domain's boundary is an important contribution to the stability of the transport solution [1, 96].

We define the following Banach space

$$W := \big\{u \in L^1(X \times V) | v \cdot \nabla_x u \in L^1(X \times V), \tau^{-1}u \in L^1(X \times V)\big\}, \tag{19}$$

with its natural norm. We recall that τ is defined below (6). We have the following trace formula [44]

$$\|f_{|\Gamma_\pm}\|_{L^1(\Gamma_\pm, d\xi)} \leq \|f\|_W, \qquad f \in W. \tag{20}$$

This allows us to introduce the following lifting operator

$$\mathcal{I}g(x,v) = \exp\Big(-\int_0^{\tau_-(x,v)} \sigma(x-sv,v)ds\Big) g(x-\tau_-(x,v)v,v). \tag{21}$$

It is proved in [44] that $\mathcal{I}$ is a bounded operator from $L^1(\Gamma_-, d\xi)$ to W.

Let us next define the bounded operators

$$\begin{aligned} \mathcal{K}u(x,v) &= \int_0^{\tau_-(x,v)} \exp\Big(-\int_0^t \sigma(x-sv,v)ds\Big) \int_V k(x-tv,v',v)u(x-tv,v')dv'dt \\ \mathcal{L}S(x,v) &= \int_0^{\tau_-(x,v)} \exp\Big(-\int_0^t \sigma(x-sv,v)ds\Big) S(x-tv,v)dt \end{aligned} \tag{22}$$

for $(x,v) \in X \times V$. Looking for solutions in W, the integro-differential equation (18) is thus recast as

$$(I - \mathcal{K})u = \mathcal{I}g + \mathcal{L}S. \tag{23}$$

Then we have the following result [17, 44].

Theorem 2.2 *Assume that*

$$(I-\mathcal{K}) \textit{ admits a bounded inverse in } L^1(X \times V, \tau^{-1}dxdv). \tag{24}$$

Then the integral equation (23) *admits a unique solution* $u \in W$ *for* $g \in L^1(\Gamma_-, d\xi)$ *and* $S \in L^1(X \times V)$. *Furthermore, the albedo operator*

$$\mathcal{A} : L^1(\Gamma_-, d\xi) \to L^1(\Gamma_+, d\xi), \qquad g \mapsto \mathcal{A}g = u_{|\Gamma_+}, \tag{25}$$

where u *solves* (23) *with* $S \equiv 0$, *is a bounded operator.*

The invertibility condition (24) *holds under either of the following assumptions*

$$\sigma - \sigma_p \geq 0 \tag{26}$$

$$\|\tau\sigma_p\|_\infty < 1. \tag{27}$$

The above theorem states that the transport equation admits a unique solution provided that $(I-\mathcal{K})$ is invertible, which is somewhat tautological. The main messages of the theorem are that (i) the albedo operator is then well-defined and that the results stated below for the inverse transport problem hold so long as $I-\mathcal{K}$ is invertible; and (ii) $I-\mathcal{K}$ is not necessarily invertible and conditions, which are not necessary in the time-dependent setting, need be imposed. It turns out that under general assumptions on $k(x,v',v)$, $\mathcal{K}^m$ is compact in the L^1 and L^2 settings for m sufficiently large [45, 96, 131, 133] so that (24) is invertible when $1 \notin \Sigma_p(\mathcal{K})$, the spectrum of $\mathcal{K}$ (see also the results in [131, 133] recalled below (141)). Note that $\mathcal{K}$ is linear in k. When $1 \in \Sigma_p(\mathcal{K})$, we thus verify that replacing k by λk with $\lambda \neq 1$ sufficiently close to 1 makes $(I-\lambda\mathcal{K})$ invertible so that the steady-state transport equation may be seen to be invertible for *generic* optical coefficients.

When $\Sigma_p(\mathcal{K}) \subset [0,1)$, the problem is said to be *subcritical.* This is the case in most applications of forward and inverse transport, with the notable exception of nuclear reactor physics, where the objective is for the reactor to be exactly critical [96]. Sufficient conditions for subcriticality are given in (26) and (27). The former is the most physical in practice. It states that particles "created" by scattering into direction v have scattered from direction v' and are thus "absorbed" for direction v'. The total absorption (intrinsic absorption plus "absorption" caused by scattering) is thus always larger than "creation" caused by scattering. The latter constraint is less physical. It states that leakage alone compensates for the creation of particles in the operator $\mathcal{K}$ even when absorption σ vanishes. Note that when the problem is subcritical, its solution may be expressed in terms of the following Neumann expansion in $L^1(X \times V)$

$$u = \sum_{m=0}^{\infty} \mathcal{K}^m (\mathcal{I}g + \mathcal{L}S). \tag{28}$$

The contribution $m = 0$ is the ballistic part of u, the contribution $m = 1$ the single scattering part of u, and so on. It is essentially this decomposition of the transport solution into orders of scatterings that allows us to stably reconstruct the optical parameters in the following sections. Note that the above Neumann series expansions has an additional benefit. Since the optical parameters are non-negative, each term in the above series is non-negative provided that g and S are non-negative so that the transport solution itself is non-negative. A little more work [45] allows us to prove the maximum principle, which states that u in $X \times V$ is bounded a.e. by the (essential) supremum of g in Γ_- when $S \equiv 0$.

When $S \equiv 0$, the transport solution may be decomposed as in (17) as a superposition of the ballistic part $\mathcal{I}g$, the single scattering part $\mathcal{K}\mathcal{I}g$, and the multiple scattering part $(I - \mathcal{K})^{-1}\mathcal{K}^2\mathcal{I}g$.

2.2. Diffusive regime

The theory outlined in the preceding section holds independent of the strength of the scattering and absorption coefficients so long as a subcriticality condition such as (26) or (27) is verified in the time independent setting. Yet the regime of interest in this paper is the regime where scattering is not too overwhelming. When scattering is strong and particles interact often with the underlying medium, the transport solution is very well approximated by the solution to simplified equations such as the diffusion equation or the Fokker-Planck equation. In such regimes, the inverse transport problem is fundamentally modified. The ballistic $\mathcal{I}g$ and single scattering $\mathcal{K}\mathcal{I}g$ components become asymptotically negligible. The techniques presented in this paper are based on using the ballistic and single scattering components, and therefore become asymptotically irrelevant as the diffusive or Fokker-Planck regimes set in.

In order to better understand the limitations of the techniques detailed in this paper, we briefly present the regimes of validity of the diffusion and Fokker-Planck approximations and mention the related inverse problems. We start with the diffusion approximation.

The diffusion approximation is valid when (i) scattering is strong, and (ii) absorption is weak. The derivation of diffusion approximations is well-developed in the physical and mathematical literatures; see e.g. [27, 28, 45, 80, 120]. What may be less well appreciated is the fact that diffusion approximations also hold in the presence of spatially varying indices of refraction.

We thus consider the transport equation with $H(x, v) = c(x)|v| = \omega$, where ω is fixed. We also assume that the Hamiltonian H is preserved by scattering (elastic scattering); see e.g. [39, 120]. Under these assumptions, the transport equation (1) takes the form

$$\frac{\partial u}{\partial t} + c(x)\frac{v}{|v|} \cdot \nabla_x u - |v|\nabla c(x) \cdot \nabla_v u + \sigma(x)u = \int_{\mathbb{R}^d} k(x, v', v)u(t, x, v')\delta(\omega - c(x)|v'|)dv', \tag{29}$$

augmented here with non-vanishing initial conditions $u(0, x, v) = u_{\rm in}(x)$ independent of v to simplify. In order to derive the diffusion approximation to the above equation, we follow the presentation in [15]. It is convenient to perform the change of variables

$$\tilde{u}(t, x, \theta) = u(t, x, \frac{\omega}{c(x)}\theta),$$

where $\theta = \frac{v}{|v|} \in \mathbb{S}^{d-1}$. After the change of variables $(x, v) \to (x, \omega = c(x)|v|, \theta = \frac{v}{|v|})$, we find that $\tilde{u}$ solves the transport equation

$$\frac{\partial \tilde{u}}{\partial t} + c(x)\theta \cdot \nabla_x \tilde{u} - \nabla c(x) \cdot (I - \theta \otimes \theta)\nabla_\theta \tilde{u} + \sigma(x)\tilde{u} = \int_{\mathbb{S}^{d-1}} \tilde{k}(x, \theta', \theta)\tilde{u}(t, x, \theta')d\theta', \tag{30}$$

with

$$\tilde{k}(x, \theta', \theta) = \frac{\omega^{d-1}}{c(x)^d} k\big(x, \frac{\omega}{c(x)}\theta', \frac{\omega}{c(x)}\theta\big).$$

In optical tomography, it is customary to introduce

$$\mu_a(x) = \frac{\sigma(x)}{c(x)} - \mu_s(x), \qquad \mu_s(x) f(x, \theta \cdot \theta') = \frac{\tilde{k}(x, \theta', \theta)}{c(x)}, \tag{31}$$

where $\mu_a(x)$ is the intrinsic absorption coefficient, $\mu_s(x)$ the scattering coefficient, and $f(x, \theta \cdot \theta')$ the phase function , which we assume depends on θ and θ' only through $\theta \cdot \theta'$ and integrates to 1 in the sense that $\int_{\mathbb{S}^{d-1}} f(x, \theta \cdot \theta')d\theta' = 1$. This models the fact that scattering is invariant by rotation.

The diffusion regime sets in when scattering is large so that μ_s is replaced by μ_s/η for $\eta \ll 1$, when absorption is small so that μ_a is replaced by $\eta\mu_a$, and when time is rescaled so that dynamics have time to develop with t replaced by t/η. In such a regime, the mean free path $\eta l(x, \omega)$ is defined as

$$\eta l(x, \omega) := \frac{\eta}{\mu_t(x, \omega)} \ll L, \qquad \frac{\mu_t(x, \omega)}{\eta} = \frac{\mu_s(x, \omega)}{\eta} + \eta\mu_a(x, \omega), \tag{32}$$

where L is the distance over which propagation is observed. The equation for $\tilde{u}_\eta$ then becomes

$$\begin{aligned} &\frac{\eta}{c(x)}\frac{\partial \tilde{u}_\eta}{\partial t} + \theta\cdot\nabla_x \tilde{u}_\eta - \frac{\nabla c(x)}{c(x)}\cdot (I-\theta\otimes\theta)\nabla_\theta \tilde{u}_\eta + \eta\mu_a(x)\tilde{u}_\eta \\ &= \frac{\mu_s(x,\omega)}{\eta}\int_{\mathbb{S}^{d-1}} f(x,\theta'\cdot\theta)\big(\tilde{u}_\eta(t,x,\theta',\omega)-\tilde{u}_\eta(t,x,\theta,\omega)\big)d\theta', \end{aligned} \tag{33}$$

with initial conditions $\tilde{u}_\eta(0,x,\theta,\omega) = \tilde{u}_{\rm in}(x,\omega)$ independent of θ to simplify.

We may then write $\tilde{u}_\eta = U_0 + \eta\tilde{u}_1 + \eta^2\tilde{u}_2$, plug the asymptotic expansion into (33) and equate like powers of η [15, 45]. The first equation shows that $U_0 = U_0(t,x)$ independent of θ since the right-hand side in (33) is a conservative operator. The second equation allows us to obtain that

$$\tilde{u}_1(t,x,\theta) = \frac{-\theta\cdot\nabla_x U_0(t,x)}{\mu_s(x)(1-\lambda_1(x))},$$

where $\lambda_1(x)$ is uniquely defined by the equation

$$\lambda_1(x)\theta = \int_{\mathbb{S}^{d-1}} f(x,\theta\cdot\theta')\theta' d\theta'.$$

The third equation corresponding to terms proportional to $O(\eta)$ admits a compatibility condition, which after some algebra, may be recast as

$$\frac{1}{c^d(x)}\frac{\partial U_0}{\partial t} - \nabla\cdot\Big(\frac{D(x)}{c^{d-1}(x)}\nabla U_0\Big) + \frac{\mu_a(x)}{c^{d-1}(x)}U_0 = 0, \tag{34}$$

with initial conditions $U_0(0,x) = \tilde{u}_{\rm in}(x)$ and where the diffusion coefficient is defined by

$$D(x) := \frac{l^*(x)}{d} = \frac{l(x)}{d(1-\lambda_1(x))} = \frac{1}{d\mu_t(x)(1-\lambda_1(x))}, \tag{35}$$

where μ_t is defined in (32) above. Here $l^*(x)$ is the rescaled transport mean free path (while $\eta l^*(x)$ is the physical transport mean free path). The transport mean free path l^* is always larger than the mean free path l. While the latter characterizes the mean distance between interactions with the underlying medium, the former characterizes the mean distance it takes of particles to significantly change direction because of scattering. When scattering is isotropic so that $f(\mu) := 1$, then $\lambda_1 = 0$ and $l = l^*$. When scattering is anisotropic, in the sense that it is primarily occurring in the forward direction, then $\lambda_1 > 0$ and $l^* > l$. The diffusive regime sets in when the transport mean free path ηl^* is sufficiently small so that the initial direction of the particles is lost because of multiple interactions with the underlying medium.

In the presence of a boundary ∂X, the above diffusion equation needs to be augmented with boundary conditions. The derivation of boundary conditions is not completely straightforward because the decomposition $u_\eta = U_0 + O(\eta)$ is not valid in the vicinity of the boundary (since U_0 does not depend on v while u_η prescribed on Γ_-

inevitably does). The proper derivation of boundary conditions involves the introduction of boundary layers and is now well-understood, even if it is seldom used in practice. The resulting boundary conditions are of Robin type $U_0 - \eta L_e D(x)\nu \cdot \nabla_x U_0 = G$ on ∂X for some boundary source term G and extrapolation length L_e. We refer the reader to e.g. [26, 28, 45, 84].

In the diffusive regime of propagation, the ballistic and single scattering components of the transport solution are negligible and of order $e^{-\frac{\sigma_0}{\eta}}$, where σ_0 is a typical value for the attenuation coefficient σ. The inverse problem for the reconstruction of the optical parameters from boundary measurements becomes an inverse elliptic or inverse parabolic problem for the diffusion coefficient $D(x)$ and the attenuation coefficient $\mu_a(x)$ assuming that the sound speed $c(x)$ is known. Such inverse problems are qualitatively quite different from the inverse transport problem. We refer the reader to e.g. [6, 7, 59, 136] for a few references on the theory and practice of such widely used inverse problems.

2.3. Highly peaked forward regime

The preceding section presented a limitation to the inverse transport theory in the regime of strong scattering, when the transport mean free path l^* is small compared to the overall distance of propagation. We now consider another limitation of the inverse transport theory when the transport mean free path l^* is not necessarily small but the mean free path l is small. This occurs in the presence of strong highly peaked forward scattering. While strong highly peaked forward scattering may not be sufficiently strong to generate significant spatial diffusion as in the diffusive regime, it may be sufficiently strong to generate diffusion in the angular variable. This angular diffusion then destroys the singularities of the transport operator and as such significantly reduce the applicability of the inverse transport methods presented later in the paper; except the approximate stability results shown in section 5. For references on the derivation of the Fokker-Planck model in this context and its applications, see e.g. [66, 81, 111].

Let us consider the transport equation (29) with $c(x) = c := 1$ constant with ω normalized to $\omega = 1$ to simplify, which we recast, using the notation in (31) above, as

$$\frac{\partial u}{\partial t} + \theta \cdot \nabla_x u + \mu_a(x) u = \mu_s(x) \int_{\mathbb{S}^{d-1}} f(\theta' \cdot \theta)\big(u(t,x,\theta') - u(t,x,\theta)\big) d\theta'. \qquad (36)$$

Here, $f(\theta \cdot \theta')$ is the phase function, which indicates how the scattered particles are distributed in the angle θ' after collision. Mie scattering theory [60] tends to show that $f(\mu)$ is constant for small size scatterers while it is significantly peaked in the vicinity of $\mu = 1$ for large size scatterers. A typical expression for the phase function is the Henyey-Greenstein phase function (when $d = 3$) [57]

$$f(\theta' \cdot \theta) = \frac{1 - g^2}{(1 + g^2 - 2g\cos(\theta' \cdot \theta))^{\frac{3}{2}}}. \qquad (37)$$

Here, $g \in [0, 1)$ is called anisotropic factor, which measures the strength of forward-peakedness of the phase function. Typical values in animal tissues for g are in the range $0.9 \le g \le 0.99$, which correspond to quite highly peaked forward scattering.

Let us assume highly peaked forward scattering. We also assume that the mean free path is sufficiently small so that enough highly peaked forward scattering occurs to have a visible effect. The right scaling for (36) is then

$$\frac{\partial u_\varepsilon}{\partial t}+\theta\cdot\nabla_x u_\varepsilon+\mu_a(x)u_\varepsilon=\frac{\mu_s(x)}{\varepsilon^{d+1}}\int_{\mathbb{S}^{d-1}}\tilde f\Big(\frac{1-(\theta'\cdot\theta)^2}{\varepsilon^2}\Big)\big(u_\varepsilon(t,x,\theta')-u_\varepsilon(t,x,\theta)\big)d\theta', \quad (38)$$

where $\tilde f(\frac{1}{\varepsilon^2}(1-\mu^2))=C_\varepsilon f(\mu)$ for the rescaled highly peaked forward phase function. Define a system of coordinates on $\mathbb{S}^{d-1}$ such that $\theta=(0,\ldots,0,1)$ and $\theta'=(\sin\phi\theta'',\cos\phi)$, where $\phi\in(0,\frac{\pi}{2})$ and θ'' is a parameterization of $\mathbb{S}^{d-2}$ so that $d\theta'=\sin\phi^{d-2}d\phi d\theta''$.

Assuming that $\tilde f(\mu)$ is rapidly decaying at infinity, the above scaling implies that $\sin\phi$ is of order ε. We define $\sin\phi=\varepsilon\psi$ and $v_\varepsilon(\sin\phi\theta'',\cos\phi)=u_\varepsilon(\theta')=v_\varepsilon(\varepsilon\psi\theta'',\sqrt{1-\varepsilon^2\psi^2})$. A Taylor expansion of v_ε in the vicinity of $\theta=(0,\ldots,0,1)$ shows that

$$\begin{aligned}&\int_{\mathbb{S}^{d-2}}(v_\varepsilon(\varepsilon\psi\theta'',\sqrt{1-\varepsilon^2\psi^2})-v_\varepsilon(0,\ldots,0,1))d\theta''\\ =\ &\frac{\varepsilon^2\psi^2c_d}{2}\big(\Delta_{d-1}v_\varepsilon-(d-1)\partial_d v_\varepsilon)(0,\ldots,0,1)+o(\varepsilon^2),\end{aligned}$$

provided that v_ε is sufficiently smooth, where Δ_{d-1} is the (Euclidean) Laplacian with respect to the first $d-1$ variables and ∂_d is derivative with respect to the last variable and where $c_d=\int_{\mathbb{S}^{d-2}}(\theta''_1)^2d\theta''=\frac{1}{d-1}\int_{\mathbb{S}^{d-2}}d\theta''$ (obtained e.g. by completing the squares).

Using the definition of the Laplace Beltrami operator $\Delta_\perp$ in spherical coordinates, we observe after some algebra that at $\theta=(0,\ldots,0,1)$,

$$\Delta_\perp u_\varepsilon=\Delta_{d-1}v_\varepsilon-(d-1)\partial_d v_\varepsilon.$$

In other words, we find that

$$\int_{\mathbb{S}^{d-1}}\tilde f\big(\frac{1-(\theta'\cdot\theta)^2}{\varepsilon^2}\big)\big(u_\varepsilon(t,x,\theta')-u_\varepsilon(t,x,\theta)\big)d\theta'=\varepsilon^{d+1}\int_{\mathbb{R}_+}\psi^d\tilde f(\psi^2)d\psi\ \frac{c_d}{2}\Delta_\perp u_\varepsilon+o(\varepsilon^{d+1}).$$

This means that $u_\varepsilon=u+o(1)$, where u is the solution to the following Fokker-Planck equation

$$\frac{\partial u}{\partial t}+\theta\cdot\nabla_x u+\mu_a(x)u=\mu_s(x)\mathfrak{D}\Delta_\perp u,\qquad \mathfrak{D}=\frac{c_d}{2}\int_{\mathbb{R}_+}\psi^d\tilde f(\psi^2)d\psi. \quad (39)$$

The above equation is mathematically quite different from the transport equation. The operator $\theta\cdot\nabla_x-D\Delta_\perp$ is known to be hypo-elliptic, which implies that the solution to the above equation is smooth, at least when μ_a and μ_s are smooth. The singularities of the albedo operator that will be useful in later sections are therefore smoothed-out by the angular diffusion operator $\Delta_\perp$. The reconstruction of $\mu_a(x)$ and $\mu_s(x)\mathfrak{D}$ from boundary measurements is an open problem to-date.

The regimes of diffusion and Fokker Planck approximations are major limitations to the theory presented in the following sections. In these regimes, the ballistic and single scattering contributions to the albedo operator are negligible because the mean free path, the mean distance between successive interactions of the particles with the underlying medium, is very small. In the diffusive regime, both the transport mean free path and the mean free path are small, and the ballistic front is extremely strongly damped. In the Fokker Planck regime, the transport mean free path may still be large while the mean free path is small. The ballistic front, however, may not be completely destroyed. It is simply blurred by the angular diffusion as a beam of particles propagates through the scattering medium. The blurring caused by the angular diffusion, when $\mathfrak{D}$ in (39) is small, may then be modeled as noise at the detector levels. Stable reconstructions are no longer feasible at those scales that have been blurred by the angular diffusion. The approximate stability estimates obtained in section 5 may then be used to understand how the reconstructions degrade as $\mathfrak{D}$ increases.

3. Decomposition of the albedo operators and Uniqueness results

In both the time dependent and the time independent settings, we may follow (17) and decompose the albedo operator as

$$\begin{aligned} \mathcal{A}g &= \mathcal{I}g\big|_{\Gamma_+} + \mathcal{K}\mathcal{I}g\big|_{\Gamma_+} + \mathcal{K}^2(I-\mathcal{K})^{-1}\mathcal{I}g\big|_{|\Gamma_+} \\ &:= \mathcal{A}_0 g + \mathcal{A}_1 g + \mathcal{A}_2 g. \end{aligned} \tag{40}$$

We denote by α the Schwartz kernel of the albedo operator $\mathcal{A}$ and use the same symbol α both for the time dependent and the time independent settings. In other words,

$$\begin{aligned} \mathcal{A}g(t,x,v) &= \int_{(0,\eta)\times\Gamma_-} \alpha(t-s,x,v,y,w)g(s,y,w)d\mu(y)dwds \\ \mathcal{A}g(x,v) &= \int_{\Gamma_-} \alpha(x,v,y,w)g(y,w)d\mu(y)dw, \end{aligned}$$

in the time dependent and time independent settings, respectively. We follow here the convention in [17, 19, 44] for the definition of α. The Schwartz kernel in [75, 129] is defined as $|\nu(y)\cdot w|^{-1}$ times the above Schwartz kernel as the integrals on Γ_- are considered with respect to the measure $d\xi(y,w) = |\nu(y)\cdot w|d\mu(y)dw$ in those references.

The Schwartz kernel is decomposed as the sum of α_0, α_1 and α_2, the contributions of the ballistic, single scattering, and multiple scattering components, respectively, as in the decomposition (40). The decomposition was first developed in [43, 44] to obtain uniqueness results for the reconstruction of the optical parameters and was later used in [17, 19, 141] to prove stability results. We mainly follow the presentation in the latter references. For other works on the decomposition of the transport solution and their use in inverse transport theory, we refer the reader to [31, 32, 79].

What makes the decomposition useful is that α_0 is more singular, as a distribution, than α_1 and α_2 in the sense that there exists a sequence of continuous functions ϕ_ε such that $\langle\alpha_0,\phi_\varepsilon\rangle \to 1$ as $\varepsilon \to 0$ while $\langle\alpha_m,\phi_\varepsilon\rangle \to 0$, $m = 1,2$. These statements will be made more precise in the next section.

3.1. Time dependent case

We have the following decompositions in the time dependent case:

$$\alpha_0(t,x,v,y,w) = \exp\Big(-\int_0^{\tau_-(x,v)} \sigma(x-sv,v)ds\Big)\,\delta_v(w)\delta_{\{x-\tau_-(x,v)v\}}(y)\delta(t-\tau_-(x,v)), \quad (41)$$

$$\begin{aligned}\alpha_1(t,x,v,y,w) &= \int_0^{\tau_-(x,v)} \exp\Big(-\int_0^s \sigma(x-\tau v,v)d\tau - \int_0^{\tau_-(x-sv,w)} \sigma(x-sv-\tau w,w)d\tau\Big) \\ &\qquad k(x-sv,w,v)\delta_{\{x-sv-\tau_-(x-sv,w)w\}}(y)\delta(t-s-\tau_-(x-sv,w))ds.\end{aligned} \quad (42)$$

Here, $\delta_{\{x\}}$ is the delta function on the surface ∂X defined by $\int_{\partial X} \delta_{\{x\}}(y)\phi(y)d\mu(y) = \phi(x)$ for $x \in \partial X$ and ϕ continuous on ∂X. The other delta functions $\delta_v(w)$ and $\delta(t)$ are defined similarly on $V = \mathbb{S}^{d-1}$ and $\mathbb{R}$. Note that α_0 and α_1 are not necessarily functions and have to be defined as distributions.

The multiple scattering contribution α_2 does not admit as simple an expression as those above. However, α_2, unlike α_0 and α_1, is always a function. It is shown in [43] that $|\nu(y)\cdot w|^{-1}\alpha_2 \in L^\infty(\Gamma_-, L^1_{\rm loc}(\mathbb{R}, L^1(\Gamma_+, d\xi)))$. Under the additional assumption that $k \in L^\infty(X \times V \times V)$, we have the following more precise estimate, shown in [19],

$$|\nu(y)\cdot w|^{-1}\alpha_2(t,x,v,y,w) \in L^\infty(\Gamma_-, L^p((-\eta, \mathrm{T}), L^p(\Gamma_+, d\xi))), \quad 1 \le p < \frac{d+1}{d}. \quad (43)$$

3.2. Time independent case

We have the following decompositions in the time independent case:

$$\alpha_0(x,v,y,w) = \exp\Big(-\int_0^{\tau_-(x,v)} \sigma(x-sv,v)ds\Big)\delta_v(w)\delta_{\{x-\tau_-(x,v)v\}}(y). \quad (44)$$

$$\begin{aligned}\alpha_1(x,v,y,w) &= \int_0^{\tau_-(x,v)} \exp\Big(-\int_0^t \sigma(x-sv,v)ds - \int_0^{\tau_-(x-tv,w)} \sigma(x-tv-sw,w)ds\Big) \\ &\qquad k(x-tv,w,v)\delta_{\{x-tv-\tau_-(x-tv,w)w\}}(y)dt.\end{aligned} \quad (45)$$

Note that the corresponding terms in the time domain are decomposed as the product of the time independent terms above and delta functions in time corresponding to arrival times. Also, we verify that

$$\alpha_m(x,v,y,w) = \int_0^\infty \alpha_m(t,x,v,y,w)dt, \qquad m = 0,1.$$

As in the time dependent setting, the multiple scattering α_2 is a function unlike the ballistic part α_0, which needs to be interpreted as a bounded distribution. The single scattering contribution α_1 is a function in dimension $d = 2$ whereas it is not necessarily a function in dimension $d \ge 3$. The reason is essentially due to the fact that two lines almost always intersect in two space dimensions whereas they almost never intersect in dimensions three and higher. Single scattering is therefore more singular than multiple scattering only in dimension $d \ge 3$.

The analysis of the multiple scattering term $\mathcal{A}_2$ is best performed by recasting $\mathcal{A}_2 = \mathcal{K}^2(I-\mathcal{K})^{-1}\mathcal{I}$. We have seen in the preceding section that $(I-\mathcal{K})^{-1}\mathcal{I}$ is a bounded operator from $L^1(\Gamma_-, d\xi)$ to $L^1(X\times V, \tau^{-1}dxdv)$. Let then $\mathfrak{k}(x,v,y,w)$ be the Schwartz kernel of $\mathcal{K}^2_{|\Gamma_+}$ with $(x,v)\in\Gamma_+$ and $(y,w)\in X\times V$. Then, it is shown in [18, Lemma 4.1] that

$$\begin{aligned}&\Big\|\int_{\Gamma_+}\psi(x,v)\mathfrak{k}(x,v,y,w)\tau(y,w)d\xi(x,v)\Big\|_{L^\infty(X_y\times V_w)}\\ &\le C\int_V\Big(\int_{\substack{x\in\partial X\\ \nu(x)\cdot v>0}}|\psi(x,v)|^{p'}(\nu(x)\cdot v)d\mu(x)\Big)^{\frac{1}{p'}}dv,\end{aligned} \tag{46}$$

for any $\psi\in L^\infty(\Gamma_+)$ and any $1<p<1+\frac{1}{d-1}$, $p^{-1}+p'^{-1}=1$. This estimate, which improves on a similar estimate obtained in [17] shows that $\mathcal{K}^2_{|\Gamma_+}$ maps $L^1(X\times V,\tau^{-1}dxdv)$ into a space that is smaller than the space $L^p(\Gamma_+,d\xi)$ for $1\le p<1+\frac{1}{d-1}$. This estimate will be useful in the analysis of approximate stability estimates in section 5. It admits a similar expression in the time dependent setting, which improves on (43) above.

3.3. Full measurement setting

Let us now consider the setting in which the full albedo operator $\mathcal{A}$ is supposed to be known as an operator from $L^1((0,\eta),L^1(\Gamma_-,d\xi))$ to $L^1((0,\mathrm{T}),L^1(\Gamma_+,d\xi))$ in the time dependent setting and from $L^1(\Gamma_-,d\xi)$ to $L^1(\Gamma_+,d\xi)$ in the time independent setting.

This is equivalent to knowing the Schwartz kernels $\alpha(t,x,v,y,w)$ on $(0,\mathrm{T})\times X\times V\times X\times V$ in the time dependent setting and $\alpha(x,v,y,w)$ on $X\times V\times X\times V$ in the time independent setting [43, 44].

Recovery of $\sigma=\sigma(x)$ independent of v. Since α_0 is more singular than α_m, $m=1,2$, knowledge of α provides knowledge of α_0, which in turns provides knowledge of $\sigma=\sigma(x)$. More precisely, considering the time independent setting, let $(x_0,v_0)\in\Gamma_+$ and define $y_0=x_0-\tau_-(x_0,v_0)v_0$ such that $(y_0,v_0)\in\Gamma_-$. Let now g_ε be a sequence of normalized L^1 functions on Γ_- converging to $\delta_{v_0}(v)\delta_{\{y_0\}}(y)$. Let ϕ_ε be a sequence of bounded functions on Γ_+ equal to 1 in the vicinity of (x_0,v_0) and with vanishing support as $\varepsilon\to0$. Then we verify [17, 44] that

$$\int_{\Gamma_+\times\Gamma_-}\alpha_m(x,v,y,w)\phi_\varepsilon(x,v)g_\varepsilon(y,w)d\mu(x)dvd\mu(y)dw\xrightarrow{\varepsilon\to0}0,\quad m=1,2,$$

so that

$$\begin{aligned}\langle\phi_\varepsilon,\mathcal{A}g_\varepsilon\rangle \quad &:= \quad \int_{\Gamma_+\times\Gamma_-}\alpha(x,v,y,w)\phi_\varepsilon(x,v)g_\varepsilon(y,w)d\mu(x)dvd\mu(y)dw\\ &\xrightarrow{\varepsilon\to0}\quad \exp\Big(-\int_0^{\tau_-(x_0,v_0)}\sigma(x_0-sv_0,v_0)ds\Big).\end{aligned} \tag{47}$$

A similar result [43] holds in the time dependent setting. As a consequence, $-\ln\langle\phi_\varepsilon, \mathcal{A}g_\varepsilon\rangle$ converges to the integral of $\sigma(x, v_0)$ along the line passing through x_0 with direction v_0. Since x_0 and v_0 are arbitrary in Γ_+, we have thus obtained that knowledge of $\mathcal{A}$ provides knowledge of the X-ray transform (hence of the Radon transform) of $\sigma(x, v)$.

It is known that the X-ray transform does not allow us to uniquely determine a general function $\sigma(x, v)$; we refer the reader to section 3.4 below for a presentation of a recent result in [131] on the obstructions to uniqueness in inverse transport theory. When $\sigma = \sigma(x)$ depends only on position however, then the X-ray transform of σ uniquely determines σ with an explicit inversion formula [59, 101]. The absorption coefficient is thus uniquely determined by knowledge of $\mathcal{A}$, both in the time dependent and time independent settings for all dimensions $d \geq 2$.

The measurements corresponding to the above choices of functions g_ε and ϕ_ε then converge to the function

$$\langle\phi_\varepsilon, \mathcal{A}g_\varepsilon\rangle \xrightarrow{\varepsilon\to 0} E(x, y) := \exp\Big(-\int_0^{|x-y|}\sigma\big(x - s\frac{x-y}{|x-y|}\big)ds\Big). \tag{48}$$

Recovery of $k(x, v, w)$. We assume that $\sigma = \sigma(x)$ is now recovered. Let $z_0 \in X$, $v_0 \in V$, and $v_0 \neq w_0 \in V$. Define $x_0 = z_0 + \tau_+(z_0, v_0)v_0$ so that $(x_0, v_0) \in \Gamma_+$ and $y_0 = z_0 - \tau_-(z_0, w_0)w_0$ so that $(y_0, w_0) \in \Gamma_-$. We formally show how the scattering coefficient may be uniquely reconstructed from full knowledge of $\mathcal{A}$.

We first consider the time independent setting. Let us define g_{ε_1} as before and ϕ_ε as a sequence of bounded functions on Γ_+ equal to a constant in the vicinity of (x_0, v_0) and with vanishing support as $\varepsilon \to 0$. Since $v_0 \neq w_0$, we find that

$$\int_{\Gamma_+\times\Gamma_-}\alpha_0(x, v, y, w)\phi_\varepsilon(x, v)g_{\varepsilon_1}(y, w)d\mu(x)dvd\mu(y)dw = 0, \quad 0 \leq \varepsilon, \varepsilon_1 < \varepsilon_0(x_0, v_0, y_0, w_0).$$

i.e., the ballistic contribution vanishes with such measurements. Let us define g_{ε_1} such that $|\nu(y_0)\cdot w_0|^{-1}g_{\varepsilon_1}(y, w)$ converges to a delta function. The factor $|\nu(y_0)\cdot w_0|^{-1}$ is here to ensure that the number of emitted particles is independent of y_0 and w_0. The ballistic part of the transport solution is then approximately concentrated on the line passing through y_0 and with direction w_0. Scattering occurs along this line and particles scattered in direction v_0 are approximately supported on the plane with directions v_0 and w_0 passing through x_0. The intersection of that plane with the boundary ∂X is a one-dimensional *curve* $\gamma(x_0, v_0, w_0) \subset X$. In two space dimensions, the curve γ has the same dimension as ∂X. As a consequence, α_1 is a function and therefore is not more singular than α_2 in the time independent setting when $d = 2$.

Let $\phi_\varepsilon(x, v)$ be a bounded test function supported in the $\varepsilon-$vicinity of γ. Because γ is of measure 0 in ∂X when $d \geq 3$, we find using (46) that

$$\int_{\Gamma_+\times\Gamma_-}\alpha_2(x, v, y, w)\phi_\varepsilon(x, v)g_{\varepsilon_1}(y, w)d\mu(x)dvd\mu(y)dw \xrightarrow{\varepsilon,\varepsilon_1\to 0} 0,$$

i.e., the multiple scattering contribution is asymptotically negligible with such measurements. Now, choosing $\phi_\varepsilon(x,v)$ properly normalized and supported in the ε_2-vicinity of (x_0,v_0) (for $\varepsilon \ll \varepsilon_2 \ll 1$), we find that

$$\langle \phi_\varepsilon, \mathcal{A} g_{\varepsilon_1} \rangle \xrightarrow{\varepsilon,\varepsilon_1,\varepsilon_2\to 0} E(y_0,z_0)E(z_0,x_0)k(z_0,w_0,v_0),$$

at each point of continuity of $k(z_0,w_0,v_0)$, where $E(x,y)$ is defined in (48). Since $\sigma(x)$ and hence $E(x,y)$ are known from knowledge of $\mathcal{A}$, then so is $k(z_0,w_0,v_0)$ at each point of continuity in $X\times V\times V$ thanks to the above formula. For more general admissible k as defined in (5), other test functions need be constructed. Such a construction was performed, using a different strategy from the one exposed here, in [44].

The same type of results hold in the time dependent setting in dimension $d\geq 2$. We consider as before a boundary condition $g_{\varepsilon_1}(s,y,w)$ supported in the ε_1-vicinity of $(0,y_0,w_0)$. The time it takes for particles scattering exactly once to travel from y_0 to z_0 and then from z_0 to y_0 is $\tau_0 = |x_0-z_0|+|z_0-y_0|$. Particles scattering more than twice will arrive at later times $t>\tau_0$. We thus define the function $\psi_\varepsilon(t,x,v)$ as the product of the test function $\phi_\varepsilon(x,v)$ defined in the time independent setting and a function $\varphi_\varepsilon(t)$ equal to 1 in the ε-vicinity of the travel time τ_0 and equal to 0 in a larger ε-vicinity of τ_0. We then find that

$$\int_{(0,\mathrm{T})\times\Gamma_+\times(0,\eta)\times\Gamma_-} \alpha_m(t-s,x,v,y,w)\psi_\varepsilon(t,x,v)g_{\varepsilon_1}(s,y,w)d\mu(x)dvdtd\mu(y)dwds \to 0, \quad m=0,2,$$

so that

$$\begin{aligned}\langle \psi_\varepsilon, \mathcal{A} g_\varepsilon\rangle \quad &:= \quad \int_{(0,\mathrm{T})\times\Gamma_+\times(0,\eta)\times\Gamma_-} \alpha(t-s,x,v,y,w)\psi_\varepsilon(t,x,v)g_{\varepsilon_1}(s,y,w)d\mu(x)dvdtd\mu(y)dwds \\ &\xrightarrow{\varepsilon,\varepsilon_1\to 0} \quad E(y_0,z_0)E(z_0,x_0)k(z_0,w_0,v_0). \end{aligned} \qquad (49)$$

The result now holds in all dimension $d\geq 2$ at all points of continuity of k.

Note that the above reconstructions of $\sigma(x)$ and $k(x,v',v)$ are explicit. The above formal results are proved in [43] for the time dependent case and in [44] for the time independent case using the decomposition of the Schwartz kernel of the albedo operators into singular components but with a different choice for the test functions $g_{\varepsilon_1}\otimes\phi_\varepsilon$. The proofs in [43, 44] also extend to arbitrary admissible optical parameters in the sense of (5). The test functions presented above follow more closely the definitions in [19] for the time dependent case and [17] for the time independent case, to which we refer the reader for additional details. We may state the uniqueness results obtained in [43, 44] as follows:

Theorem 3.1 ([43, 44]) *Let (σ,k) and $(\tilde\sigma,\tilde k)$ be two admissible pairs of optical parameters associated with the same albedo operator $\mathcal{A}$ and such that σ and $\tilde\sigma$ are independent of the velocity variable. Then $\sigma=\tilde\sigma$ in dimension $d\geq 2$ both in the time dependent and time independent settings. Moreover, $k=\tilde k$ in dimension $d\geq 2$ in the time dependent setting and in dimension $d\geq 3$ in the time independent setting.*

3.4. Angularly dependent absorption and non-uniqueness results

We have seen that $\mathcal{A}$ uniquely characterizes $\sigma(x)$ independent of v. When $\sigma = \sigma(x, v)$ and $k = 0$, then only $\int_{\mathbb{R}} \sigma(x + tv, v)dt$ may be reconstructed [44] and this is not enough to uniquely characterize $\sigma(x, v)$ for $x \in X$ and $v \in \mathbb{S}^{d-1}$. The presence of scattering when $k \not\equiv 0$ might help us reconstruct more information about σ. A recent result in [131] shows that this is sometimes but now always the case and rather that $\sigma(x, v)$ and $k(x, v', v)$ are uniquely characterized up to an arbitrary gauge transformation of the form

$$\tilde{\sigma} = \sigma - v \cdot \nabla_x \ln \phi(x, v), \qquad \tilde{k}(x, v', v) = \frac{\phi(x, v)}{\phi(x, v')} k(x, v', v), \tag{50}$$

where $\phi(x, v) > 0$ is an arbitrary bounded function such that $v \cdot \nabla_x \phi$ is also bounded and $\phi(x, v) = 1$ on $\partial X \times \mathbb{S}^{d-1}$. A simple calculation shows that $\mathcal{A} = \tilde{\mathcal{A}}$, where $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are the albedo operators associated with the coefficients (σ, k) and $(\tilde{\sigma}, \tilde{k})$, respectively. The striking result is that the above gauge transformation is the only obstruction to reconstructing the optical parameters:

Theorem 3.2 ([131]) *Let* (σ, k) *and* $(\tilde{\sigma}, \tilde{k})$ *be two admissible pairs such that the corresponding transport problem* (18) *is well-posed. Then* $\mathcal{A} = \tilde{\mathcal{A}}$ *if and only if there is a positive bounded function* ϕ *such that* $v \cdot \nabla_x \phi$ *is also bounded,* $\phi(x, v) = 1$ *on* $\partial X \times \mathbb{S}^{d-1}$*, and* (50) *holds.*

The above theorem is proved using the decomposition of the albedo operator (40) into singular components. It admits the uniqueness result stated in Theorem 3.1 in the stationary case as a corollary. Indeed, when σ and $\tilde{\sigma}$ are independent of v, then $\phi \equiv 1$ [131] so that $(\sigma, k) = (\tilde{\sigma}, \tilde{k})$. It also allows us to derive uniqueness results in cases where symmetries are present. For instance, we have the following result:

Corollary 3.3 ([131]) *Let* (σ, k) *and* $(\tilde{\sigma}, \tilde{k})$ *be two admissible pairs such that the corresponding transport problem* (18) *is well-posed and* $\mathcal{A} = \tilde{\mathcal{A}}$*. Assume moreover that* k *and* $\tilde{k}$ *are positive and such that* $k(x, v, v') = k(x, v', v)$ *and* $\tilde{k}(x, v, v') = \tilde{k}(x, v', v)$*.*

Then $k = \tilde{k}$ *and* $a = \tilde{a} + v \cdot \nabla w(x)$ *for some function* w *vanishing on* ∂X*.*

A corollary of the corollary is that when $a(x, v) = a(x, -v)$ and $\tilde{a}(x, v) = \tilde{a}(x, -v)$, then $v \cdot \nabla w(x) = 0$ above and $a = \tilde{a}$. In anisotropic media with the physical reciprocity relations $\sigma(x, v) = \sigma(x, -v)$ and $k(x, v', v) = k(x, v, v')$, we thus obtain the important result that $\mathcal{A}$ uniquely determines the optical parameters (σ, k).

3.5. Angularly averaged measurements

In practice of inverse transport, full knowledge of the albedo operator is rarely available. There are two primary reasons for this. First, acquiring the full albedo operator requires many experiments in order for Γ_- to be sampled accurately. Second, it is difficult to measure the angular dependency of particles on Γ_+. Such a dependency is typically achieved by using collimators, which absorb particles away from a narrow solid angle in the vicinity of a direction w_0 of interest. However, good angular accuracy often results in very low particle counts and is therefore very much affected by noise. In practice, neither the incoming conditions on Γ_- nor the measured particles on Γ_+ are angularly resolved. We may however assume that the spatial resolution is adequate. We may also assume, although this is significantly more problematic in practice, that the temporal resolution is also adequate.

We consider the situation where the incoming boundary conditions are of the form

$$g(t,y,w) = g(t,y)\mathfrak{h}(y,w), \tag{51}$$

where the weight $\mathfrak{h}(y,w)$ is known on Γ_- and $g(t,x)$ is arbitrary in $L^1((0,\eta)\times\partial X)$. Similarly, we assume that angularly averaged measurements of the form

$$J_{\mathfrak{w}}(t,x) = \int_{V_{x,+}} u_{|\Gamma_+}(t,x,v)\mathfrak{w}(x,v)dv, \tag{52}$$

are available for some known weight $\mathfrak{w}(x,v)$ modeling how detectors located at x capture particles exiting X at x with velocity $v \in V = \mathbb{S}^{d-1}$. Here, we have defined for $x \in \partial X$:

$$V_{x,\pm} = \{v \in V;\ \pm v\cdot\nu(x) > 0\}. \tag{53}$$

A typical choice for $\mathfrak{w}(x,v)$ is $\mathfrak{w}(x,v) = \nu(x)\cdot v$ for $(x,v)\in\Gamma_+$. Then $J_{\mathfrak{w}}(t,x)$ the flux of particles leaving X at $x \in \partial X$ and time $t \in (0,\mathrm{T})$.

The resulting angularly averaged albedo operators are therefore now of the form

$$\mathcal{B} : g(t,x) \in L^1((0,\eta)\times\partial X) \mapsto \mathcal{B}g(t,x) = J_{\mathfrak{w}}(t,x) \in L^1((0,\mathrm{T})\times\partial X) \tag{54}$$

for the time dependent case and

$$\mathcal{B} : g(x) \in L^1(\partial X) \mapsto \mathcal{B}g(x) = J_{\mathfrak{w}}(x) \in L^1(\partial X) \tag{55}$$

for the time independent case. We may introduce the Schwartz kernels $\beta(t,x,y)$ and $\beta(x,y)$ in the time dependent and time independent settings, respectively, of the above albedo operators. The kernel of $\mathcal{B}$ may be related to that of $\mathcal{A}$ as follows

$$\beta(t,x,y) = \int_{V_{x,+}\times V_{y,-}} \alpha(t,x,v,y,w)\mathfrak{w}(x,v)\mathfrak{h}(y,w)dvdw \tag{56}$$

$$\beta(x,y) = \int_{V_{x,+}\times V_{y,-}} \alpha(x,v,y,w)\mathfrak{w}(x,v)\mathfrak{h}(y,w)dvdw, \tag{57}$$

in the time dependent and time independent settings, respectively.

Let us consider first the time independent setting. It is not difficult to see that the singularities in $\alpha(x,v,y,w)$ are no longer present in $\beta(x,y)$ after integration in $V_{x,+}\times V_{y,-}$ for integrable functions $\mathfrak{w}(x,v)$ and $\mathfrak{h}(y,w)$. No uniqueness results on the reconstruction of $\sigma(x)$ or $k(x,v',v)$ are available in this setting. The stability results obtained in [21] with $\mathfrak{w}(x,v) = \nu(x)\cdot v$ and $\mathfrak{h}(y,w) = |\nu(y)\cdot w|$, which will be made more explicit in the following section, show that the reconstruction of $k = k(x)$ when $\sigma(x)$ is known and small is severely ill-posed, in the sense that the error in the reconstruction of the

mth Fourier coefficient of k grows exponentially with m. This behavior is quite similar to what is observed in the reconstruction of a diffusion coefficient from Cauchy data in the inverse conductivity problem [59] or in the diffusion approximation to the transport equation (34).

The situation is more favorable in the time dependent setting. Although the spatial singularities of $\alpha(t,x,v,y,w)$ are no longer present in $\beta(t,x,y)$, the temporal singularities survive after angular averaging. Let us decompose

$$\begin{aligned}\beta(t,x,y) &= \sum_{m=0}^{2} \beta_m(t,x,y), \\ \beta_m(t,x,y) &= \int_{V_{x,+}\times V_{y,-}} \alpha_m(t,x,v,y,w)\mathfrak{w}(x,v)\mathfrak{h}(y,w)dvdw, \quad 0\le m\le 2.\end{aligned}$$

We assume that $\mathfrak{w}(x,v)$ and $\mathfrak{h}(y,w)$ are continuous functions on Γ_+ and Γ_-, respectively.

It is shown in [20] that the ballistic component $\beta_0(t,x,y)$ is more singular than the remaining components and that $\beta_1(t,x,y)$ is also more singular than $\beta_2(t,x,y)$ in an appropriate sense and that all components are supported on the domain $t\ge|x-y|$. We have $\beta_m(t,x,y)=0$ for $t<|x-y|$ and from now on provide expressions for the coefficients $\beta_m(t,x,y)$ over the domain $t\ge|x-y|$. The singularities are however of a different nature than the singularities observed with full, angularly-resolved, measurements. More precisely, the first two terms in the Schwartz kernel take the form

$$\beta_0(t,x,y) = \frac{E(x,y)}{|x-y|^{d-1}}\delta(t-|x-y|)\big[\mathfrak{w}(x,v)\mathfrak{h}(y,v)|\nu(y)\cdot v|\big]_{|v=\frac{x-y}{|x-y|}} \tag{58}$$

$$\begin{aligned}\beta_1(t,x,y) = \chi_{(0,+\infty)}(t-|x-y|)\int_{V_{x,+}} \mathfrak{w}(x,v)\frac{(t-(x-y)\cdot v)^{d-3}}{|x-y-tv|^{2d-4}} \\ \big[E(x,x-sv)E(x-sv,y)k(x-sv,w,v)\mathfrak{h}(y,w)\big]_{|w=\frac{x-sv-y}{|x-sv-y|},s=\frac{t^2-|x-y|^2}{2(t-v\cdot(x-y))}}dv.\end{aligned} \tag{59}$$

We recall that $V_{x,+}$ is defined in (53).

Let $x_0\in\partial X$ and $y_0\in\partial X$ and let $\tau_0(x_0,y_0)=|x_0-y_0|$ the travel time between x_0 and y_0. Assume that $g_\varepsilon(s,y)$ is a function that concentrates in the vicinity of $s=0$ and $y=y_0$ (and converges to $\delta(s)\delta_{\{y_0\}}(y)$). Assume also that $\phi_\varepsilon(t,x)$ concentrates in the vicinity of $t=\tau_0(x_0,y_0)$ and $x=x_0$. Then it is shown in [20] that

$$\langle\langle\beta_m,\phi_\varepsilon\otimes g_\varepsilon\rangle\rangle \xrightarrow{\varepsilon\to 0} 0, \qquad m=1,2, \tag{60}$$

where we have defined

$$\langle\langle\beta,\phi\otimes g\rangle\rangle = \int_{\partial X\times(0,\eta)\times\partial X\times(0,\mathrm{T})} \beta(t-s,x,y)\phi(t,x)g(s,y)dtdxdsdy. \tag{61}$$

The ballistic term, however, converges to

$$\langle\langle\beta_0,\phi_\varepsilon\otimes g_\varepsilon\rangle\rangle \xrightarrow{\varepsilon\to 0} \frac{E(x_0,y_0)}{|x_0-y_0|^{d-1}}\big[\mathfrak{w}(x_0,v_0)\mathfrak{h}(y_0,v_0)|\nu(y_0)\cdot v_0|\big], \tag{62}$$

where $v_0 = \frac{x_0-y_0}{|x_0-y_0|}$. The angularly averaged measurements therefore provide $E(x,y)$ for all $x \in \partial X$ and $y \in \partial X$. This is again equivalent to knowing the X-ray transform of σ so that $\sigma = \sigma(x)$ is uniquely determined by the measurement operator $\mathcal{B}$.

The reconstruction of the scattering coefficient is significantly more difficult. Because of the angular averaging in the construction of $\mathcal{B}$, only temporal singularities survive. Yet, it is apparent from (59) that $\beta_1(t,x,y)$ is a function. Its behavior in time for $t - |x-y|$ small however allows us to distinguish this single scattering contribution from the multiple scattering component.

Let us assume that $k \in L^\infty(X \times V \times V)$ and that k vanishes in the $0 < \delta$-vicinity of an analytic boundary ∂X. Under these assumptions, it is shown in [20] that the following holds:

$$\begin{aligned} \beta_2(t,x,y) &\in L^\infty((0,\mathrm{T}) \times V \times V), \qquad d = 2 \\ \frac{1}{(t-|x-y|)^{\frac{d-1}{2}}}\beta_2(t,x,y) &\in L^\infty((0,\mathrm{T}) \times V \times V), \quad d \geq 3. \end{aligned} \tag{63}$$

In the vicinity of $t = \tau$, with $\tau := |x-y|$, single scattering is larger than the multiple scattering contributions. We also define $v = \frac{x-y}{|x-y|}$. We have [20] in dimension $d \geq 2$,

$$\begin{aligned} \beta_1(t,x,y) &= (t-\tau)^{\frac{d-3}{2}} |\mathbb{S}^{d-2}| \Big(\frac{2}{\tau}\Big)^{\frac{d-1}{2}} \mathfrak{w}(x,v)\mathfrak{h}(y,v)|\nu(y)\cdot v| \\ &\quad E(x,y) \int_0^\tau \frac{k(x-sv,v,v)}{(s(\tau-s))^{\frac{d-1}{2}}} ds + o\big((t-\tau)^{\frac{d-3}{2}}\big), \end{aligned} \tag{64}$$

where $|\mathbb{S}^{d-2}|$ is the volume of the unit sphere $\mathbb{S}^{d-2}$ and $|\mathbb{S}^0| = 2$.

Comparing the asymptotic behavior of β_1 and β_2 for small values of $t-\tau$, we observe that $\beta_1(t)$ is arbitrary larger than $\beta_2(t)$ as $t \to \tau^+$. By appropriately choosing the test function $g_\varepsilon(s,y)$ to concentrate in the vicinity of $(0,y_0)$ and $\phi_\varepsilon(t,x)$ to concentrate in the vicinity of $\tau_0 + \varepsilon_2$ and x_0, and then sending ε_2 to 0, we gain knowledge of

$$R_2 k(x_0,y_0) := \int_0^{\tau_0} \frac{k(x_0-sv_0,v_0,v_0)}{(s(\tau_0-s))^{\frac{n-1}{2}}} ds. \tag{65}$$

We observe that $R_2k(x_0,y_0)$ is the weighted line integral of $k(x,v_0,v_0)$ along the line passing through x_0 and v_0. This does not provide enough information about $k(x,w,v)$ to uniquely determine it. However, in the simplified setting where $k(x,v',v) = k_0(x)f(v',v)$, where $f(v',v)$ is known in advance and normalized so that $f(v,v) = 1$, then we have that

$$R_2 k(x_0,y_0) := \int_0^{\tau_0} \frac{k_0(x_0-sv_0)}{(s(\tau_0-s))^{\frac{n-1}{2}}} ds, \tag{66}$$

is a weighted X-ray transform of k_0 along the segment (x_0,y_0). Since $k_0(x)$ is assumed to vanish in the vicinity of ∂X, the weight $(s(\tau_0-s))^{-\frac{n-1}{2}}$ is real-analytic on an open set including the support of $k(x)$. A recent result in [56] shows that knowledge of the weighted X-ray transform (for all $x_0 \in \partial X$ and $y_0 \in \partial X$) uniquely determines $k(x)$ under appropriate analyticity assumptions on the weight; see Theorem 7.2 below. We summarize the results as follows.

Theorem 3.4 ([20]) *Let (σ, k) and $(\tilde{\sigma}, \tilde{k})$ be admissible functions corresponding to the same time dependent, angularly averaged measurement operator $\mathcal{B}$. Let us assume that σ and $\tilde{\sigma}$ are independent of the angular variable v and that $k(x, v', v) = k_0(x) f(v', v)$ and $\tilde{k}(x, v', v) = \tilde{k}_0(x) f(v', v)$ for a known function $f(v', v)$. We also assume that $k_0(x)$ and $\tilde{k}_0(x)$ are continuous and supported away from the boundary ∂X, which we assume to be analytic.*

Then in all dimensions $d \geq 2$, $\sigma(x) = \tilde{\sigma}(x)$ and $k_0(x) = \tilde{k}_0(x)$.

The uniqueness result for the reconstruction of $k_0(x)$ is based on the invertibility of the weighted X-ray transform (66). When X is an open sphere in $\mathbb{R}^d$, it turns out that the weighted X-ray transform is in fact a classical X-ray transform, which admits explicit inversion formulas; see e.g. [101]. Up to rescaling and translation, we may assume that $X = B_d(0, 1)$, the unit sphere in $\mathbb{R}^d$ centered at 0 and of radius 1. Let us then define the X-ray transform of $f(x)$ for $(x_0, y_0) \in (\partial X)^2$ and $f \in L^2(X)$ extended by 0 outside of X:

$$R_0[f](x_0, y_0) = \int_{\mathbb{R}} f\Big(x_0 + t\frac{y_0 - x_0}{|y_0 - x_0|}\Big) dt. \qquad (67)$$

Then we have the result:

Theorem 3.5 ([20]) *Assume that $X = B_d(0, 1)$. Then we have*

$$R_2 k_0(x_0, y_0) = R_0[\rho k_0](x_0, y_0), \qquad \textit{for } (x_0, y_0) \in (\partial X)^2,$$

where $\rho(x) = \dfrac{1}{(1 - |x|^2)^{\frac{d-1}{2}}}$ for $x \in X$.

This shows that the reconstruction of $k_0(x)$ is quite straightforward in the geometry where X is a ball (which requires one to place the source terms and the detectors around a sphere).

4. Stability in inverse transport

The results stated in the preceding section show which parameters may be uniquely reconstructed from the singularities of the albedo operators $\mathcal{A}$ or $\mathcal{B}$. In this section, we consider the problem of stability estimates, which addresses the question of the errors committed in the reconstruction of the optical parameters based on a given error in the measurements. A typical estimate assumes the existence of two types of measurements $\mathcal{A}$ and $\tilde{\mathcal{A}}$, say, corresponding to the optical parameters (σ, k) and $(\tilde{\sigma}, \tilde{k})$, respectively. The question is to bound the errors $\sigma - \tilde{\sigma}$ and $k - \tilde{k}$ as a function of $\mathcal{A} - \tilde{\mathcal{A}}$.

We first consider the case of full measurements and look at the stability of the reconstruction of $\sigma(x)$ and $k(x, v', v)$ from knowledge of $\mathcal{A}$, first in the time dependent setting and second in the time independent setting. We next consider the stability of the reconstruction in the two dimensional setting under additional smallness assumptions on scattering.

We then consider reconstructions from angularly averaged measurements, first in the time independent setting assuming angularly resolved sources and then in the time dependent setting assuming isotropic sources.

4.1. Time dependent angularly resolved measurements

We consider here X a bounded open convex subset of $\mathbb{R}^d$, $d \geq 2$ with C^1 boundary and assume that $\sigma \in C^0(X)$ and $k \in C^0(X \times V \times V)$. We assume that $(\tilde{\sigma}, \tilde{k})$ is another admissible pair of optical parameters satisfying the same assumptions. Let $\mathcal{A}$ and $\tilde{\mathcal{A}}$ be the associated albedo operators in the time dependent setting. We want to find error estimates for $\sigma - \tilde{\sigma}$ and $k - \tilde{k}$ in terms of $\mathcal{A} - \tilde{\mathcal{A}}$. In order to do so, we revisit the decomposition of the albedo operator and follow a strategy first proposed in [141]. We follow the presentation in [19].

The strategy that allows one to obtain stability estimates is the following. The difference of albedo operators $\mathcal{A} - \tilde{\mathcal{A}}$ is a bounded operator from incoming radiation bounded in L^1 to outgoing radiations bounded in L^1. We thus construct sequences of probing source terms on $(0, \mathrm{T}) \times \Gamma_-$ (see $\phi_{\varepsilon_1,\varepsilon_2}(t, x, v)$ in (68) below) that are uniformly bounded in the L^1 sense and whose support is increasingly concentrated in the vicinity of time $t = 0$ and point $(x_0, v_0) \in \Gamma_-$. We then construct uniformly bounded test functions on $(0, \mathrm{T}) \times \Gamma_+$ (see $\psi(t, x, v)$ (68) below) whose support converges either to the support of the outgoing ballistic part in order to obtain stability estimates for the attenuation coefficient or to the support of the exiting single scattering part in order to obtain stability estimates for the scattering coefficient. More precisely, the constructions are performed as follows.

Let (x_0, v_0) be a fixed point in Γ_-. Let $\varepsilon_1 > 0$ and $\eta > \varepsilon_2 > 0$ and $0 \leq f_{\varepsilon_1} \in C^1(\Gamma_-)$ and $0 \leq g_{\varepsilon_2} \in C^\infty(\mathbb{R})$ be such that f_{ε_1} is supported in the ε_1 vicinity of (x_0, v_0) (in the sense that $f(x, v) = 0$ for $(x, v) \in \Gamma_-$ such that $|x - x_0| + |v - v_0| > \varepsilon_1$), $g_{\varepsilon_2}(t)$ is supported in the ε_2 vicinity of 0 (in the sense that $g_{\varepsilon_2}(t) = 0$ for $t > \varepsilon_2$), and both f_{ε_1} and $g_{\varepsilon_2}(t)$ are normalized so that $\int_{\Gamma_-} f_{\varepsilon_1} d\xi = 1$ and $\int_0^\eta g_{\varepsilon_2}(t)dt = 1$.

Define now the incoming source term $\phi_{\varepsilon_1,\varepsilon_2}(t, x, v) = g_{\varepsilon_2}(t) f_{\varepsilon_1}(x, v)$. The above normalizations show that $|\nu(x) \cdot v| \phi_{\varepsilon_1,\varepsilon_2}(t, x, v)$ is a smooth approximation of the delta function on $\mathbb{R} \times \Gamma_-$ which normalizes to one after integration on $(0, \eta) \times \Gamma_-$.

Let now ψ be a compactly support continuous function on $(0, \mathrm{T}) \times \Gamma_+$ such that $\|\psi\|_\infty \leq 1$. Then we obtain that

$$\begin{aligned} \Big| \int_{(0,\mathrm{T})\times\Gamma_+} \psi(t,x,v)\big((\mathcal{A}-\tilde{\mathcal{A}})\phi_{\varepsilon_1,\varepsilon_2}\big)(t,x,v)dtd\xi(x,v) \Big| &\leq \|(\mathcal{A}-\tilde{\mathcal{A}})\phi_{\varepsilon_1,\varepsilon_2}\|_{L^1((0,\mathrm{T}),L^1(\Gamma_+,d\xi))} \\ &\leq \|(\mathcal{A}-\tilde{\mathcal{A}})\|_{\mathcal{L}(L^1)}, \end{aligned} \qquad (68)$$

where we have defined $\|\cdot\|_{\mathcal{L}(L^1)} = \|\cdot\|_{\mathcal{L}(L^1((0,\eta),L^1(\Gamma_-,d\xi)),L^1((0,\mathrm{T}),L^1(\Gamma_+,d\xi)))}$.

Using the decomposition of the albedo operator (40), we introduce

$$I_m(\psi, \varepsilon_1, \varepsilon_2) = \int_{(0,\mathrm{T})\times\Gamma_+} \psi(t,x,v)\big((\mathcal{A}_m - \tilde{\mathcal{A}}_m)\phi_{\varepsilon_1,\varepsilon_2}\big)(t,x,v)dtd\xi(x,v), \qquad m = 0, 1, 2.$$

We may thus recast (68) as

$$\begin{array}{rcl} |I_0(\psi,\varepsilon_1,\varepsilon_2)| &\leq& \|(\mathcal{A}-\tilde{\mathcal{A}})\|_{\mathcal{L}(L^1)} + |I_1(\psi,\varepsilon_1,\varepsilon_2)| + |I_2(\psi,\varepsilon_1,\varepsilon_2)| \\ |I_1(\psi,\varepsilon_1,\varepsilon_2)| &\leq& \|(\mathcal{A}-\tilde{\mathcal{A}})\|_{\mathcal{L}(L^1)} + |I_0(\psi,\varepsilon_1,\varepsilon_2)| + |I_2(\psi,\varepsilon_1,\varepsilon_2)|. \end{array} \tag{69}$$

The above upper bound is independent of the choice of the test function ψ such that $\|\psi\|_\infty \leq 1$ and of the values of ε_m, $m = 1, 2$. Let us define $y_0 = x_0 + \tau_+(x_0, v_0)v_0$ such that $(y_0, v_0) \in \Gamma_+$. We first send ε_m, $m = 1, 2$ to 0 and obtain [19] that when $\mathrm{T} > \mathrm{diam}(X)$, we have

$$\lim_{\varepsilon_2\to 0^+}\lim_{\varepsilon_1\to 0^+} I_0(\psi,\varepsilon_1,\varepsilon_2) \;=\; \psi\big(\tau_+(x_0,v_0),y_0,v_0\big)\Big(E(x_0,y_0)-\tilde{E}(x_0,y_0)\Big), \tag{70}$$

where $E(x, y)$ is defined in (48) and $\tilde{E}$ is defined similarly with σ replaced by $\tilde{\sigma}$.

When $\mathrm{T} > 2\mathrm{diam}(X)$, then we find [19] that

$$\lim_{\varepsilon_2\to 0^+}\lim_{\varepsilon_1\to 0^+} I_1(\psi,\varepsilon_1,\varepsilon_2) \;=\; I_1(\psi), \tag{71}$$

where

$$\begin{array}{rl} I_1(\psi) = & \displaystyle\int_V \int_0^{\tau_+(x_0,v_0)} \psi(s+\tau_+(x_0+sv_0,v_0), x_0+sv_0+\tau_+(x+sv_0,v)v, v) \\ & \qquad \times \big(E_+k - \tilde{E}_+\tilde{k}\big)(x_0+sv_0,v_0,v)\,ds\,dv \end{array} \tag{72}$$

and where we have defined

$$E_+(x,v,w) = \exp\Big(-\int_0^{\tau_-(x,v)}\sigma(x-sv)ds - \int_0^{\tau_+(x,w)}\sigma(x+sw)ds\Big), \tag{73}$$

the total attenuation factor on the broken line $(x - \tau_-(x, v)v, x, x + \tau_+(x, w)w)$.

Let $\psi(t, x, v)$ be extended by 0 on $(0, \mathrm{T}) \times \Gamma_-$ so that it is now defined on $(0, \mathrm{T}) \times X \times V$. A similar estimate to (46) for the Schwartz kernel of $\mathcal{K}^2$ shows that

$$\big|I_2(\psi,\varepsilon_1,\varepsilon_2)\big| \leq C\int_0^T\int_V \Big(\int_{\partial X}|\psi(t,x,v)|^{p'}dx\Big)^{\frac{1}{p'}}dv\,dt, \tag{74}$$

for all $p' > d$. In other words, $I_2 \to 0$ as the support of ψ on $(0, \mathrm{T}) \times \Gamma_-$ tends to 0. This shows that multiple scattering contributions are negligible on the measurements when the support of the bounded function ψ tends to 0, and hence are also negligible on the reconstruction of the optical parameters using (69).

Two sequences of test functions $\psi := \psi_\lambda$ are then considered. In the first sequence, we choose ψ_λ to have a small support concentrated in the vicinity of $(y_0, v_0) \in \Gamma_+$. In such a situation, the explicit expression for I_1 allows us to show that $I_1(\psi_\lambda) \to 0$ as $\lambda \to 0$. The first inequality in (69) then provides a stability estimate for the total attenuation coefficient between x_0 and y_0.

The second sequence of test functions is more involved and is constructed as follows. Let w_0 be fixed in V. We have already mentioned that the single scattering contribution for a fixed (x_0, v_0) source term emitting at time 0 and a fixed direction of outgoing particles $w_0 \neq v_0$ is concentrated on a curve $s \mapsto \gamma(s, x_0, v_0, w_0) = x_0 + sv_0 + \tau_+(x_0 + sv_0)w_0$. Moreover, such particles exit the domain at time $s + \tau_+(x_0 + sv_0)$. The corresponding contribution in $I_1(\psi)$ involves an integration along the curve $\gamma(s)$ of a term proportional to $(E_+k - \tilde{E}_+\tilde{k})(x_0 + sv_0, v_0, w_0)$. We therefore choose the sequence of functions ψ_λ such that they are concentrated in the vicinity of the curve $\gamma(s)$ on Γ_+, that they concentrate in the vicinity of the time $s + \tau_+(x_0 + sv_0)$ along that curve, and that they approximately take the value $\mathrm{sign}(E_+k - \tilde{E}_+\tilde{k})(x_0 + sv_0, v_0, w_0)$ along that curve. Such functions are indeed bounded uniformly and have small support on $(0, \mathrm{T}) \times \Gamma_+$ in all dimension $d \geq 2$. Since $v_0 \neq w_0$, we verify that $I_0(\psi_\lambda) \to 0$ for such a sequence of functions. Their construction is detailed in [17, 19]. This allows us to state the following result.

Theorem 4.1 ([19]) *Assume that $\sigma(x)$ and $k(x, v', v)$ are continuous on $\bar{X}$ and $\bar{X} \times V \times V$, respectively and that $(\tilde{\sigma}, \tilde{k})$ satisfy the same hypotheses. Let $(x_0, v_0) \in \Gamma_-$ and $y_0 = x_0 + \tau_+(x_0, v_0)v_0$. Then we have for $\mathrm{T} > 2\mathrm{diam}(X)$ the following estimates*

$$\begin{aligned} \big|E(x_0, y_0) - \tilde{E}(x_0, y_0)\big| &\leq \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)} \\ \int_V \int_0^{\tau_+(x_0,v_0)} \big|E_+k - \tilde{E}_+\tilde{k}\big|(x_0 + sv_0, v_0, v)ds dv &\leq \|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}. \end{aligned} \tag{75}$$

We obtain a stability estimate for the X-ray transform of σ and for the scattering coefficient k weighted by the total absorption along each broken geodesic taken by particles that have scattered only once. Such are the terms that appear in the decomposition of the albedo operator. In order to obtain direct stability estimates on $\sigma(x)$ and $k(x, v', v)$, additional regularity assumptions on σ are necessary since the inverse X-ray transform is an unbounded operation. Before addressing this issue, we consider the quite similar case of time independent measurements.

4.2. Time independent angularly resolved measurements when $d \geq 3$

The same stability estimates as before may be obtained in the setting of time independent measurements in dimension $d \geq 3$. We follow the presentation in [17, 141]. For other stability results in this context, we refer the reader to [116, 117].

Our hypotheses of regularity on the optical parameters are the same as in the time dependent setting. The construction of the incoming source $\phi_\varepsilon(x, v)$ is the same as that of f_{ε_1} above: $\phi_\varepsilon \in C^1(\Gamma_-)$ is supported in the ε_1 vicinity of (x_0, v_0) and normalized so that $\int_{\Gamma_-} \phi_\varepsilon d\xi = 1$. Let ψ be a compactly support continuous function, which models the array of detectors, on Γ_+ such that $\|\psi\|_\infty \leq 1$. Then

$$\Big|\int_{\Gamma_+} \psi(x, v)\big((\mathcal{A} - \tilde{\mathcal{A}})\phi_\varepsilon\big)(x, v) d\xi(x, v)\Big| \leq \|(\mathcal{A} - \tilde{\mathcal{A}})\|_{\mathcal{L}(L^1)}, \tag{76}$$

where now $\|\cdot\|_{\mathcal{L}(L^1)} = \|\cdot\|_{\mathcal{L}(L^1(\Gamma_-, d\xi), L^1(\Gamma_+, d\xi))}$. We still introduce

$$I_m(\psi, \varepsilon) = \int_{\Gamma_+} \psi(x, v)\big((\mathcal{A}_m - \tilde{\mathcal{A}}_m)\phi_\varepsilon\big)(x, v) d\xi(x, v), \qquad m = 0, 1, 2,$$

and obtain that

$$\begin{aligned} \lim_{\varepsilon\to 0+} I_0(\psi,\varepsilon) &= \psi(y_0,v_0)\Big(E(x_0,y_0)-\tilde{E}(x_0,y_0)\Big) \\ \lim_{\varepsilon\to 0+} I_1(\psi,\varepsilon) &= \int_V\int_0^{\tau_+(x_0,v_0)} \psi(x(s)+\tau_+(x(s),v)v,v)\big(E_+k-\tilde{E}_+\tilde{k}\big)(x(s),v_0,v)dsdv \end{aligned} \tag{77}$$

where we have introduced $x(s)=x_0+sv_0$.

With $\psi(t,x,v)$ extended by 0 on $(0,\mathrm{T})\times\Gamma_-$ as before, the estimate (46) allows us to show that

$$\big|I_2(\psi,\varepsilon)\big| \le C\int_V\Big(\int_{\partial X}|\psi(t,x,v)|^{p'}dx\Big)^{\frac{1}{p'}}dv, \qquad p'>d. \tag{78}$$

Multiple scattering is therefore still negligible when the support of $\psi:=\psi_\lambda$ tends to 0 when $\lambda\to 0$.

The first sequence of functions ψ_λ is chosen as in the time dependent setting: we choose ψ_λ to have a small support concentrated in the vicinity of $(y_0,v_0)\in\Gamma_+$. Then the single scattering contribution $I_1(\psi_\lambda)\to 0$ as $\lambda\to 0$ as before. The second sequence of test functions is also similar to the situation in the time dependent setting, except that no concentration in the time variable is possible. For w_0 fixed in V, we choose the sequence of functions ψ_λ such that they are concentrated in the vicinity of the curve $\gamma(s)$ on Γ_+ and that they approximately take the value $\mathrm{sign}(E_+k-\tilde{E}_+\tilde{k})(x_0+sv_0,v_0,w_0)$ along that curve. Since $v_0\neq w_0$, we verify that $I_0(\psi_\lambda)\to 0$ for such sequence of functions; see [17]. Now, the function ψ_λ has a small support only in dimension $d\ge 3$. Indeed, in dimension $d=2$, the curve γ has the same dimensionality as the boundary $\partial\Omega$. When $d=2$, multiple scattering may no longer be separated from single scattering by using the singular structure of the albedo operator $\mathcal{A}$. This allows us to state the result:

Theorem 4.2 ([17]) *Assume that $\sigma(x)$ and $k(x,v',v)$ are continuous on $\bar{X}$ and $\bar{X}\times V\times V$, respectively and that $(\tilde{\sigma},\tilde{k})$ satisfy the same hypotheses. Let $(x_0,v_0)\in\Gamma_-$ and $y_0=x_0+\tau_+(x_0,v_0)v_0$. Then we have for $d\ge 2$ that*

$$\big|E(x_0,y_0)-\tilde{E}(x_0,y_0)\big| \;\le\; \|\mathcal{A}-\tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}, \tag{79}$$

while in dimension $d\ge 3$, we have

$$\int_V\int_0^{\tau_+(x_0,v_0)} \big|E_+k-\tilde{E}_+\tilde{k}\big|(x_0+sv_0,v_0,v)dsdv \;\le\; \|\mathcal{A}-\tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)}. \tag{80}$$

The stability estimates are the same in the time dependent and time independent settings. The only difference is that we have no stability result on the reconstruction of the scattering coefficient in dimension $d=2$ in the time independent setting.

4.3. Regularity assumptions and stability

The stability obtained above for the X-ray transform of the absorption coefficient is not sufficient to obtain any stability of σ itself without a priori regularity assumptions on σ.

This results from the well known fact that the X-ray transform is a smoothing (compact) operator so that the inverse X-ray transform is an unbounded operator. Let us assume that σ belongs to some space $H^s(\mathbb{R}^d)$ for s sufficiently large and that σ_p defined in (5) is bounded. More precisely, define

$$\mathcal{M} = \big\{(\sigma, k) \in C^0(\bar{X})\times C^0(\bar{X}\times V\times V) | \sigma \in H^{\frac{d}{2}+r}(X), \|\sigma\|_{H^{\frac{d}{2}+r}(X)} + \|\sigma_p\|_\infty \leq M\big\}, \quad (81)$$

for some $r > 0$ and $M > 0$. Then using Theorem 4.1 in the time dependent setting and Theorem 4.2 in the time independent setting, we have the following result.

Theorem 4.3 ([17, 19]) *Let $d \geq 2$ and assume that $(\sigma, k) \in \mathcal{M}$ and that $(\tilde{\sigma}, \tilde{k}) \in \mathcal{M}$. Then the following is valid:*

When T $>$ diam(X) *in the time dependent setting, and without additional hypotheses in the time independent setting, we have*

$$\|\sigma - \tilde{\sigma}\|_{H^s(X)} \leq C\|\mathcal{A} - \tilde{\mathcal{A}}\|^{\kappa}_{\mathcal{L}(L^1)}, \quad (82)$$

where $-\frac{1}{2} \leq s < \frac{d}{2} + r$ *and* $\kappa = \frac{d+2(r-s)}{d+1+2r}$.

When T $>$ 2diam(X) *in the time dependent setting and when* $d \geq 3$ *in the time independent setting, we have*

$$\|k - \tilde{k}\|_{L^1(X\times V\times V)} \leq \|\mathcal{A} - \tilde{\mathcal{A}}\|^{\kappa'}_{\mathcal{L}(L^1)}\big(1 + \|\mathcal{A} - \tilde{\mathcal{A}}\|^{1-\kappa'}_{\mathcal{L}(L^1)}\big), \quad (83)$$

where $\kappa' = \frac{2(r-r')}{d+1+2r}$ *and* $0 < r' < r$.

Such estimates show that under additional regularization assumptions on σ, we have explicit stability expression of Hölder type on σ and k. The first stability result (82) was first established in [141].

4.4. Time independent angularly resolved measurements when $d = 2$

In dimension $d = 2$, the single scattering component is a function that cannot be distinguished from higher orders of scattering solely based on its singular structure. What makes the recovery of k possible is that multiple scattering is at least quadratic in k and therefore smaller than the single scattering contribution when k is sufficiently small. We follow the presentation in [132] and refer the reader to [115, 134] for other inversion results in two dimensions of space.

The uniqueness of the reconstruction of the absorption coefficient and its stability was obtained in Theorem 4.3 above. In the two-dimensional context, we may rewrite the scattering component α_1 as [132]

$$\alpha_1(x, v, y, w) = \chi(x, v, y, w)\frac{E(x, z)k(z, v, w)E(z, y)|\nu(y)\cdot w|}{|v \cdot w|}, \quad (84)$$

where $\chi(x, v, y, w) = 1$ when $z := z(x, v, y, w) = x - tv = y + sw \in X$ for some $s > 0$ and $t > 0$ (i.e., when the half lines $\{x - tv, t > 0\}$ and $\{y + sw, s > 0\}$ meet at $z \in X$)

and $\chi(x,v,y,w)=0$ otherwise. We thus obtain an explicit expression for $k(z,v,w)$ from the single scattering kernel.

In a similar spirit to the bound in (43), it is shown in [132] that multiple scattering is bounded as follows:

$$0 \le \alpha_2(x,v,y,w) \le C\|k\|_{L^\infty}^2 \Big(1+\log\frac{1}{|v\cdot w|}\Big). \tag{85}$$

Let us now assume that $\alpha=\tilde\alpha$. Since $\alpha_0=\tilde\alpha_0$, hence $\sigma=\tilde\sigma$, we have

$$\alpha_2-\tilde\alpha_2=\tilde\alpha_1-\alpha_1=\chi(x,v,y,w)\frac{E(x,z)(\tilde k-k)(z,v,w)E(z,y)|\nu(y)\cdot w|}{|v\cdot w|},$$

from which we deduce that

$$|k-\tilde k|(z,v,w) \le C \sup_{\Gamma_+\times\Gamma_-} |v\cdot w||\alpha_2-\tilde\alpha_2|. \tag{86}$$

The expansion (40) may be recast as $\alpha_1=\mathcal{K}\alpha_0$ and $\alpha_2=(I-\mathcal{K})^{-1}\mathcal{K}^2\alpha_0$ so that

$$\begin{aligned}\alpha_2-\tilde\alpha_2 &= (I-\mathcal{K})^{-1}\mathcal{K}^2\alpha_0-(I-\tilde{\mathcal{K}})^{-1}\tilde{\mathcal{K}}^2\alpha_0\\ &= (I-\mathcal{K})^{-1}\big(\mathcal{K}(\mathcal{K}-\tilde{\mathcal{K}})+(\mathcal{K}-\tilde{\mathcal{K}})\tilde{\mathcal{K}}\big)\alpha_0+(I-\tilde{\mathcal{K}})^{-1}(\mathcal{K}-\tilde{\mathcal{K}})(I-\mathcal{K})^{-1}\tilde{\mathcal{K}}^2\alpha_0,\end{aligned}$$

from which it is possible to show that

$$|\alpha_2-\tilde\alpha_2| \le C\varepsilon\|k-\tilde k\|_{L^\infty}\Big(1+\log\frac{1}{|v\cdot w|}\Big), \qquad \varepsilon=\sup(\|k\|_{L^\infty},\|\tilde k\|_{L^\infty}). \tag{87}$$

Since $x(1-\ln x)$ is bounded on $[0,1]$, we deduce from (86)-(87) that

$$\|k-\tilde k\|_{L^\infty} \le C\varepsilon\|k-\tilde k\|_{L^\infty} \tag{88}$$

so that $k-\tilde k=0$ when ε is sufficiently small.

A similar analysis allows us to obtain stability estimates for the attenuation and scattering coefficients. Following [132], let us define

$$a(x,v)=\int_0^{\tau_+(x,v)}\sigma(x+tv)dt, \qquad (x,v)\in\Gamma_-$$

and

$$\delta_1=\|a-\tilde a\|_{H^1(\Gamma_-)}, \qquad \delta_2=\Big\|\big((\alpha_1+\alpha_2)-(\tilde\alpha_1+\tilde\alpha_2)\big)\frac{|v\cdot w|}{|\nu(y)\cdot w|}\Big\|_{L^\infty(\Gamma_-\times\Gamma_+)}. \tag{89}$$

Then we have the following uniqueness and stability result:

Theorem 4.4 ([132]) *Define*

$$\mathcal{V}=\Big\{(\sigma(x),k(x,v,w))\in H^s(X)\times C(X\times V\times V);\ \|\sigma\|_{H^s}\le\Sigma,\ \|k\|_{L^\infty}\le\varepsilon\Big\}.$$

Then for $s>1$ and $\Sigma>0$, there is $\varepsilon=\varepsilon(s,\Sigma)>0$ such that for any $(\sigma,k)\in\mathcal{V}$ and $(\tilde\sigma,\tilde k)\in\mathcal{V}$ and $0<\mu<1-\frac{1}{s}$, there is $C>0$ such that

$$\|\sigma-\tilde\sigma\|_{L^\infty}\le C\delta_1^{1-\frac{1}{s}-\mu}, \qquad \|k-\tilde k\|_{L^\infty}\le C(\delta_1^{1-\frac{1}{s}-\mu}+\delta_2). \tag{90}$$

The above stability estimates are of a slightly different nature than the ones presented above as the definition of δ_1 and δ_2 directly involves the kernels α and $\tilde{\alpha}$ rather than the operators $\mathcal{A}$ and $\tilde{\mathcal{A}}$. The stability results that we obtain in the following section are of a similar type. Because single scattering can no longer be separated from multiple scattering by the analysis of singularities, the operator norm in $\mathcal{L}(L^1)$ is no longer adapted. Note that the above stability results still show that σ and k may be reconstructed stably (with Hölder stability) from knowledge of the kernel of the albedo operator.

4.5. Time independent averaged measurements with angularly resolved sources

Still for time independent measurements, we now consider the case of angularly averaged measurements of the form (52) but for sources $g(x,v)$ that are *angularly resolved*. The problem was addressed in [75]. We are in an intermediate situation between the full measurement operator $\mathcal{A}$ defined in (40) and the fully averaged measurement operator $\mathcal{B}$ defined in (55), which will be treated in section 4.7 below. We define

$$\mathcal{C} : g(x,v) \in L^1(\Gamma_-) \mapsto \mathcal{C}g(x) = J_{\mathfrak{w}}(x) \in L^1(\partial X). \tag{91}$$

The kernel $\gamma(x,y,w)$ of $\mathcal{C}$ is related to that of $\mathcal{A}$ in the sense that

$$\gamma_m(x,y,w) = \int_{V_{x,+}} \alpha_m(x,v,y,w)\mathfrak{w}(x,v)dv, \qquad m = 0,1,2. \tag{92}$$

This shows that

$$\gamma_0(x,y,w) = \exp\Big(- \int_0^{\tau_-(x,v)} \sigma(x-sv,v)ds\Big)\mathfrak{w}(x,w)\delta_{\{x-\tau_-(x,v)v\}}(y),$$

while it may be seen that γ_1 and γ_2 are functions. The ballistic part of $\mathcal{C}$ is therefore more singular than the scattering part. This allows us to reconstruct the X-ray transform of σ, whence $\sigma = \sigma(x)$, uniquely from knowledge of $\mathcal{C}$. The single scattering component is however a function, like the multiple scattering component. Proofs of unique reconstruction of k then hinge on the fact that k is sufficiently small. This is of a similar nature to the reconstruction seen in the preceding section. The main difference is that the angular average in V no longer allows for point-wise estimates of k from the single scattering kernel as in (84). What we obtain instead is

$$\gamma_1(x,y,w) = \int_0^{\tau_+(y,w)} \frac{(E_+k)(y+sw,v(s),w)}{|x-y-sw|^{d-1}}|w\cdot\nu(y)|\mathfrak{w}(x,v(s))ds, \tag{93}$$

with $v(s) = \frac{x-y-sw}{|x-y-sw|}$. In other words, γ_1 provides a weighted integral of k on the line passing through y with direction w. The weight depends on $x \in \partial X$. Yet, it is apparent that $x \mapsto \gamma_1(x,y,w)$ is a smooth function, provided that ∂X is smooth, independent of the smoothness of k. In other words, the information obtained by varying $x \in \partial X$ cannot be used to reconstruct information about k in a stable manner. For a fixed

x, varying $(y, w) \in \Gamma_-$ however provides all weighted line integrals of k. An arbitrary function $k(x, v, w)$ cannot be reconstructed from line integrals and we therefore assume that $k(x, v, w) = k(x)\phi(x, v, w)$, where $\phi(x, v, w)$ is a *known* phase function.

At a fixed value of $x \in \partial X$, $\gamma_1(x, y, w)$ may then be seen as a weighted X-ray transform of $k(x)$. Injectivity for such weighted X-ray transforms was established in [56] under generic assumptions. These assumptions impose that the weight in (93) be close to real analytic. Because of the singular behavior of $|x - y - sw|^{1-d}$ as $y - sw$ is close to the boundary, we assume that $k(x)$ is supported away from ∂X. Then, provided that $\sigma(x)$ and $\phi(x, v, w)$ are analytic, or close to being analytic, then the result in [56] ensures injectivity of the weighted X-ray transform $k(\cdot) \mapsto \alpha_1(x, \cdot)$ at $x \in \partial X$ fixed. This allows us to uniquely reconstruct $k = k(x)$ from single scattering measurements.

Provided that $k(x)$ is sufficiently small, an analysis of multiple scattering similar to that performed in the preceding paragraph also leads to (88) so that $k(x)$ may be uniquely reconstructed provided that it is sufficiently small. The method is also amenable to Hölder stability estimates. Let us assume that absorption is known to simplify. Then we have:

Theorem 4.5 ([75]) *Assume that $(\sigma, \phi, \mathfrak{w})$ are known real analytic functions, that $\|k\|_{L^\infty}, \|\tilde{k}\|_{L^\infty} \le \varepsilon$ for ε sufficiently small and k and $\tilde{k}$ vanish in the $D-$vicinity of ∂X for some $D > 0$. Let $x \in \partial X$ fixed. Then there is a constant C such that*

$$\|k - \tilde{k}\|_{L^2(X)} \le C \Big\| \frac{((\alpha_1 + \alpha_2) - (\tilde{\alpha}_1 + \tilde{\alpha}_2))(x, y, w)}{|\nu(y) \cdot w|} \Big\|_{H^1(\Gamma_-)}. \qquad (94)$$

4.6. Time dependent angularly averaged measurements

We return to the problem of the reconstruction of $\sigma(x)$ and the spatial structure of scattering $k(x)$ from angularly averaged time dependent measurements. Uniqueness was established in Theorem 3.4. Under the hypotheses stated in that theorem, we want to obtain stability estimates for $\sigma - \tilde{\sigma}$ and $k - \tilde{k}$ in terms of the distribution kernel of $\mathcal{B} - \tilde{\mathcal{B}}$. Unlike the previous cases, the estimates no longer necessarily involve operator norms of $\mathcal{B} - \tilde{\mathcal{B}}$.

The stability estimate for the absorption term σ is similar to the case of angularly resolved measurements. We use the fact that a source term concentrated in the vicinity of $y_0 \in \partial X$ and time $s = 0$ has a ballistic part in the vicinity of $x \in \partial X$ concentrated in the vicinity of time $t = |y_0 - x|$. We construct a source term $g_\varepsilon(s, y)$ concentrating in the vicinity of $s = 0$ and $y = y_0$. We now construct a bounded function $\phi_\varepsilon(t, x)$ defined for all $x \in \partial X$ and with a support in time concentrated in the vicinity of the time $t = |y_0 - x|$. Because β_1 and β_2 are functions, we find [20] that (60) holds. Once ϕ_ε is properly normalized, we find that for each $y_0 \in \partial X$, we have

$$\int_{\partial X} \frac{|E - \tilde{E}|(x, y_0)}{|x - y_0|^{d-1}} \big[\mathfrak{w}(x, v_0)\mathfrak{h}(y_0, v_0)|\nu(y_0) \cdot v_0|\big]_{|v_0 = \frac{x - y_0}{|x - y_0|}} d\mu(x) \le \|\mathcal{B} - \tilde{\mathcal{B}}\|_{\mathcal{L}(L^1)}, \qquad (95)$$

where here $\mathcal{L}(L^1) = \mathcal{L}(L^1((0, \eta) \times \partial X); L^1((0, \mathrm{T}) \times \partial X))$. We assume that $\mathrm{T} > \eta + \mathrm{diam}(X)$ in order to define the test function $\phi_\varepsilon(t, x)$. This provides us with a stability estimate for the attenuation coefficient σ. The result is however weaker than for angularly resolved measurements as the above stability involves a weighted integral of the X-ray transform over ∂X.

As we have seen in (64) and (63), the non-ballistic contributions $\beta_m(t,x,y)$, $m=1,2$, to the Schwartz kernel of the albedo operator are functions. They may be separated in a stable manner only via their behavior in the vicinity of the ballistic time $t=|x-y|$. We still construct $g_\varepsilon(s,y)$ as above. In the limit $\varepsilon\to 0$, g_ε is concentrated at $s=0$ and $y=y_0$. We thus need $\phi_\varepsilon(t,x)$ to concentrate in time in the vicinity of $t=|x-y|_+$ while it is equal to 0 on the surface $t=|x-y|$. Such a function $\phi_\varepsilon(t,x)$ ensures that $\langle\langle\beta_0,\phi_\varepsilon\otimes g_\varepsilon\rangle\rangle=0$, where the duality product is defined in (61). We need to ensure that $\langle\langle\beta_1,\phi_\varepsilon\otimes g_\varepsilon\rangle\rangle=0(1)$ while $|\langle\langle\beta_2,\phi_\varepsilon\otimes g_\varepsilon\rangle\rangle|\ll 1$. The function ϕ_ε thus cannot be bounded and have arbitrary small support at the same time since β_1 is an integrable function.

The optimal choice of ϕ_ε depends on the behavior of β_1 in the vicinity of $t=|x-y|_+$. We need to ensure that $\beta_1(t,x,y)\phi_\varepsilon(t,x)$ converges to a delta function at $t=|x-y|_+$ in the time variable for fixed x,y on the boundary ∂X. Since the time of singularity $\tau:=|x-y|$ depends on both “entrance” point y and the “exit” point x, it is difficult to obtain estimates that depend on functional norms of $\mathcal{B}-\tilde{\mathcal{B}}$ of the form $\mathcal{L}(Y_1,Y_2)$. Rather, the estimates for the scattering coefficient involve estimates directly on the coefficients $\beta_m-\tilde\beta_m$. Note however that these estimates are obtained as limits of expressions of the form $\langle\langle\beta,\phi_\varepsilon\otimes g_\varepsilon\rangle\rangle$ and as such may be obtained as limits of physically realizable experiments with well-defined source term g_ε and measurement moment ϕ_ε.

We thus obtain the following result:

Theorem 4.6 ([20]) *Under the hypotheses of Theorem 3.4, we find that* (95) *above holds as a stability result for the attenuation coefficient. Under the simplifying assumptions that* $\mathfrak{w}(x,v_0)\geq\mathfrak{w}_0>0$ *and* $\mathfrak{h}(y,v)\geq\mathfrak{h}_0>0$ *and recalling the definition of the half sphere* $V_{y,-}$ *in* (53)*, we obtain that there exists a constant* C_1 *such that*

$$\int_{V_{y,-}}\big|(E-\tilde E)(y,y+\tau_+(y,w)w)\big||\nu(y)\cdot w|dw\leq C_1\|\mathcal{B}-\tilde{\mathcal{B}}\|_{\mathcal{L}(L^1)}. \tag{96}$$

For the scattering coefficient, we obtain the following dimension-dependent stability results for each fixed points $x_0,y_0\in\partial X$ *with* $v_0=\frac{x_0-y_0}{|x_0-y_0|}$,

$$\begin{aligned}&\int_0^{|x_0-y_0|}\frac{|E_+k-\tilde E_+\tilde k|(x_0-sv_0,v_0,v_0)}{s^{\frac{d-1}{2}}(|x_0-y_0|-s)^{\frac{d-1}{2}}}ds\\ \leq\;& C_2\Big\|\sum_{m=1}^2(\tau-|z-z'|)^{\frac{3-d}{2}}(\beta_m(\tau,z,z')-\tilde\beta_m(\tau,z,z'))\Big\|_{L^\infty((0,\mathrm{T})\times\partial X\times\partial X)}.\end{aligned} \tag{97}$$

The total absorption coefficient along broken geodesics E_+ *is defined in* (73)*.*

The result in (96), which is equivalent to (95), shows that a weighted average over the half sphere $V_{y,-}$ of the difference of X-ray transforms is bounded by the error in the measurements. In other words, a weighted L^1 norm of the X-ray transform of σ

is stably determined by the measurement operator in the $\mathcal{L}(L^1)$ norm. The result in (97) shows that the measurements uniquely determine scattering. However, stability is with respect to the kernel $\beta_1 + \beta_2$ and not with respect to $\mathcal{B}$ as we discussed earlier. In dimension $d = 3$, the result may be reformulated as follows. Let $\mathcal{B}_0$ be the operator with kernel β_0 and $\mathcal{B}_s = \mathcal{B} - \mathcal{B}_0$ the operator with kernel $\beta_1 + \beta_2$. Then (97) is equivalent to

$$\int_0^{|x_0-y_0|} \frac{|E_+ k - \tilde{E}_+ \tilde{k}|(x_0 - sv_0, v_0, v_0)}{s(|x_0 - y_0| - s)} ds \leq C \|\mathcal{B}_s - \tilde{\mathcal{B}}_s\|_{\mathcal{L}(L^1;L^\infty)}. \tag{98}$$

In other words, the scattering contribution maps L^1 functions into L^∞ functions, which is not true for the ballistic term. It is in this stronger norm that we can separate the single scattering contribution from the multiple scattering contribution and obtain the stability estimate (98). Because of the weight on the right-hand-side of (97), we are not able to write the latter estimate in terms of a norm for the operator $\mathcal{B}_s$ in dimension $d \neq 3$. The above estimates combined with regularity assumptions on σ and k then yield Hölder estimates similar to those obtained in Theorem 4.3; we refer the reader to [20] for precise expressions.

4.7. Time independent angularly averaged measurements

We come back to the angularly averaged, time independent, measurements defined in (55) in the simplified setting where $\mathfrak{h}(y, w) = 1$ on Γ_- and $\mathfrak{w}(x, v) = \nu(x) \cdot v$ on Γ_+. We then have the measurements

$$\begin{aligned} &\mathcal{B} : g(x) \in L^1(\partial X) \mapsto \mathcal{B}g(x) = J(x) \in L^1(\partial X) \\ &J(x) = \int_{V_{x,+}} \nu(x) \cdot v \, u_{|\Gamma_+}(x, v) dv, \end{aligned} \tag{99}$$

where u solves the transport equation (18) with $S = 0$ and $u_{|\Gamma_-}(x, v) = g(x)$. There is no proof of unique reconstruction of σ and k from knowledge of $\mathcal{B}$ in this restricted setting. The analogy with the diffusion approximation tends to indicate that only one of the coefficients may be reconstructed provided that $\sigma = \sigma(x)$ and $k = k(x)$ [7].

We summarize here results obtained in [21] and follow the presentation in [16], which consider the case of a known, sufficiently small absorption coefficient $\sigma(x)$ and an unknown, sufficiently small, scattering coefficient $k(x)$. Because $k(x)$ is small, the mapping from k to $\mathcal{B}$ may be linearized, which is equivalent to considering only the single scattering $\mathcal{B}_1$ in $\mathcal{B}$. The linearized problem may then be inverted to give a first approximation of $k(x)$. It turns out that the mapping $k \mapsto \mathcal{B}_1$ is infinitely smoothing in the sense that the inverse operator mapping $\mathcal{B}_1$ to k exponentially increases high frequency Fourier modes of k.

The reconstructions are based on the following decomposition of the measurement operator:

$$\mathcal{M}(g, f) = \langle g \otimes f, T_0 \rangle_{L^2((\partial X)^2)} + \sum_{m=1}^{\infty} \langle g \otimes f, T_m(k) \rangle_{L^2((\partial X)^2)}, \tag{100}$$

with the notation $\langle g\otimes f, T\rangle = \int_{(\partial X)^2}\bar T(x,y)g(x)f(y)d\mu(x)d\mu(y)$, where $\bar T$ is complex conjugation of T. Here, T_0 corresponds to the ballistic part of the measurements obtained by setting $k=0$ in (18). The kernels $T_m(k)$ are then multilinear of order m in $k(x)$ so that $T_1(k)$ corresponds to single scattering, $T_2(k)$ double scattering, and so on.

We recall that $E(x,y)$ is defined in (48) and define by induction

$$E(x_1,\dots,x_{n-1},x_n) = E(x_1,\dots,x_{n-1})E(x_{n-1},x_n), \tag{101}$$

the total attenuation on the broken path $[x_1,\dots,x_n]$. Then we have

$$\begin{aligned} T_0(x_0,x) &= \frac{E(x_0,x)|\nu_x\cdot v||\nu_{x_0}\cdot v|}{|x_0-x|^{d-1}}\Big|_{v=\frac{x_0-x}{|x_0-x|}}, \\ T_m(k)(x_0,x) &= \int_{X^m} k(x_1)\cdots k(x_m)\frac{E(x_0,\dots,x_m,x)}{|x_0-x_1|^{d-1}\cdots|x_m-x|^{d-1}} \\ &\quad \times|\nu_{x_0}\cdot\tfrac{x_0-x_1}{|x_0-x_1|}||\nu_x\cdot\tfrac{x_m-x}{|x_m-x|}|dx_1\dots dx_m. \end{aligned} \tag{102}$$

Note that T_0 and $T_m(k)$, taken at points x and x_0, are the measurements given source $g=\delta_{\{x_0\}}$, and weight $f=\delta_{\{x\}}$, where we recall that $\delta_{\{x\}}$ is the surface delta function such that $\int_{\partial X}\delta_{\{x\}}(y)\phi(y)d\mu(y)=\phi(x)$. In other words $\sum_{m=0}^\infty T_m(k)(x_0,x)$ is formally the Schwartz kernel of the operator $\mathcal{B}$:

$$\mathcal{B}g(x) = \int_{\partial X}\Big(T_0(x_0,x)g(x_0)+\sum_{m=1}^\infty T_m(k)(x_0,x)g(x_0)\Big)dx_0. \tag{103}$$

We refer the reader to [21] for the derivation of (100).

Because $\sigma(x)$ is known, then so is $\langle g\otimes f, T_0\rangle$ in (100). For k sufficiently small, $\mathcal{M}(g,f)-\langle g\otimes f,T_0\rangle$ is then equal to $\langle g\otimes f, T_1(k)\rangle$ up to a small term that is quadratic in k. The first objective is therefore to reconstruct k from the linearized measurements $\langle g\otimes f, T_1(k)\rangle$. This may be done explicitly when σ vanishes. Specifically, we have that

$$\langle g\otimes f, T_1(k)\rangle_{L^2((\partial X)^2)} = \langle Ag\,Af, k\rangle_{L^2(X)},$$

where the so-called half-adjoint operator A is defined as

$$Af(y) = \omega_d\int_{\partial X} f(x)E(x,y)\partial_{\nu_x}N(x,y)\,d\mu(x). \tag{104}$$

Here ω_d is the measure of the unit sphere $\mathbb{S}^{d-1}$ and $N(x,y)=N(x-y)$ is the Newton potential

$$N(x,y) := \frac{1}{c_d|x-y|^{d-2}}\ \ (d>2);\qquad \frac{1}{c_2}\log|x-y|\ \ (d=2),$$

where $c_d=(2-d)\omega_d$ and $c_2=2\pi$. Indeed, we verify that $\partial_{\nu_x}N(x,y)=\frac{\nu_x\cdot(x-y)}{\omega_d|x-y|^d}$. Let A_0 be the operator defined as A in (104) with $\sigma=0$ so that $E(x,y)\equiv 1$. We thus draw the following conclusion: $A_0f(y)$ is a *harmonic* function on D because $y\mapsto N(x,y)$ is.

Moreover, each sufficiently smooth harmonic function v on D may be constructed as $v = A_0 f_v$ for some function f_v on ∂X, which we assume is of class C^2. The implicit construction goes as follows. Let us define

$$\tilde{A}_0 f(y) = \omega_d \int_{\partial X} f(x) \partial_{\nu_x} N(x,y)\, d\mu(x),$$

the classical double layer potential for $y \in \partial X$. It is a classical result that $\frac{1}{2}I + \omega_n^{-1}\tilde{A}_0$ is an isomorphism on $L^2(\partial X)$. We may then define the operator

$$A_0^\dagger u := (\frac{1}{2}I + \omega_d^{-1}\tilde{A}_0)^{-1}(\omega_d^{-1} u|_{\partial X}), \tag{105}$$

and verify that $A_0 A_0^\dagger u = u|_X$ for all harmonic function $u \in H^{\frac{1}{2}}(X)$.

We are now ready to use the same Complex Geometrical Optics (CGO) solutions as in the Calderón problem [34]. Let $\mathbb{C}^d \ni \rho = \frac{1}{2}(\xi + i\eta)$, where $\xi, \eta \in \mathbb{R}^d$, $\xi \cdot \eta = \sum_{i=1}^d \xi_i \eta_i = 0$, and $|\xi| = |\eta|$. Then the functions $e^{i\rho\cdot x}$, and $e^{i\bar{\rho}\cdot x}$ are harmonic, and $e^{i\rho\cdot x}e^{i\bar{\rho}\cdot x} = e^{i\xi\cdot x}$. Define the boundary conditions

$$g_\xi(x) := A_0^\dagger e^{-i\rho\cdot x} \text{ and } f_\xi(x) := A_0^\dagger e^{-i\bar{\rho}\cdot x}, \quad x \in \partial X. \tag{106}$$

Then we find that

$$\langle g_\xi \otimes f_\xi, T_1^0(k)\rangle_{L^2((\partial X)^2)} = \langle A_0 g_\xi\, A_0 f_\xi, k\rangle_{L^2(X)} = \langle e^{-i(\xi,\cdot)}, k\rangle_{L^2(X)} := \hat{k}(\xi), \tag{107}$$

where $T_1^0(k)$ is defined as $T_1(k)$ with $\sigma = 0$, whence $E \equiv 1$.

In other words, we obtain an explicit reconstruction of $\hat{k}(\xi)$ from the single scattering measurements provided that absorption $\sigma \equiv 0$. We have thus two sources of error: one is from the higher scattering contributions $T_m(k)$ for $m \geq 2$, and one is from the error coming from the non-zero absorption $T_1^\sigma(k) := T_1(k) - T_1^0(k)$. The approximation $\hat{k}_l(\xi)$ (l for low-frequency) of $\hat{k}(\xi)$ obtained by this linearization algorithm is thus given by

$$\hat{k}_l(\xi) = \mathcal{M}(g_\xi, f_\xi) = \hat{k}(\xi) + \langle g_\xi \otimes f_\xi, T_1^\sigma(k)\rangle + \sum_{m=2}^\infty \langle g_\xi \otimes f_\xi, T_m(k)\rangle. \tag{108}$$

The error made by the linearization is given by

$$|\hat{k}_l(\xi) - \hat{k}(\xi)| \leq \Big(\|T_1^\sigma(k)\|_{L^2} + \sum_{m\geq 2} \|T_m(k)\|_{L^2}\Big)\|f_\xi\|_{L^2}\|g_\xi\|_{L^2}. \tag{109}$$

Under smallness assumptions on k and σ, we obtain that

$$\|T_1^\sigma(k)\|_{L^2} \leq C\|\sigma\|_\infty \|k\|_\infty, \quad \sum_{m\geq 2} \|T_m(k)\|_{L^2} \leq C\|k\|_\infty^2.$$

The bound on $\|f_\xi\|_{L^2}$ and $\|g_\xi\|_{L^2}$ is however exponentially large as ξ increases:

$$\|f_\xi\|_{L^2(\partial X)}, \|g_\xi\|_{L^2(\partial X)} \leq Ce^{\alpha|\xi|}, \qquad \alpha = \frac{1}{2}\text{diam}(X). \tag{110}$$

for some constant C independent of ξ. This shows that

$$|\hat{k}_l(\xi) - \hat{k}(\xi)| \leq C\|k\|_\infty(\|\sigma\|_\infty + \|k\|_\infty)e^{2\alpha|\xi|}. \tag{111}$$

Errors on the reconstructions of the Fourier modes of $k(x)$ grow exponentially with wavenumber ξ. The reconstruction of $k(x)$ is severely ill-posed.

Because the operator T_1^0 is highly smoothing, which is responsible for the above severe ill-posedness, we introduce the following approximate inverse. Let $\chi(\xi)$ be a compactly supported (to simplify) function in $\mathbb{R}^n$. Typically $\chi(\xi) = 1$ for $|\xi| < M$ and $\chi(\xi) = 0$ for $|\xi| > M$ if one wants to reconstruct all frequencies $|\xi| < M$ of $k(x)$. Let then $P_\chi k$ be the operator

$$P_\chi k := \int_{\mathbb{R}^d} \hat{k}(\xi)\chi(\xi)e^{i\xi\cdot x}\,\frac{d\xi}{(2\pi)^d},$$

and let $k_\chi = P_\chi k$. We define

$$T^\chi h(x) := \int_{\mathbb{R}^d} \langle g_\xi \otimes f_\xi, h\rangle_{L^2((\partial X)^2)}\chi(\xi)e^{i\xi\cdot x}\,\frac{d\xi}{(2\pi)^d}, \tag{112}$$

as the regularized inverse of T_1^0 (since $T^\chi T_1^0 = P_\chi$). We find that

$$\|T^\chi\|_{L^2\to L^\infty} \leq C\|\chi(\xi)e^{2\alpha|\xi|}\|_{L^1(\mathbb{R}^d)},$$

so that eventually,

$$\|k_\chi(x) - P_\chi k_l(x)\|_\infty \leq C\|k\|_\infty(\|\sigma\|_\infty + \|k\|_\infty)\|\chi(\xi)e^{2\alpha|\xi|}\|_{L^1(\mathbb{R}^d)}, \tag{113}$$

where $P_\chi k_l(x)$ is the regularized inverse Fourier transform of $\hat{k}_l(\xi)$ defined in (108).

We have obtained an error estimate for the low-frequency component of $k(x)$. Note however that the estimate is very large, even for small values of ξ, unless σ and k are extremely small. Such an estimate may be improved by using an iterative scheme. Once $P_\chi k_l(x)$ has been obtained, it may be used to estimate the error term in (108), which may be used to modify the measured data $\mathcal{M}(g, f)$ and use the linearized inverse one more time to obtain a better estimate of $k_\chi(x)$. More precisely, let us define iteratively:

$$\begin{aligned} k_\chi^0 &= T^\chi \sum_{m=1}^\infty T_m(k) \\ k_\chi^{\nu+1} &= T^\chi\left(\sum_{m=1}^\infty T_m(k)\right) - T^\chi\left(T_1^\sigma(k_\chi^\nu) + \sum_{m=2}^\infty T_m(k_\chi^\nu)\right). \end{aligned} \tag{114}$$

Here, $T_m(k)$ are seen as integral operators with Schwartz kernel $T_m(k)(x, y)$. Note that $k_\chi^0 = P_\chi k_l$, the linearized reconstruction. Under appropriate smallness conditions on σ and k, which are given explicitly in [21], a Picard fixed-point theorem shows that the above iterative scheme converges to k_χ^∞ and that

$$\|k_\chi - k_\chi^\infty\|_\infty \leq C\|k - k_\chi\|_\infty. \tag{115}$$

In other words, the smooth part of k_χ is well reconstructed provided that the non-smooth part $k - k_\chi$ of k is small. The result is not very satisfactory because the smallness hypotheses on k and σ depend on the norm of T^χ. The smallness constraints on k and σ become exponentially stronger as the maximal wavenumber we want to reconstruct increases. Nonetheless, the result shows that some reconstructions of optical parameters are indeed theoretically feasible from diffusion type measurements, but that problems are severely ill-posed.

We may formally write the stability results obtained in [21] as follows:

Theorem 4.7 ([21]) *Assume that $\sigma(x)$ is known, continuous, and sufficiently small in the uniform norm and that $k(x)$ is continuous, compactly supported in X, and sufficiently small in the uniform norm.*

Then $\mathcal{B}$ determines an approximation $k_l(x)$ of $k(x)$ obtained by linearization of $\mathcal{B}$ and we have in the Fourier domain that

$$|\hat{k}_l(\xi) - \hat{k}(\xi)| \le C\|k\|_\infty(\|\sigma\|_\infty + \|k\|_\infty)e^{\mathrm{diam}(X)|\xi|}. \tag{116}$$

Let $k_\chi = P_\chi k$ be a smooth approximation of k such that $\hat{k_\chi}(\xi) = 0$ for $|\xi| > M$. Then, for σ and k sufficiently small (and the smallness depends on M), $\mathcal{B}$ determines an approximation k_χ^∞ of k_χ and we have that

$$\|k_\chi - k_\chi^\infty\|_\infty \le C\|k - k_\chi\|_\infty, \tag{117}$$

for some constant that depends on M.

The linear approximation of the time independent inverse transport problem with angularly averaged measurements is of a similar nature to the inverse conductivity problem and is thus equally ill-posed. How the inverse of the linearized approximation applies to the nonlinear part of $\mathcal{B}$ is not understood and it is still unclear whether $\mathcal{B}$ uniquely determines $k(x)$. The above result however shows that the reconstruction of k from knowledge of $\mathcal{B}$ is a severely ill-posed problem, unlike the reconstruction of the optical parameters from angularly resolved measurements or time dependent measurements.

5. Approximate Stability Estimates

Let $\mathcal{F}$ be the functional mapping the optical parameters (σ, k) to $\mathcal{A} = \mathcal{F}(\sigma, k)$. Assume that $\mathcal{A}$ is the "true", noise-free, measurement, and that $\tilde{\mathcal{A}}$ is the available, noise-corrupted, measurement. Then the stability estimates seen in the last section give an upper bound for the error in the reconstruction of (σ, k) since ideally $(\tilde{\sigma}, \tilde{k})$ can be reconstructed from the available $\tilde{\mathcal{A}}$. The main drawback of such estimates is that they *assume* that the noisy measurements are in the *range* of $\mathcal{F}$. When the available measurement $\tilde{\mathcal{A}}$ is not in the range of $\mathcal{F}$, then $\|\tilde{\mathcal{A}} - \mathcal{A}\|$ for an appropriate metric on the albedo operators may still make sense, but may often provide an extremely pessimistic, thus useless, upper bound for the error in the reconstruction.

Let us assume that the measurements have a limiting scale η, for instance because the detectors have finite resolution, and that $\tilde{\mathcal{A}}$ may be seen as a smoothing approximation of identity R_η applied to $\mathcal{A}$, i.e., $\tilde{\mathcal{A}} = R_\eta \mathcal{A}$. A reconstruction based on the measurements $\tilde{\mathcal{A}}$ is feasible as we saw: the optical parameters may be explicitly reconstructed from measurements of the form $\langle \phi, \tilde{\mathcal{A}} f_\varepsilon \rangle$ in expressions such as (48) and (49). Yet, independent of $\eta \ll 1$, we easily find that $\|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)} = O(1)$ when R_η is smoothing at the scale η. It suffices for this to consider functions ϕ that concentrate on domains that are small compared to η in an angular, temporal, or spatial variable. To understand this, we may consider the following simplified example. Let $A = I$ be the identity operator and A_η be the convolution by $\eta^{-d}\phi(\frac{x}{\eta})$ for a smooth, compactly supported, function $\phi(x) \geq 0$ such that $\int_{\mathbb{R}^d} \phi(x)dx = 1$. Then A_η converges to A strongly but not uniformly and it is straightforward to obtain that $\|A - A_\eta\|_{\mathcal{L}(L^1(\mathbb{R}^d))} = 2$ independent of η. That $\|\mathcal{A} - \tilde{\mathcal{A}}\|_{\mathcal{L}(L^1)} = O(1)$ renders estimates of the type (75),(79) useless in some practical situations.

Other estimates, which we call here approximate stability estimates, need to be developed to understand the role of a smoothing operator on the reconstructions. Such estimates are still based on the decomposition of the albedo operator and are refined versions of the estimates presented thus far. The goal of such estimates is to help us understand which blurred version of the optical parameters may be stably reconstructed, and which error is made by neglecting higher orders of scattering in the reconstruction.

We follow the presentation in [18] and consider the time independent case only to simplify the presentation. We refer the reader to [18] for generalizations to the time dependent case.

We are interested in two types of noise generated by limits in the resolution of the source term and of the detectors. The source resolution is quantified by the scale $\varepsilon = (\varepsilon_1, \varepsilon_2)$, where ε_1 measures the minimal spatial extension of the source and ε_2 the minimal angular extension. In other words, the source term may be written as a function of the form $(\varepsilon_1\varepsilon_2)^{1-d}\phi(\frac{x-x_0}{\varepsilon_1})\psi(\frac{v-v_0}{\varepsilon_2})$. The detector resolution is quantified by the scale $\eta = (\eta_1, \eta_2)$, where again η_1 is related to spatial resolution and η_2 to angular resolution.

The smoothing of the detectors is quantified by a kernel $\phi_\eta \in C^1(\Gamma_+ \times \Gamma_+, \mathbb{R})$ such that

$$\phi_\eta \geq 0, \tag{118}$$

$$\text{supp}\phi_\eta \subset \{(x, v, y, w) \in \Gamma_+ \times \Gamma_+ \mid |x - y| < \eta_1 \text{ and } |v - w| < \eta_2\}, \tag{119}$$

$$\int_{\Gamma_+} \phi_\eta(x, v, y, w)d\xi(y, w) = 1 \text{ for all } (x, v) \in \Gamma_+, \tag{120}$$

$$\int_{\Gamma_+} \phi_\eta(x, v, y, w)d\xi(x, v) \leq C \text{ for all } (y, w) \in \Gamma_+, \tag{121}$$

where C is a constant. We denote by R_η the bounded operator from $L^1(\Gamma_+, d\xi)$ to $L^1(\Gamma_+, d\xi)$ defined by

$$R_\eta g(x, v) = \int_{\Gamma_+} \phi_\eta(x, v, y, w)g(y, w)d\xi(y, w), \tag{122}$$

for a.e. $(x,v) \in \Gamma_+$ and for $g \in L^1(\Gamma_+, d\xi)$. Note that R_η is a smoothing operator at the spatial scale η_1 and the angular scale η_2. The details of the optical coefficients at scales smaller than η are thus not recoverable in a stable manner.

We now model some limitations of the source term. Let $(x_0', v_0') \in \Gamma_-$. The point $(x_0', v_0') \in \Gamma_-$ models the incoming condition and is fixed in the analysis that follows. For $\varepsilon := (\varepsilon_1, \varepsilon_2) \in (0, +\infty)^2$, let $f_\varepsilon \in C_0^1(\Gamma_-)$ such that

$$\|f_\varepsilon\|_{L^1(\Gamma_-, d\xi)} = 1, \qquad f_\varepsilon \geq 0 \tag{123}$$

$$\mathrm{supp} f_\varepsilon \subset \{(x', v') \in \Gamma_- \mid |x' + \tau_+(x', v')v' - x_0' - \tau_+(x_0', v_0')v_0'| < \varepsilon_1 \text{ and } |v' - v_0'| < \varepsilon_2\}. \tag{124}$$

The condition for ε_1 is written as a constraint on Γ_+ rather than a constraint on Γ_-. Yet f_ε above is easily seen as a smooth approximation of the delta function on Γ_- at (x_0', v_0') as $\varepsilon_1 \to 0^+$ and $\varepsilon_2 \to 0^+$ and is thus an admissible incoming condition in $L^1(\Gamma_-, d\xi)$.

Now that the source term has resolution limited by ε and the measurements are convolved measurements at the scale η, we need to select measurements that capture the singularities of the albedo operator while eliminating multiple scattering as efficiently as possible. Since the source term and detector resolution is limited, the separation between different orders of scattering based on the singularities of the albedo operator is no longer feasible exactly. The role of approximate stability estimates is to show what may still be reconstructed stably and with which error. The selection is performed by means of a function ψ whose support indicates which measurements are selected or not. Such a function is different for the selection of the ballistic and the single scattering components as we shall see.

Assume that $(k, \tilde{k}) \in L^\infty(X \times V \times V)^2$. Let $\psi \in L^\infty(\Gamma_+)$ such that $\|\psi\|_{L^\infty(\Gamma_+)} \leq 1$. Then using the decomposition of the albedo operator (40) and (122), we obtain that

$$\int_{\Gamma_+} \psi(x,v) R_\eta(\mathcal{A} - \tilde{\mathcal{A}}) f_\varepsilon(x,v) d\xi(x,v) = I_0(\psi,\eta,\varepsilon) + I_1(\psi,\eta,\varepsilon) + I_2(\psi,\eta,\varepsilon), \tag{125}$$

where, thanks to the estimate on the multiple scattering contribution given in (46), we have

$$\begin{aligned} I_0(\psi,\eta,\varepsilon) &= \int_{\Gamma_+} \psi(x,v) \int_{\Gamma_+} \phi_\eta(x,v,y,w) \Big(e^{-\int_0^{\tau_-(y,w)} \sigma(y-sw,w)ds} \\ &\quad - e^{-\int_0^{\tau_-(y,w)} \tilde{\sigma}(y-sw,w)ds} \Big) f_\varepsilon(y - \tau_-(y,w)w, w) d\xi(y,w) d\xi(x,v), \end{aligned} \tag{126}$$

$$\begin{aligned} I_1(\psi,\eta,\varepsilon) &= \int_{\Gamma_+} \psi(x,v) \int_{V \times \Gamma_+} \phi_\eta(x,v,y,w) \\ &\quad \int_0^{\tau_-(y,w)} \Big((kE_+)(y - tw, w', w) - (\tilde{k}\tilde{E}_+)(y - tw, w', w) \Big) \\ &\quad f_\varepsilon(y - tw - \tau_-(y - tw, w')w', w') dt d\xi(y,w) dw' d\xi(x,v), \end{aligned} \tag{127}$$

$$I_2(\psi,\eta,\varepsilon) \leq C \int_V \Big(\int_{\substack{y \in \partial X \\ \nu(y)\cdot w > 0}} \Big| \int_{\Gamma_+} \phi_\eta(x,v,y,w) \psi(x,v) d\xi(x,v) \Big| \nu(y) \cdot w dy \Big)^{\frac{1}{p'}} dw, \tag{128}$$

and $C = C(p, X, V, \sigma, k, \tilde{\sigma}, \tilde{k})$ for $1 < p < \frac{d}{d-1}$ and $p^{-1} + p'^{-1} = 1$. Here, I_0 corresponds to the ballistic part we aim at selecting while I_1 and I_2 correspond to the single and multiple scattering components. The objective is to find some stability for I_0.

The theorems in this paper have been presented so far for $V = \mathbb{S}^{d-1}$ the unit sphere although they generalize to other velocity spaces. Because the approximate stability estimates depend on the dimension on V, we define $\dim V$ as

$$\dim V := \begin{cases} d-1, & \text{when } V := \mathbb{S}^{d-1}, \\ d, & \text{when } V \text{ is an open subset of } \mathbb{R}^d. \end{cases} \tag{129}$$

The main approximate stability result is as follows.

Theorem 5.1 ([18]) *Assume that* $(k, \tilde{k}) \in L^\infty(X \times V \times V)^2$. *Let* $1 < p < \frac{d}{d-1}$ *and let* $p' = \frac{p}{p-1} > d$. *Then the following statements are valid:*

i. There exists a constant $C_1 = C_1(X, V, p, \sigma, k, \tilde{\sigma}, \tilde{k})$ *such that*

$$|I_0(\psi, \eta, \varepsilon)| \le \|R_\eta(\mathcal{A} - \tilde{\mathcal{A}})f_\varepsilon\|_{L^1(\Gamma_+, d\xi)} + C_1(\eta_2 + \rho)^{\dim(V)}, \tag{130}$$

for $(\rho, \varepsilon_1, \varepsilon_2, \eta_1, \eta_2) \in (0, +\infty)^5$ *and for* $\psi \in L^\infty(\Gamma_+)$, $\|\psi\|_{L^\infty(\Gamma_+)} \le 1$,

$$\mathrm{supp}\psi \subset \{(x, v) \in \Gamma_+ \mid |v - v_0'| < \rho\}. \tag{131}$$

ii. There exists a constant $C_2 = C_2(X, V, p, \sigma, k, \tilde{\sigma}, \tilde{k})$ *such that*

$$\begin{aligned} |I_1(\psi, \eta, \varepsilon)| &\le \|R_\eta(\mathcal{A} - \tilde{\mathcal{A}})f_\varepsilon\|_{L^1(\Gamma_+, d\xi)} + C_2\left(\rho_1 + \eta_1 + \frac{2\mathrm{diam}(X)\check{\eta}_2}{\sqrt{1 - \check{\eta}_2}}\right)^{\frac{d-2}{p'}}, \\ \check{\eta}_2 &= \frac{2\eta_2}{\mathrm{v}_0(1 - \rho_2^2)^{\frac{1}{2}}}, \end{aligned} \tag{132}$$

for $(\rho_1, \rho_2, \varepsilon_1, \varepsilon_2, \eta_1, \eta_2) \in (0, +\infty)^6$, $\check{\eta}_2 < 1$, *and for* $\psi \in L^\infty(\Gamma_+)$, $\|\psi\|_{L^\infty(\Gamma_+)} \le 1$,

$$\mathrm{supp}\psi \subset \{(x, v) \in \Gamma_+ \mid |x - x_0' - \tau_+(x_0', v_0')v_0'| > \eta_1 + \varepsilon_1 \text{ or } |v - v_0'| > \eta_2 + \varepsilon_2\}, \tag{133}$$

$$\mathrm{supp}\psi \subset \{(x, v) \in \Gamma_+ \mid \inf_{(s, s') \in \mathbb{R}^2} |x - sv - x_0' + s'v_0'| < \rho_1\}, \tag{134}$$

$$\mathrm{supp}\psi \subset \{(x, v) \in \Gamma_+ \mid |\hat{v} \cdot \hat{v}_0'| < \rho_2\}. \tag{135}$$

Similar though slightly different estimates may also be obtained in the time dependent setting; see [18, Theorem 3.2].

The first result (130) applies to all functions ψ supported in the velocity variable in the ρ-vicinity of v_0 as indicated in (131). Not all such test functions are of interest. When ρ is much smaller than ε_2 or η_2, then $I_0(\psi, \eta, \varepsilon)$ does not capture the whole ballistic part. This renders the estimate (130) useless. The support of ψ thus needs to be sufficiently large so that it captures the ballistic part. With our assumptions on f_ε and ϕ_η, this means that ψ should have a support of size $\varepsilon_1 + \eta_1$ in the vicinity of x_0 and of size $\varepsilon_2 + \eta_2$ in the vicinity of v_0.

Once the support of ψ is sufficiently large as indicated above, then $I_0(\psi, \eta, \varepsilon)$ captures the ballistic part of the signal up to an error caused by single and multiple scattering as indicated in Theorem 5.1. This is the error made on the X-ray transform of σ averaged over the support of ψ. We then have to invert the X-ray transform from these smoothed out measurements at the scale of the support of ψ. This is a task that needs to be performed carefully and whose analysis will be carried out elsewhere. At a qualitative level, we expect to reconstruct $\sigma = \sigma(x)$ at the scale limited by the support of ψ. The latter should therefore be sufficiently large in order to capture the ballistic front and yet sufficiently small so as to guarantee the best available resolution for the reconstruction of σ. All spatial scales in σ smaller than ε and η cannot be reconstructed stably. What our results says is that all scales larger than these numbers can indeed be reconstructed stably from transport measurements.

The second result (132) in Theorem 5.1 addresses the reconstruction of the scattering coefficient. The test function ψ should be supported away from the ballistic part, have a support that is sufficiently large so that it captures all of the single scattering contribution, and yet not too large so that the multiple scattering contribution is small over the support and so that resolution is not compromised in the reconstruction of $k(x, v', v)$. Note the role of $\check{\eta}_2$ as a combination of η_2 and ρ_2. The term involving $\check{\eta}_2$ shows that the reconstruction of $k(x, v_0, v)$ involves an error of order $\check{\eta}_2 \sim \eta_2$ when v_0 and v are not close to being parallel (i.e., when $v_0 \cdot v$ bounded away from 1). When v_0 and v become parallel, it becomes harder to separate the ballistic part from the single scattering part and $\check{\eta}_2 \gg \eta_2$ when ρ_2 approaches 1.

6. Varying indices of refraction and non-Euclidean geometries

Many of the results presented above have been extended to the case of spatially varying indices of refraction and more general non-Euclidean metric. Let us return to the transport problem with $H(x, v) = c(x)|v|$. The trajectories (bi-characteristics) corresponding to this Hamiltonian, solving Hamilton's equations

$$\dot{x} = \nabla_v H(x, v) = c(x)\frac{v}{|v|}, \qquad \dot{v} = -\nabla_x H(x, v) = -\nabla c(x)|v|,$$

are seen to the geodesics for the metric $g_{ij} = c^2(x)\delta_{ij}$ propagating with speed $c(x)$. By fixing the Hamiltonian of the trajectories to $H^2(x, v) = 1$, we observe that the particles propagate with normalized speed in the metric g_{ij} since $g_{ij}v^iv^j = c^2(x)\delta_{ij}v^iv^j = c^2(x)|v|^2 = 1$. Since the metric $g = g(x)$ is independent of direction v, only Hamiltonians that may be written (up to a nonlinear transform $H \mapsto \varphi \circ H$ since H, whence $\varphi \circ H$ is preserved along bi-characteristics) in the form $H(x, v) = g_{ij}(x)v^iv^j$ have bi-characteristics whose projections in the physical domain are geodesics for a Riemannian metric. More general Hamiltonians, for instance of the form $H(x, v) = \frac{1}{2}|v|^2 + V(x)$, are not treated by the Riemannian case and would require generalizations of the results presented below to a more general framework involving e.g. Finslerian metrics [25].

When scattering is *elastic* so that the Hamiltonian $H(x,v)$ is preserved through scattering events as in e.g. (29), then the stationary transport equation may be generalized as

$$\mathcal{D}u(x,v) + \sigma(x)u(x,v) - \int_{\Omega_x M} k(x,v',v)u(x,v')dv' = 0, \tag{136}$$

where $\mathcal{D}$ is the geodesic vector field associated to the metric g defined on a Riemannian manifold (M,g) and where $\Omega_x M$ is the unit tangent sphere at $x \in M$. The metric g is assumed to be *simple*, in the sense that M is strictly convex and for $x \in \bar{M}$, the closure of M, the exponential map $\exp_x : \exp_x^{-1}(\bar{M}) \to \bar{M}$ is a diffeomorphism. The hypotheses of simplicity of the metric prevents e.g. the crossing of several geodesics starting at a same point x with different directions.

The incoming and outgoing boundary conditions are defined as

$$\Gamma_\pm = \{(x,v) |\, x \in \partial M, \quad \pm g(v,\nu_x) > 0\},$$

where ν_x is the outer normal to M at $x \in \partial M$. As a generalization of the results obtained in section 2, the transport equation with incoming boundary conditions $u_{|\Gamma_-} = g$ admits a unique solution [91] under appropriate generalizations of the subcriticality conditions in (27) and we may define the albedo operator $\mathcal{A} : u_{|\Gamma_-} \mapsto \mathcal{A}u_{\Gamma_-} = u_{\Gamma_+}$.

We assume here that (M,g) is known. For reconstructions of the metric g from various boundary measurements, we refer the reader to e.g. [98, 107, 123, 130]. For a reconstruction of the whole Riemannian manifold (M,g) from knowledge of the length of all broken geodesics, we refer the reader to [74]. It is also shown in the latter reference that the albedo operator $\mathcal{A}$ defined above uniquely determines the length of all broken geodesics under appropriate smoothness and positivity constraints on the scattering coefficient k. We refer the reader to [74] for the details. Assuming the manifold (M,g) known, uniqueness of the reconstruction of the optical parameters from full or partial knowledge of the albedo operator $\mathcal{A}$ in the Riemannian setting was studied in [76, 91, 92]. We briefly mention these results and refer the reader to the recent review paper [93] for more details.

The generalization of Theorem 3.1 is as follows:

Theorem 6.1 ([91]) *Let $M \subset \mathbb{R}^d$, $d \geq 2$ be a bounded domain with smooth boundary and g be a known simple Riemannian metric on M. Let $0 \leq \sigma(x) \in L^\infty(M)$ and $0 \leq k \in L^\infty(M, \Omega_x M, \Omega_x M)$ such that the subcriticality condition* (27) *holds. Then $\mathcal{A}$ uniquely determines σ. When $d \geq 3$, then $\mathcal{A}$ uniquely determines k.*

The proof is also based on the decomposition of the albedo operator (40) as in the Euclidean case. In dimension $d = 2$, the scattering coefficient may be stably reconstructed under appropriate smallness conditions on k and some geometric assumptions. More precisely, let κ_0 be the maximal sectional curvature of (M,g). We assume that $2\sqrt{\kappa_0}\mathrm{diam}(M,g) < \pi$. Also, for every Jacobi field $J(t)$ define on $t \in [a,b]$ along a geodesic γ with $J(a) = 0$, we assume that $\|J(t)\|_g$ is strictly increasing on $[a,b]$. Then we have:

Theorem 6.2 ([92]) *Define* $\mathcal{V} = \{(\sigma(x), k(x, v, w));\ \|\sigma\|_{L^\infty} \le \Sigma,\ \|k\|_{L^\infty} \le \varepsilon\}$*. Under the above hypotheses and for a given* Σ*, there exists* $\varepsilon > 0$ *such that any pair* $(\sigma(x), k(x, v', v)) \in V$ *is uniquely determined by knowledge of the albedo operator* $\mathcal{A}$*.*

We conclude this section with a result extending Theorem 4.5 to the Riemannian setting. We define the angularly averaged measurements with angularly resolved source as

$$\mathcal{C} : g(x, v) \in L^1(\Gamma_-) \mapsto \mathcal{C}g(x) = J_{\mathfrak{w}}(x) := \int_{\Omega_{x,+}M} u_{|\Gamma_+}(x, v)\mathfrak{w}(x, v)d\mu(v) \in L^1(\partial X), \quad (137)$$

where $\Omega_{x,+}M = \{v \in \Omega_x M;\ g(v, \nu(x)) > 0\}$. Then we have the following result:

Theorem 6.3 ([76]) *Let* $\|k\|_{L^\infty}$ *be sufficiently small. Then the attenuation coefficient* σ *is uniquely determined by* $\mathcal{C}$*.*

Suppose in addition that $k(x, v, w) = k(x)\phi(x, v, w)$ *with* $(g, \mathfrak{w}, \sigma, \phi)$ *known and real analytic. Suppose that* $\tilde{k}(x, v, w) = \tilde{k}(x)\phi(x, v, w)$*, that* $\|k\|_{L^\infty}, \|\tilde{k}\|_{L^\infty} \le \varepsilon$ *for* ε *sufficiently small and* k *and* $\tilde{k}$ *vanish in the* $D-$*vicinity (for the metric* g*) of* ∂M *for some* $D > 0$*. Then* $k(x)$ *is uniquely determined by* $\mathcal{C}$*.*

7. Inverse source problem

We now consider the stationary source problem (18) with $u_{|\Gamma_-}(x, v) = g(x, v) = 0$. We assume that the optical coefficients $\sigma(x, v)$ and $k(x, v', v)$ are known. The simultaneous reconstruction of optical parameters and source terms is difficult to justify theoretically, although some positive results in this direction were obtained numerically [99]. The inverse problem consists of reconstructing the source term $S(x)$ from knowledge of $u_{|\Gamma_+}$. Note that an arbitrary source $S(x, v)$ is unlikely to be uniquely determined from $u_{|\Gamma_+}(x, v)$ since Γ_+ is a manifold of dimension smaller than that of $X \times V$ by one. Note that angularly averaged measurements such as those considered in (54) would also generate data on a manifold ∂X whose dimension is smaller than that of X by one again. Angularly resolved measurements are therefore required to reconstruct general, spatially dependent, sources $S(x)$.

We present here two different results. The first result is based on an extension of the inversion of the attenuated Radon transform in [104] (see also [5, 33]) to the case of weak scattering as it is presented in [24]. The second result is based on extension of injectivity results for generic X-ray transforms [56] to the setting of scattering media [133]. The first result applies to spatially dependent attenuation coefficients $\sigma = \sigma(x)$ and assumes a smallness constraint on the anisotropy of the scattering coefficient. The second result removes the smallness constraint on scattering and works "generically", i.e., for a dense set of general optical parameters $\sigma(x, v)$ and $k(x, v', v)$.

Other important contributions in the theory of the inverse transport source problem include [77, 106, 126]. For related works on the attenuated Radon transform in Euclidean and non-Euclidean geometries, we refer the reader to [3, 12, 14, 30, 73, 100, 103, 106, 118, 124].

Let us first consider the case $d = 2$ with isotropic scattering and the equation

$$v \cdot \nabla u + \sigma(x) u = f(x) := k(x) \int_V u(x, v') dv' + S(x), \tag{138}$$

with $u_{|\Gamma_-} = 0$. Then $u_{|\Gamma_+}(x, v) = \mathcal{L} f(x, v)$ with $\mathcal{L}$ defined in (22) may be identified with the attenuated Radon transform of $f(x)$. Under mild regularity assumptions on σ, the attenuated Radon transform was shown in [5, 104] to be injective and in [104] (see also [33]) to admit an explicit inversion formula. In other words, there exists an operator N such that formally, $N\mathcal{L} = I$, the identity operator in $L^2(X)$.

This shows that $\mathcal{L} f(x, v)$ uniquely determines $f(x)$. Upon solving (138), $u(x, v)$ is thus known so that $S(x) = f(x) - k(x) \int_V u(x, v') f(v') dv'$ is also uniquely determined from $\mathcal{L} f(x, v)$ on Γ_+ [134]. The result generalizes to dimension $d = 3$ (and higher dimensions) by assuming that two-dimensional measurements are available in each plane orthogonal to a fixed vector e_3 in $\mathbb{R}^3$.

In the presence of anisotropic scattering, the inversion operator may still be applied to the measurements $u_{|\Gamma_+}(x, v)$ and yields an equation of the form

$$N u_{|\Gamma_+}(x) = (I - N_K) S(x), \tag{139}$$

where N_K is an operator that is linear in the scattering coefficient $k(x, v', v)$. Under appropriate smallness assumptions on a scattering coefficient of the form $k(x, v', v) = k(x, v' \cdot v)$, it is shown in [24] that N_K is a contraction in $\mathcal{L}(L^2(X))$. More precisely, in dimension $d = 2$, we define $\tilde{k}(x, \theta) = k(x, \mu)$, $k_n(x) = \int_0^{2\pi} \tilde{k}(x, \theta) e^{-in\theta} d\theta$ and $\tilde{k}_n(\xi)$ the Fourier transform of $k_n(x)$. In dimension $d = 3$, we define $k(x, t) = \sum_{n \geq 0} k_n(x) P_n(t)$, where $P_n(t)$ is the $n-$th Legendre polynomial on $(-1, 1)$. Then $\hat{k}_n(\xi, t)$ is the Fourier transform of $k_n(x', x_3)$ with respect to the first two variables. Let $Y_{mn}(v)$ be the spherical harmonics on $\mathbb{S}^2$. Then we have the following result:

Theorem 7.1 ([24]) *Let $S(x) \in L^2(\mathbb{R}^d)$ be of compact support in X and $a(x) \in C_0^2(X)$ an absorption of compact support. In dimension $d = 2$, there is $\varepsilon = \varepsilon(X, a) > 0$ such that when*

$$\max_{n \in \mathbb{Z}} n^\alpha \| \hat{k}_n \|^2_{L^1(\mathbb{R}^2)} \leq \varepsilon$$

for some $\alpha > 1$, then the measurement $u_{|\Gamma_+}(x, v)$ uniquely determine S and from (139)*, there exists a constant C such that*

$$\| S \|_{L^2(X)} \leq C \| N u_{\Gamma_+} \|_{L^2(X)}. \tag{140}$$

In dimension $d = 3$, the same result of uniqueness and stability holds provided that

$$\max_{n \in \mathbb{N}^*} \Big(n^{\alpha - 1} \max_{|m| \leq n} \max_{\theta \cdot e_3 = 0} |Y_{nm}(\theta)|^2 \int_{\mathbb{R}} \| \hat{k}_n(\cdot, z) \|^2_{L^1(\mathbb{R}^2)} dz \Big) \leq \varepsilon.$$

Remark that N_K in (139) is a compact operator as results in e.g. [96] show. The equation (139) is therefore invertible so long as 1 is not an eigenvalue of the compact operator N_K. When N_K is sufficiently small in operator norm, then (139) may be solved by Neumann series expansion $S(x) = \sum_{m=0}^{\infty} N_K^m N u_{|\Gamma_+}$.

We now revisit the inverse source problem following the presentation in [133]. The forward transport equation is of the form

$$v \cdot \nabla u + \sigma(x, v)u - \int_V k(x, v', v)u(x, v')dv' = S(x, v) \quad (x, v) \in X \times V, \tag{141}$$

with $u_{|\Gamma_-} = 0$ and $V = \mathbb{S}^{d-1}$. We may formally recast the above problem as

$$(I - \mathcal{K})u = \mathcal{L}S$$

Solving for u is therefore equivalent to showing that $I - \mathcal{K}$ is invertible. When $\mathcal{K}$ is a compact operator, this is equivalent to 1 not being an eigenvalue of $\mathcal{K}$. It turns out that the L^1 setting used in Theorem 2.2 is not the most convenient to obtain compactness results because $\mathcal{K}$ is only weakly compact; see [96]. It is more convenient to work in the L^2 setting though the results presented below also generalize to the L^1 setting; see e.g. [131], where it is shown that $\mathcal{K}^4$ is compact in $\mathcal{L}(L^1(X \times V))$ when the optical parameters are continuous. That $\mathcal{K}$ is compact as an operator in $\mathcal{L}(L^2(X \times V))$ has been known under some restrictions on the optical parameters since the work in [139]; see [96] for a comprehensive presentation. It is shown in [133] that $\mathcal{K}^2$ is compact in $\mathcal{L}(L^2(X \times V))$ when the optical coefficients are continuous functions. The transport equation may then be recast as

$$(I - \mathcal{K}^2)u = (I + \mathcal{K})\mathcal{L}S,$$

and a solution u exists when 1 is not an eigenvalue of $\mathcal{K}^2$. This is always the case by replacing k by λk if necessary and this allows [133] to show that (141) admits a unique solution $u \in L^2(X \times V)$. Moreover, a classical trace estimate shows in this setting that $u_{|\Gamma_+} \in L^2(\Gamma_+, d\xi)$.

As in the proof of Theorem 7.1, the strategy followed in [133] to solve the inverse source problem is based on seeing $\mathcal{M} := (I - \mathcal{K})^{-1}\mathcal{L}$ as a (compact) perturbation of $\mathcal{L}$. We have seen that $\mathcal{L}$ was injective when $\sigma = \sigma(x)$ as $\mathcal{L}$ is identified with the attenuated Radon transform. The injectivity of $\mathcal{L}$ for more general, velocity dependent, attenuation coefficients $\sigma(x, v)$ may be obtained using the tools developed in [56]. We now recall these results in their full generality although they are only necessary here for integrations along straight lines. Let M be a compact manifold with boundary and Γ an open family of smooth curves on M with end points on ∂M and extended outside of M into a larger manifold M_1 with end points in M_1^{int}, the interior of M_1. The curves are parameterized locally by $\gamma_{x,\xi}(t)$ with $\gamma_{x,\xi}(0) = x$ and $\dot{\gamma}_{x,\xi}(0) = \mu\xi$ with $\mu > 0$ for $(x, \xi) \in TM$, the tangent bundle of M. Only one curve in Γ passes through $(x, \xi) \in TM$ and $\gamma_{x,\xi}$ is assumed to be smooth in (x, ξ). Let us define the generator $G(x, \xi) = \ddot{\gamma}_{x,\xi}(0)$, which uniquely determines Γ. Additional structure on the curves make Γ a smooth manifold. The family of curves Γ is said to be *analytic* when $G(x, \xi)$ and $\mu(x, \xi)$ are analytic.

Finally, define the exponential map $\exp_x(t,\xi) := \gamma_{x,\xi}(t)$. Then $x = \gamma(0)$ and $y = \gamma(t_0)$ are called conjugate along γ if $D_{t,\xi}\exp_x(t,\xi)$ does not have full rank at $(t_0, \dot\gamma(0))$.

Γ is defined in [56] as a *regular* family of curves if for any $(x,\zeta) \in T^*M$, the cotangent bundle of M, there exists $\gamma \in \Gamma$ through x normal to ζ without conjugate points. Then the authors in [56] prove the following result:

Theorem 7.2 ([56]) *Let Γ be an analytic regular family of curves in M_1 and let $w(x,\xi)$ be analytic and non-vanishing in M_1. Then*

$$\mathcal{L}_{\Gamma,w}S(\gamma) := \int w(\gamma(t),\dot\gamma(t))S(\gamma(t))dt, \qquad \gamma \in \Gamma, \tag{142}$$

is injective in $L^1(M)$. Moreover let (G_0, w_0, μ_0) be such a example of curves and weights. Then $\mathcal{L}_{\Gamma,w}$ is still injective for all (G, w, μ) such that (G,μ) is in the C^2 vicinity of (G_0,μ_0) and w is in the C^1 vicinity of w_0. (The vicinity is $(G_0, w_0, \mu_0)-$dependent.)

Moreover, the regularization properties of $\mathcal{L}_{\Gamma,w}$ are established in [56]. Let α be an appropriate weight function and define $\mathcal{L}_{\Gamma,w,\alpha} = \alpha\mathcal{L}_{\Gamma,w}$. Then it is proved in [56] that

$$\|\mathcal{L}^*_{\Gamma,w,\alpha}\mathcal{L}_{\Gamma,w,\alpha}\|_{H^1(M_1)} \sim \|f\|_{L^2(M)}, \tag{143}$$

where $a \sim b$ means $C^{-1}a \le b \le Ca$ for some $C > 0$. In other words, $\mathcal{L}_{\Gamma,w,\alpha}$ is a smoothing operator by $\frac{1}{2}$ a derivative as for the X-ray transform.

The above result is used in [133] to solve the inverse source problem as follows. Let us first assume that f is compactly supported in X_2 with $\bar X_2 \subset X$. Then we have the result:

Theorem 7.3 ([133]) *There exists a dense set of pairs*

$$(\sigma, k) \in C^2(\bar X \times V) \times C^2(\bar X \times V_{v'}; C^{d+1}(V_v)),$$

such that $\mathcal{M} = (I-\mathcal{K})^{-1}\mathcal{L}$ is injective in $L^2(X_2)$ with the estimate

$$\|f\|_{L^2(X_2)} \le C\|\mathcal{M}^*\mathcal{M}f\|_{H^1(X)}. \tag{144}$$

The result is based on showing that $\mathcal{L}$ is injective using Theorem 7.2 and on showing that $\mathcal{M}^*\mathcal{M} = \mathcal{L}^*\mathcal{L} + \mathcal{N}$, where $\mathcal{N}$ is shown to be compact from $H^1(X)$ to $L^2(X)$.

8. Practical and numerical aspects of forward and inverse transport

Solving the stationary or evolution transport equations numerically is not an easy task. Numerical methods have to deal with a large number of dimensions (typically three spatial dimensions plus at least two velocity dimensions) and the fact that solving hyperbolic equations, in which singularities are allowed to propagate without damping, is notoriously difficult on grids. For a brief illustration of the main numerical methods, including the discrete ordinate method and the spherical harmonics method, and their analysis, we refer the reader to e.g. [39, 61, 62, 64, 82, 86, 105, 108]. For a catastrophic effect of the spurious modes that develop in a discrete transport equation, see also [11].

Many techniques have been devised to solve the inverse transport problem. These techniques typically do not provide uniqueness or stability results but they are amenable to numerical simulations and provide practical reconstructions for the optical parameters. We briefly review such methods.

Energy balance techniques and transport equations in divergence form. One such method uses clever integrations by parts involving the direct and adjoint transport equations to obtain specific constraints on the optical parameters involving the available boundary measurements. Such methods typically deal with spatially independent optical parameters. Following [78], let u and $\tilde{w}$ be solutions of

$$v \cdot \nabla u + \sigma(u - \bar{u}) = 0, \qquad X \times V,$$

where $\bar{u} = |V|^{-1} \int_V u(x,v)dv$ is the velocity average of u and σ is constant. Let $w(x,v) = \tilde{w}(x,-v)$, which we verify is solution of the adjoint equation

$$-v \cdot \nabla w + \sigma(w - \bar{w}) = 0, \qquad X \times V.$$

Let us multiply the first equation by ∇w, the second equation by ∇u, and sum the two equalities to obtain

$$v \cdot \nabla(u \nabla w) - \nabla(uv \cdot \nabla w) + \sigma \nabla(uw) - \sigma(\bar{u}\nabla v + \bar{v}\nabla u) = 0.$$

We integrate the above equality over $X \times V$ and obtain

$$\int_{\partial X \times V} \Big((v \cdot \nu u \nabla w - (uv \cdot w)\nu) + \sigma(uw - \bar{u}\bar{w})\nu \Big) d\mu(x) dv = 0.$$

In other words,

$$\sigma = \frac{h \cdot \displaystyle\int_{\partial X \times V} u(v \cdot \nu \nabla_{\|} w - v_{\|} \cdot \nabla_{\|} w \nu) d\mu(x) dv}{h \cdot \displaystyle\int_{\partial X \times V} (uw - \bar{u}\bar{w})\nu d\mu(x) dv}, \tag{145}$$

where $Z_{\|} = Z - Z \cdot \nu\nu$ for $Z = \nabla$ and $Z = v$ and h is an arbitrary vector in $\mathbb{R}^d$. This provides an expression for σ by using two measurements (or one measurement with $w(x,v) = u(x,-v)$) provided that the above denominator does not vanish. The method generalizes to scattering coefficients $k = k(v' \cdot v)$ and provides systems of equations to solve for appropriate discretizations of $k(\mu)$. The main drawback of such methods is that they cannot handle spatially varying optical coefficients. For more details on the method and its applications in nuclear engineering and radiative transfer in the ocean, we refer the reader to [78, 90, 121, 143] and the references there.

Least square formulations and the adjoint method. Generalized least square formulations are ubiquitous in the numerical simulation of inverse problems. Such methods have been implemented in radiative transfer in e.g. [42, 48, 49, 71, 112] in the medical imaging context of optical tomography. The method is very similar to the inverse problem for the diffusion equation (34) (with $c(x) = c$ constant) as it is used in optical tomography [6].

The methodology for such methods is as follows. Let $(g_q)_{1\leq q\leq Q}$ be Q sources on Γ_- and let e.g. $\mathcal{C}g_q$ be the unperturbed measurements on ∂X, where $\mathcal{C}$ is defined in (91) as measurements are typically angularly averaged in practice. The sources g_q are also typically functions of x only. Let $(z_q)_{1\leq q\leq Q}$ be the corresponding perturbed (noisy) measurements. The least square error is therefore given by

$$\mathcal{F}(\sigma, k) = \frac{1}{2}\sum_{q=1}^{Q} \|\mathcal{C}g_q - z_q\|^2_{L^2(\partial X)}. \tag{146}$$

The objective is to minimize the above functional to retrieve σ and k from available measurements. Because the inverse problem is ill-posed, the parameters we want to reconstruct need to be regularized to avoid over-fitting of the data [53]. An example of regularization used in [112] is

$$\mathcal{F}_\beta(\sigma, k) = \mathcal{F}(\sigma, k) + \beta\mathcal{I}(\sigma, k), \qquad \mathcal{I}(\sigma, k) = \|\sigma - \sigma_0\|_{H^m(X)} + \epsilon\|k - k_0\|_{H^m(X\times V\times V)}, \tag{147}$$

where $m \geq 0$, $\epsilon > 0$ and σ_0 and k_0 are *reasonable* guesses for σ and k.

The minimization of $\mathcal{F}_\beta(\sigma, k)$ is computationally intensive and fraught with difficulties, the major one being the possibility of a large number of local minimizers. Minimizations are typically performed by using a Gauss-Newton type algorithm [102], which requires that one compute Fréchet derivatives of $\mathcal{F}_\beta$. The adjoint (co-state) method is computationally efficient to do so; see e.g. [140] and references there. Let $\delta\sigma$ a variation in the parameter σ. We then verify that

$$\mathcal{F}'_\beta \cdot \delta\sigma = \sum_{q=1}^{Q} \big(\varphi_q, \delta\sigma u_q\big)_{L^2(X\times V)} + \beta\big(\sigma - \sigma_0, \delta\sigma\big)_{H^1(X)},$$

where u_q is the solution of the equation $Tu_q = 0$ in (18) with source term $u_q = g_q$ on Γ_- and φ_q is the solution of the *adjoint* equation $T^*\varphi_q = 0$ with boundary condition $\varphi_q = -\mathcal{C}^*(\mathcal{C}f_q - z_q)$ on Γ_+. In other words, the calculation of the Fréchet derivative of $\mathcal{F}_\beta$ in a particular direction $\delta\sigma$ (with a similar expression for variations δk) involves the calculation of (only) one forward and one adjoint transport equations. This, in combination with Gauss-Newton type algorithms (such as the BFGS) algorithm allows one to obtain reasonable reconstructions of the optical parameters from a limited (though still relatively large) number of measurements.

Other theoretical results. We briefly mention several other noteworthy results in the theory of inverse transport. For the one-dimensional inverse transport problem, where optical parameters should be reconstructed from angularly resolved and possibly time resolved measurements, we refer the reader to [9, 10, 50] for inversions based on the invariant embedding technique and the derivation of Riccati equations for the reflection and transmission operators.

Most of the results presented here rely on multiple measurements. In some specific situations, one time-dependent measurement may be sufficient to obtain information about the optical coefficient. For such results based on the use of Carleman estimates, we refer the reader to [69, 70].

The inverse source problem mentioned above corresponds to the case of an active optical source as it arises in bioluminescence. The closely problem of fluorescence requires an external excitation of the optical (passive) source. For theoretical and numerical results on the fluorescence problem, we refer the reader to [40, 67, 72].

Finally, let us mention recent results on the linearization of the inverse transport problem about spatially homogeneous optical coefficients and the use of explicit expressions, written in term of infinite series, for the fundamental solutions to the radiative transfer equation. Such methods have been used to efficiently recover large-contrast small-volume or large-volume low-contrast inclusions in the optical tomography setting. We refer the reader to [87] for the details.

Inversions with limited data. The reconstructions presented so far require that the measurement operator be available with sufficient accuracy. In many practical situations, the amount of information is much smaller than what explicit reconstruction algorithms require. In such situations, prior information needs to be added into the problem in order to select among all the possible parameters that equally fit the unresolved available measurements. Adding a priori information may be performed by e.g. modifying the constraints in the least square method (147), for instance by replacing the regularization functional $\mathcal{I}(\sigma, k)$ by other functionals that favor minimization of the total variation or other sparsity constraints. A fairly versatile methodology to handle such prior constraints is the Bayesian framework. We refer the reader to e.g. [8, 54, 63, 94] for a few, very incomplete, references on this active research area.

9. Perspectives and open problems

The theory presented in this paper analyzes the singularities of the albedo operator in several practical settings of application of inverse transport. Several other singularities have not been analyzed in detail, for instance those emanating from scattering contributions in the time-dependent setting with angularly averaged measurements when the scattering coefficient does not vanish in the vicinity of the boundary; see e.g. [20], where such singularities are partially analyzed. We also refer the reader to [129] for additional open problems in this direction. The singularities of the albedo operator are the main tools used in the derivation of stability estimates. Stability estimates have not yet been obtained in several frameworks, such as in non-Euclidean geometry or in the setting of non-uniqueness results stated in section 3.4. Several groups are pursuing research in this direction. As we mentioned in section 6, many Hamiltonian dynamics may not be accounted for in the Riemannian setting. The unique and stable reconstruction of the optical parameters in a sufficiently general setting to account for such practical Hamiltonians is still an open problem.

The inverse problems considered in this paper decouple the reconstruction of the optical parameters and that of the source terms. This corresponds to the practical setting in most applications in medical imaging. This is also the practical setting in active remote sensing in the Earth atmosphere. However, this is not the practical setting in passive remote sensing in the atmosphere. The latter problem is typically concerned with the reconstruction of gas concentrations in the atmosphere from satellite or terrestrial measurements. The inverse problem then takes the form of the time independent source problem (18) with V an open subset of $\mathbb{R}^d$. Frequency measurements (for a large number of values of $|v|$) are crucial in remote sensing and are an important difference with respect to most applications in medical imaging. The major difference mathematically, however, is that both the source term S and the attenuation coefficient σ are unknown and that they are coupled. Indeed, the source term is essentially generated by black-body radiation, which is itself very much dependent on the absorbing properties of the gas. The resulting inverse transport problem, which may be seen as a nonlinear inverse source problem, is still mostly open mathematically. We refer the reader to [83] for a presentation of the passive remote sensing problem and to [23] for a related theoretical study in the extremely simplified one-dimensional, scattering-free, setting.

In the above presentation, polarization effects have been neglected and polarized light has been replaced by a scalar description. Yet polarization measurements are known to greatly enhance the reconstruction of optical parameters in many practical settings of inverse transport; see e.g. [22, 41, 47, 95, 97, 138, 144]. The mathematical theory of matrix-valued inverse transport (see e.g. [13, 120]) as it accounts for polarization effects has not been done to-date.

Finally, the analysis of the singularities of the albedo operator has primarily remained a theoretical tool so far. The method has rarely been tested numerically, with the exception of the work in [4], and in practical inversions, with the exception of the analysis of scattering corrections in several areas of applied sciences; see e.g. [51, 119] and references there. The theory of inverse transport presented in this paper applies in specific cases: scattering has to be significant to allow for the reconstruction of the scattering coefficient, and yet not too large so that multiple scattering does not overwhelm measurements. In other words, we are concerned with transport setting that are not well approximated by the diffusion of Fokker-Planck regimes mentioned in section 2. The transport regime described in this paper is an accurate model in remote sensing in the presence of thin clouds and in medical imaging of optically thin domains such as e.g. small animals [29] or finger joints [58]. A careful analysis of the reconstruction of the optical properties using the singularities of the albedo operator with practical noise models remains to be done.

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