Inverse Diffusion Theory of Photoacoustics

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Abstract. This paper analyzes the reconstruction of diffusion and absorption parameters in an elliptic equation from knowledge of internal data. In the application of photoacoustics, the internal data are the amount of thermal energy deposited by high frequency radiation propagating inside a domain of interest. These data are obtained by solving an inverse wave equation, which is well-studied in the literature.

We show that knowledge of two internal data based on well-chosen boundary conditions uniquely determines two constitutive parameters in diffusion and Schrödinger equations. Stability of the reconstruction is guaranteed under additional geometric constraints of strict convexity. No geometric constraints are necessary when $2n$ internal data for well-chosen boundary conditions are available, where n is spatial dimension. The set of well-chosen boundary conditions is characterized in terms of appropriate complex geometrical optics (CGO) solutions.

Keywords. Photoacoustics, Optoacoustics, diffusion equation, inverse problems, internal data, stability estimates, complex geometrical optics (CGO) solutions.

1. Introduction

Photoacoustic tomography (PAT) is a recent hybrid medical imaging modality that combines the high resolution of acoustic waves with the large contrast of optical waves. When a body is exposed to short pulse radiation, typically emitted in the near infra-red region in PAT, it absorbs energy and expands thermo-elastically by a very small amount; this is the photoacoustic effect. Such an expansion is sufficient to emit acoustic pulses, which travel back to the boundary of the domain of interest where they are measured.

A first step in PAT is therefore to reconstruct the amount of deposited energy from time-dependent boundary measurement of acoustic signals. Acoustic signals propagate in fairly homogeneous domains as the sound speed varies very little from one tissue to the next. The reconstruction of the amount of deposited energy is therefore quite accurate in many practical settings. For references on the practical and theoretical aspects of PAT, we refer the reader to e.g. [1, 5, 6, 7, 9, 8, 10, 12, 13, 14, 15, 16, 17, 20, 21].

Once the amount of deposited energy has been reconstructed, a second step consists of inferring the optical properties in the body. Although this second step is less studied mathematically, it has received significant attention in the biomedical literature; see e.g. [5, 6, 7]. The reconstruction of the optical parameters is very useful in practice because the attenuation properties of healthy and unhealthy tissues are extremely different [7, 20]. The combination of high spatial resolution of the acoustic signals and large contrast of the optical parameters allows us e.g. to image blood vessels at high resolution, which is in turn important for assessing the status of cancerous tissues.

Near infra-red light is best modeled by radiative transfer equations. What we can reconstruct on the optical coefficients, seen as constitutive parameters in the radiative transfer equation, from measurements of heat deposition for all possible illuminations of the domain has recently been analyzed in [4]. The latter paper considers the continuous-illumination setting. In other words, heat deposition is measured for all possible incoming radiation condition, which can be controlled at the boundary of the domain.

In this paper, we consider the diffusion approximation to radiative transfer, which is accurate for radiation propagation in highly scattering media. Such an approximation is typically valid for propagations of radiation over one centimeter or more [2, 3]. In such a simplified setting, the unknown optical parameters are the spatially varying diffusion and attenuation coefficients. Provided that the diffusion coefficient is known at the domain's boundary, we show that these two coefficients are uniquely determined by two well-chosen illuminations at the boundary of the domain. An explicit reconstruction procedure is presented, which involves solving a first-order equation with a vector field explicitly constructed from the internal data (the amount of deposited energy).

The stability of the reconstruction of the optical coefficients from two internal data is established under geometric conditions of strict convexity on the domain of interest. In the presence of $2n$ well-chosen boundary conditions, where n is spatial dimension, such geometric constraints can be removed and constructions are possible on essentially arbitrary domains of interest with well-defined boundary.

Mathematically, the inverse problem is an inverse diffusion problem with two internal measurements. By using the standard Liouville change of variables, the diffusion equation is replaced by a Schrödinger equation with unknown potential and with internal measurements involving a second unknown function. By adapting the theory of complex geometrical optics solutions to this setting, we are able to obtain uniqueness and stability results for the inverse Schrödinger problem. By means of the inverse Liouville change of variables, we are able to conclude on the uniqueness and stability of the reconstruction of the optical parameters for the photoacoustic tomography problem in the diffusive regime.

The rest of the paper is structured as follows. Section 2 presents the photoacoustic tomography problem and the main results we obtain in this paper. The inverse

Schrödinger problem and the explicit reconstruction algorithms are addressed in section 3. How the latter results are used to solve the inverse diffusion problem is explained in section 4.

2. Photoacoustic tomography and main results

The propagation of radiation in scattering media is modeled by the following diffusion equation

$$\begin{aligned} &\frac{1}{c}\frac{\partial}{\partial t}u - \nabla\cdot D(x)\nabla u + \sigma_a(x)u = 0, \qquad x\in X\subset\mathbb{R}^n,\ t\geq 0\\ &u = g \qquad x\ \in\partial X,\ t\geq 0\end{aligned}\tag{1}$$

where X is an open, bounded, connected domain in $\mathbb{R}^n$ with C^1 boundary ∂X (embedded in $\mathbb{R}^n$), where n spatial dimension; c is light speed in tissues; $D(x)$ is a (scalar) diffusion coefficient; and $\sigma_a(x)$ is an attenuation coefficient. The source of incoming radiation is prescribed by $g(t,x)$ on the boundary ∂X and is assumed to be a very short pulse supported on an interval of time $(0,\eta)$ with $c\eta$ of order $O(1)$. The amount of energy deposited is proportional to attenuation and is given by

$$H(t,x) = \sigma_a(x)u(t,x).$$

A thermal expansion (assumed to be proportional to H) results and emits acoustic waves. Such waves are modeled by

$$\frac{1}{c_s^2(x)}\frac{\partial^2 p}{\partial t^2} - \Delta p = \beta\frac{\partial}{\partial t}H(t,x),\tag{2}$$

with c_s the sound speed and β a coupling coefficient assumed to be constant and known. The pressure (potential) $p(t,x)$ is then measured on ∂X as a function of time. Using as in [4] the difference of time scales $c_s \ll c$, which shows that radiation propagation occurs at a much faster time scale than acoustic wave propagation, we can show that

$$H(t,x)\sim H_0(x)\delta_0(t), \qquad H_0(x)=\sigma_a(x)\int_{\mathbb{R}^+}u(t,x)dt.$$

We now have a well-posed inverse wave problem, where measurement of $n-$dimensional information $p(t,x)$ for $t>0$ and $x\in\partial X$ allows us to reconstruct the $n-$dimensional spatial map $H_0(x)$. We assume this step, which has been analyzed in great detail in the literature, completed; see e.g. [9, 14, 15, 17, 21].

Denoting by $g(x)=\int_{\mathbb{R}^+}g(t,x)dt$ and $u(x)=\int_{\mathbb{R}^+}u(t,x)dt$, we thus observe that the photoacoustic problem in the diffusive regime amounts to reconstructing $(D(x),\sigma_a(x))$ from knowledge of

$$H_j(x)=\sigma_a(x)u_j(x), \qquad 1\leq j\leq J,\tag{3}$$

for $J\in\mathbb{N}^*$ illumination maps $g_j(x)$, where u_j is the solution to the steady-state equation

$$\begin{aligned} &-\nabla\cdot D(x)\nabla u_j + \sigma_a(x)u_j = 0, \qquad x\in X\subset\mathbb{R}^n,\\ &u_j = g_j \qquad x\in\partial X.\end{aligned}\tag{4}$$

The set of internal data is given by:

$$d = (d_j)_{1\le j\le J}, \qquad d_j(x) = \sigma_a(x)u_j(x). \tag{5}$$

The main problem of interest in this paper is the uniqueness and the stability of the reconstruction for the Inverse Diffusion problem with Internal Data:

ISID: Reconstruction of $(D(x), \sigma_a(x))$ from knowledge of $d = (d_j)_{1\le j\le J}$ on X for a *fixed* collection of illuminations $g = (g_j)_{1\le j\le J}$ prescribed on ∂X.

We start with the case of two real-valued measurements $J = 2$. The reconstructions are based on the construction of vector fields that are well defined only when the optical coefficients are sufficiently smooth. More precisely, let $k \ge 1$ and assume that

$$\sqrt{D} \in Y = H^{\frac{n}{2}+k+2+\varepsilon}(X) \subset C^{k+2}(\bar X), \quad \sigma_a \in C^{k+1}(\bar X), \qquad \varepsilon > 0.$$

Let then $\mathfrak{k} \in \mathbb{R}^n$ be a constant vector with $|\mathfrak{k}|$ sufficiently large.

Let $g = (g_1, g_2)$ be a given illumination and $\mathfrak{d} = d_1 + id_2 = \sigma_a(x)(u_1 + iu_2)$ the corresponding internal data, where the real-valued solutions u_j solve (4) with boundary conditions g_j for $j = 1, 2$. We introduce the vector field and scalar quantity

$$\beta \;=\; \frac{e^{-2\mathfrak{k}\cdot x}}{2|\mathfrak{k}|}\Im\big(\mathfrak{d}\nabla\bar{\mathfrak{d}} - \bar{\mathfrak{d}}\nabla\mathfrak{d}\big), \qquad \gamma \;=\; \frac{e^{-2\mathfrak{k}\cdot x}}{4|\mathfrak{k}|}\Im\big(\bar{\mathfrak{d}}\Delta\mathfrak{d} - \mathfrak{d}\Delta\bar{\mathfrak{d}}\big). \tag{6}$$

Here, $\bar{\mathfrak{d}} = d_1 - id_2$ is the complex conjugate of $\mathfrak{d}$. Then we will see in section 3 that for $\mu = D^{-\frac12}\sigma_a$, we have

$$\beta\cdot\nabla\mu + \gamma\mu = 0. \tag{7}$$

Since μ is known on ∂X as D and $\sigma_a = d_j/g_j$ are known on ∂X, the above equation is a well-posed equation for μ. When the integral curves of the vector field β map any point inside x to a point $x_0(x) \in \partial X$, then (7) uniquely characterizes μ inside the domain.

Let $v = \sqrt{D}\Re u$ and define $q = -v^{-1}\Delta v$, which we will show is well-defined. Then it turns out that

$$-\Delta\sqrt{D} - q\sqrt{D} = \mu, \qquad x \in X.$$

This provides an elliptic equation for $\sqrt{D}$ that admits a unique solution since $\sqrt{D}$ is known on ∂X.

The principal difficulty is to ensure that β is constructed in such a way that (7) allows for a unique solution μ. We will show that there exists an open set of boundary conditions g on ∂X that allows us to do so. We define the set of coefficients $(D, \sigma_a) \in \mathcal{M}$ as:

$$\mathcal{M} = \{(D, \sigma_a) \text{ such that } (\sqrt{D}, \sigma_a) \in Y \times C^{k+1}(\bar X),\ \|\sqrt{D}\|_Y + \|\sigma_a\|_{C^{k+1}(\bar X)} \le M\}. \tag{8}$$

The main results for the Inverse Diffusion problem with Internal Data (IDID) are then as follows. We start with a uniqueness results:

Theorem 2.1 *Let X be an open, bounded, domain with C^2 boundary ∂X. Assume that (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ are in $\mathcal{M}$ with $D_{|\partial X} = \tilde{D}_{|\partial X}$. Let d and $\tilde{d}$ be the internal data in* (5) *for the coefficients (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$, respectively and with boundary conditions $(g_j)_{j=1,2}$. Then there is an open set of illuminations $g \in (C^{1,\alpha}(\partial X))^2$ for some $\alpha > \frac{1}{2}$ such that if $d = \tilde{d}$, then $(D, \sigma_a) = (\tilde{D}, \tilde{\sigma}_a)$.*

The above result shows uniqueness of the reconstruction of the optical coefficients but does not imply stability. Without additional geometric information about ∂X, the above procedure may yield unstable reconstructions. However, provided that $2n$ well-chosen measurements d_j for $1 \leq j \leq J = 2n$ are available, then n vector fields similar to β above may be constructed in order to form locally a basis of vectors in $\mathbb{R}^n$. In such a setting, the following stability result holds.

Theorem 2.2 *Let $k \geq 2$ and let X be an arbitrary bounded domain with boundary ∂X of class C^{k+1}. Assume that (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ are in $\mathcal{M}$ with $D_{|\partial X} = \tilde{D}_{|\partial X}$. Let $d = (d_1, \ldots, d_{2n})$ and $\tilde{d} = (\tilde{d}_1, \ldots, \tilde{d}_{2n})$ be the internal data constructed in* (5) *for the coefficients (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$, respectively and with boundary conditions $g = (g_j)_{1 \leq j \leq 2n}$. Then there is an open set of illuminations $g \in (C^{k,\alpha}(\partial X))^{2n}$ and a constant C such that*

$$\|D - \tilde{D}\|_{C^k(X)} + \|\sigma_a - \tilde{\sigma}_a\|_{C^k(X)} \leq C\|d - \tilde{d}\|_{(C^{k+1}(X))^{2n}}. \tag{9}$$

Stability is therefore ensured without geometric constraints provided that enough measurements are available. When only two measurements are available and are of the form described in Theorem 2.1, we can still get a stability result under the following geometric hypothesis:

Hypothesis 2.3 *There exists $R < \infty$ such that for each $x_0 \in \partial X$, which we assume is of class C^2, we have $X \subset B_{x_0}(R)$ where $B_{x_0}(R)$ is a ball of radius R that is tangent to ∂X at $x_0 \in \partial X$.*

Then we can show the following result.

Theorem 2.4 *Let $k \geq 3$. Let X satisfy Hypothesis 2.3 with boundary ∂X of class C^{k+1}. Assume that (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ are in $\mathcal{M}$ with $D_{|\partial X} = \tilde{D}_{|\partial X}$. Let d and $\tilde{d}$ be internal data as above for the coefficients (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$, respectively and with boundary conditions $g = (g_j)_{j=1,2}$. Then there is an open set of illuminations $g \in (C^{k,\alpha}(\partial X))^2$ and a constant C such that*

$$\|D - \tilde{D}\|_{C^{k-1}(X)} + \|\sigma_a - \tilde{\sigma}_a\|_{C^{k-1}(X)} \leq C\|d - \tilde{d}\|_{(C^k(X))^2}. \tag{10}$$

The above three theorems are proved in the following two sections. They show that the internal data d for illuminations g with $J \geq 2$ provide stable reconstructions of the optical coefficients under the assumption that the illuminations are well-chosen. How these illuminations are chosen will become more explicit in the next two sections. The characterization of the open set of illuminations is however not very precise. The main features of the result are as follows. For coefficients in $\mathcal{M}$, there is a minimal value of $|\mathfrak{k}|$ that ensures that we can construct a vector field β with Property **P**, which means its integral curves map any point in X to an point in ∂X. Such a vector field is constructed by means of complex geometric optics solutions, which depend on the unknown optical parameters. The illuminations must then be chosen sufficiently close to the trace on ∂X of the above vector field to ensure that they generate another vector field with Property **P**. Closedness is therefore not characterized in very explicit means. It remains an interesting question to obtain a priori constraints on the illumination that will ensure that the resulting vector field satisfies Property **P**.

In the next section, we consider a similar problem for the Schrödinger equation. How the latter results are used to prove the theorems stated above is described in section 4.

3. Inverse Schrödinger with Internal Data

Let X be an open, bounded, connected, domain in $\mathbb{R}^n$, where n is spatial dimension, with smooth boundary ∂X. We consider the Schrödinger equations

$$\begin{aligned} \Delta u_j + q u_j &= 0 \qquad && X \\ u_j &= g_j && \partial X, \end{aligned} \tag{11}$$

for $1 \leq j \leq J$. Here, $J \in \mathbb{N}^*$ is the number of illuminations and q is an unknown potential. We assume that the homogeneous problem with $g_j = 0$ admits the unique solution $u \equiv 0$ so that $\lambda = 0$ is not in the spectrum of $\Delta + q$. We assume that q on X is the restriction to X of a function $\tilde{q}$ compactly supported on $\mathbb{R}^n$ and such that $\tilde{q} \in H^{\frac{n}{2}+k+\varepsilon}(\mathbb{R}^n)$ with $\varepsilon > 0$ for $k \geq 1$. Moreover we assume that the extension is chosen so that

$$\|q\|_{H^{\frac{n}{2}+k+\varepsilon}(X)} \leq C(X,k,n)\|\tilde{q}\|_{H^{\frac{n}{2}+k+\varepsilon}(\mathbb{R}^n)}, \tag{12}$$

for some constant $C(X,k,n)$ independent of q. That such a constant exists may be found e.g. in [18, Chapter VI, Theorem 5].

We assume that $g_j \in C^{k,\alpha}(X)$ with $\alpha > \frac{1}{2}$ and ∂X is of class C^{k+1} so that (11) admits a unique solution $u_j \in C^{k+1}(X)$ [11, Theorem 6.19]. The internal data are of the form

$$d_j(x) = \mu(x) u_j(x), \qquad X, \quad 1 \leq j \leq J. \tag{13}$$

Here $\mu \in C^{k+1}(\bar{X})$ verifies $0 < \mu_0 \leq \mu(x) \leq \mu_0^{-1}$ for a.e. $x \in X$.

The *inverse Schrödinger problem with internal data* (ISID) consists of reconstructing (q,μ) in X from knowledge of $d = (d_1,\ldots,d_J) \in \big(C^{k+1}(X)\big)^J$ for a given illumination $g = (g_j)_{1\leq j\leq J}$. We will mostly be concerned with the case $J = 2$ and $J = 2n$ with g_j, and hence d_j real-valued measurements.

3.1. Complex Geometrical Optics Solutions

The analysis of ISID carried out in this paper is based on the construction of complex geometrical optics (CGO) solutions. When $q = 0$, CGOs are harmonic solutions of the form $e^{\rho\cdot x}$ for $\rho \in \mathbb{C}^n$ such that $\rho\cdot\rho = 0$. When $q \not\equiv 0$, CGOs are solutions of the following problem

$$\Delta u_\rho + q u_\rho = 0, \qquad u_\rho \sim e^{\rho\cdot x} \text{ as } |x| \to \infty. \tag{14}$$

More precisely, we say that u_ρ is a solution of the above equation with $\rho \cdot \rho = 0$ and the proper behavior at infinity when it is written as

$$u_\rho(x) = e^{\rho\cdot x}\big(1 + \psi_\rho(x)\big), \tag{15}$$

for $\psi_\rho \in L^2_\delta$ a weak solution of

$$\Delta\psi_\rho + 2\rho \cdot \nabla\psi_\rho = -q(1 + \psi_\rho). \tag{16}$$

The space L^2_δ for $\delta \in \mathbb{R}$ is defined as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{L^2_\delta}$ defined as

$$\|u\|_{L^2_\delta} = \Big(\int_{\mathbb{R}^n} \langle x\rangle^{2\delta}|u|^2 dx\Big)^{\frac12}, \qquad \langle x\rangle = (1 + |x|^2)^{\frac12}. \tag{17}$$

Let $-1 < \delta < 0$ and $q \in L^2_{\delta+1}$ and $\langle x\rangle q \in L^\infty$. One of the main results in [19] is that there exists $\eta = \eta(\delta)$ such that the above problem admits a unique solution with $\psi \in L^2_\delta$ provided that

$$\|\langle x\rangle q\|_{L^\infty} + 1 \le \eta|\rho|.$$

Moreover, $\|\psi\|_{L^2_\delta} \le C|\rho|^{-1}\|q\|_{L^2_{\delta+1}}$ for some $C = C(\delta)$. In the analysis of ISID, we need smoother CGOs than what was recalled above. We introduce the spaces H^s_δ for $s \ge 0$ as the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm $\|\cdot\|_{H^s_\delta}$ defined as

$$\|u\|_{H^s_\delta} = \Big(\int_{\mathbb{R}^n} \langle x\rangle^{2\delta}|(I-\Delta)^{\frac{s}{2}}u|^2 dx\Big)^{\frac12}. \tag{18}$$

Here $(I-\Delta)^{\frac{s}{2}}u$ is defined as the inverse Fourier transform of $\langle\xi\rangle^s\hat{u}(\xi)$, where $\hat{u}(\xi)$ is the Fourier transform of $u(x)$. Then we have the following

Proposition 3.1 *Let $-1 < \delta < 0$ and $k \in \mathbb{N}^*$. Let $q \in H_1^{\frac{n}{2}+k+\varepsilon}$ and hence in $H_{\delta+1}^{\frac{n}{2}+k+\varepsilon}$ and ρ be such that*

$$\|q\|_{H_1^{\frac{n}{2}+k+\varepsilon}} + 1 \le \eta|\rho|. \tag{19}$$

Then ψ_ρ the unique solution to (16) belongs to $H_\delta^{\frac{n}{2}+k+\varepsilon}$ and

$$|\rho|\|\psi_\rho\|_{H_\delta^{\frac{n}{2}+k+\varepsilon}} \le C\|q\|_{H_{\delta+1}^{\frac{n}{2}+k+\varepsilon}}, \tag{20}$$

for a constant C that depends on δ and η.

Proof. We recall [19] that for $|\rho| \geq c > 0$ and $f \in L^2_{\delta+1}$ with $-1 < \delta < 0$, the equation

$$(\Delta + 2\rho \cdot \nabla)\psi = f \tag{21}$$

admits a unique weak solution $\psi \in L^2_\delta$ with

$$\|\psi\|_{L^2_\delta} \leq C(\delta, c)|\rho|^{-1}\|f\|_{L^2_{\delta+1}}.$$

Now since $(\Delta + 2\rho\cdot\nabla)$ and $(I - \Delta)^s$ are constant coefficient operators and hence commute, we deduce that when $f \in H^s_{\delta+1}$ for any $s > 0$, then

$$\|\psi\|_{H^s_\delta} \leq C(\delta, c)|\rho|^{-1}\|f\|_{H^s_{\delta+1}}. \tag{22}$$

The solution to (16) is known to admit the decomposition

$$\psi_\rho = \sum_{j=0}^{\infty} \psi_j, \qquad \Delta\psi_j + 2\rho \cdot \nabla\psi_j = -q\psi_{j-1},$$

with $\psi_{-1} = 1$. Let $s = \frac{n}{2} + k + \varepsilon$. Assume $q\psi_{j-1} \in H^s_{\delta+1}$ which is true for $j = 0$ by assumption on q. Then $\psi_j \in H^s_\delta$. Since H^s is an algebra, we want to prove that

$$\|q\psi_j\|_{H^s_{\delta+1}} \leq \|q\|_{H^s_1}\|\psi_j\|_{H^s_\delta}. \tag{23}$$

Indeed, decompose $\mathbb{R}^n$ into cubes. On each cube B, $\langle x\rangle^{2s}$ is more or less constant up to a $C^{\pm 2s}$. Now $H^s(B)$ is an algebra so that $\|qu\|_{H^s(B)} \leq \|q\|_{H^s(B)}\|u\|_{H^s(B)}$. Since $\langle x\rangle$ is more or less constant and equal to $\langle x_B\rangle$,

$$\langle x_B\rangle^{2\delta+2}\|qu\|^2_{H^s(B)} \leq C\|\langle x\rangle(I-\Delta)^{\frac{s}{2}}q\|^2_{L^2(B)}\|\langle x\rangle^\delta(I-\Delta)^{\frac{s}{2}}u\|^2_{L^2(B)}$$

It remains to sum over all the cubes B to get the result. When the size of the cubes tends to 0, the constant C tends to 1, which yields (23). This and (22) show that

$$\|\psi_j\|_{H^s_\delta} \leq C|\rho|^{-1}\|q\|_{H^s_1}\|\psi_{j-1}\|_{H^s_\delta}.$$

By selecting η such that $C|\rho|^{-1}\|q\|_{H^s_1} < \frac{1}{2}$, we obtain

$$\|\psi_j\|_{H^s_\delta} \leq \frac{1}{2^j}C|\rho|^{-1}\|q\|_{H^s_{\delta+1}}.$$

It remains to sum the geometric series to obtain the result. □

We now want to obtain estimates for ψ_ρ and u_ρ restricted to X. We have the following result.

Corollary 3.2 *Let us assume the regularity hypotheses of the previous proposition. Then we find that*

$$|\rho|\|\psi_\rho\|_{H^{\frac{n}{2}+k+\varepsilon}(X)} + \|\psi_\rho\|_{H^{\frac{n}{2}+k+1+\varepsilon}(X)} \leq C\|q\|_{H^{\frac{n}{2}+k+\varepsilon}(X)}. \tag{24}$$

Proof. On the bounded domain X, $\langle x\rangle$ is bounded above and below by positive constants. Since q is compactly supported on $\mathbb{R}^n$, we obtain thanks to (12) that

$$|\rho|\|\psi_\rho\|_{H^{\frac{n}{2}+k+\varepsilon}(X)} \leq C\|q\|_{H^{\frac{n}{2}+k+\varepsilon}(\mathbb{R}^n)} \leq C(X)\|q\|_{H^{\frac{n}{2}+k+\varepsilon}(X)}.$$

Now we have

$$\Delta\psi_\rho = -2\rho\cdot\nabla\psi_\rho - q(1+\psi_\rho).$$

By elliptic regularity, with X' a smooth domain in $\mathbb{R}^n$ such that $\bar{X}\subset X'$, we find for all $s=\frac{n}{2}+k+\varepsilon$, that

$$\|\psi_\rho\|_{H^{s+1}(X)} \leq C\|2\rho\cdot\nabla\psi_\rho - q(1+\psi_\rho)\|_{H^{s-1}(X')} + \|\psi_\rho\|_{H^s(X')}.$$

The latter is bounded by $|\rho|\|\psi_\rho\|_{H^s(X')} + \|q\|_{H^{s-1}(X')}\|\psi_\rho\|_{H^{s-1}(X')}$ since $s-1>\frac{n}{2}$ so that $H^{s-1}(X')$ is a Banach algebra. By using the above bound on $\|\psi_\rho\|_{H^s(X')}$, with $C(X)$ replaced by the larger $C(X')$, we get the result. □

By Sobolev embedding, we have just proved the:

Proposition 3.3 *Under the hypotheses of Corollary 3.2, the restriction to X of the CGO solution verifies that*

$$|\rho|\|\psi_\rho\|_{C^k(\bar{X})} + \|\psi_\rho\|_{C^{k+1}(\bar{X})} \leq C\|q\|_{H^{\frac{n}{2}+k+\varepsilon}(X)}. \tag{25}$$

We recall that q satisfies the constraint (19).

We are now in a position to prove the main result of this section.

Theorem 3.4 *Let u_{ρ_j} for $j=1,2$ be CGO solutions with q such that (19) holds for both ρ_j and $k\geq 1$ and with $c_0^{-1}|\rho_1|\leq|\rho_2|\leq c_0|\rho_1|$ for some $c_0>0$. Then we have*

$$\hat{\beta} := \frac{1}{2|\rho_1|}e^{-(\rho_1+\rho_2)\cdot x}\Big(u_{\rho_1}\nabla u_{\rho_2} - u_{\rho_2}\nabla u_{\rho_1}\Big) = \frac{\rho_1-\rho_2}{2|\rho_1|} + \hat{h}, \tag{26}$$

where the vector field $\hat{h}$ satisfies the constraint

$$\|\hat{h}\|_{C^k(\bar{X})} \leq \frac{C_0}{|\rho_1|}, \tag{27}$$

for some constant C_0 independent of $\rho_{1,2}$.

Proof. Some algebra shows that

$$\hat{h} = \frac{(\rho_1-\rho_2)}{2|\rho_1|}(\psi_{\rho_1}+\psi_{\rho_2}+\psi_{\rho_1}\psi_{\rho_2}) + \frac{\nabla\psi_{\rho_2}(1+\psi_{\rho_1}) - \nabla\psi_{\rho_1}(1+\psi_{\rho_2})}{2|\rho_1|}. \tag{28}$$

We know from Proposition 3.3 that $|\rho_j||\psi_{\rho_j}|$ and $|\nabla\psi_{\rho_j}|$ are bounded in $C^k(\bar{X})$ for $j=1,2$. This concludes the proof of the theorem. □

3.2. Construction of vector fields and uniqueness result

Let us consider two internal complex-valued data $d_{1,2}(x)$ obtained as follows. We assume that we can impose the complex-valued boundary conditions $g_{1,2} \in C^{k,\alpha}(\partial X;\mathbb{C})$ and define the solution $u_{1,2}$ of

$$\Delta u_j + q u_j = 0, \quad X, \qquad u_j = g_j, \quad \partial X, \qquad j = 1,2. \tag{29}$$

Note that the real and imaginary parts of $u_{1,2}$ may be solved independently since (29) is a linear equation. We then assume that we have access to the complex-valued internal data $d_j = \mu u_j$ on X for $j = 1,2$, where $u_{1,2}$ are the solutions of (29) with boundary conditions $g_{1,2}$. We recall that $\mu \in C^{k+1}(\bar{X})$ and is bounded above an below by positive constants. We verify that

$$u_1 \Delta u_2 - u_2 \Delta u_1 = 0.$$

Introducing $\nu = \frac{1}{\mu}$, which is well defined since μ is bounded away from 0, and using $u_j = \nu d_j$, we obtain that

$$2(d_1 \nabla d_2 - d_2 \nabla d_1) \cdot \nabla \nu + (d_1 \Delta d_2 - d_2 \Delta d_1)\nu = 0.$$

This is equivalent to

$$\check{\beta}_d \cdot \nabla \mu + \check{\gamma}_d \mu = 0, \tag{30}$$

where

$$\begin{aligned} \check{\beta}_d &:= \chi(x)(d_1 \nabla d_2 - d_2 \nabla d_1) \\ \check{\gamma}_d &:= \frac{1}{2}\chi(x)(d_2 \Delta d_1 - d_1 \Delta d_2) = \frac{-\check{\beta}_d \cdot \nabla \mu}{\mu}. \end{aligned} \tag{31}$$

Here, $\chi(x)$ is a smooth known complex-valued function with $|\chi(x)|$ uniformly bounded from below by a positive constant on $\bar{X}$. Note that by assumption on μ, we have that $\check{\beta}_d \in (C^k(\bar{X};\mathbb{C}))^n$ and $\check{\gamma}_d \in C^k(\bar{X};\mathbb{C})$.

A methodology for the reconstruction of (μ, q) is therefore as follows: we first reconstruct μ using the real part *or* the imaginary part of (30) for then $\Re\check{\beta}_d$ and $\Im\check{\beta}_d$ are real-valued vector fields since $\mu = d/g$ is known on ∂X. When μ is reconstructed, this gives us explicit reconstructions for $u_{1,2} = d_{1,2}/\mu$ and we may then reconstruct q from the Schrödinger equation. Such a method provides a unique reconstruction provided that the integral curves of (the real part or the imaginary part of) $\check{\beta}_d$ join any point in x to a point $x_0(x) \in \partial X$, where μ is known. We thus need the vector field $\check{\beta}$ to satisfy such properties. CGO solutions will allow us to construct families of vector fields $\check{\beta}_d$ with the required properties.

Let us consider two CGOs $u_{\rho_{1,2}}$with parameters $\rho_{1,2}$. Let $d_{1,2}$ be the complex-valued corresponding internal data. Let us decompose as before

$$u_{\rho_j}(x) = e^{\rho_j \cdot x}(1 + \psi_{\rho_j}(x)), \qquad \nabla u_{\rho_j}(x) = e^{\rho_j \cdot x}\big((1 + \psi_{\rho_j})\rho_j + \nabla \psi_{\rho_j}\big).$$

Let us choose $\chi(x) = e^{-(\rho_1+\rho_2)\cdot x}$ in (31). Then we find after some algebra that $\check{\beta}_d$ in (31) is given by

$$\check{\beta}_d = \mu^2\Big((\rho_1 - \rho_2)(1 + \psi_{\rho_1})(1 + \psi_{\rho_2}) + \nabla \psi_{\rho_2}(1 + \psi_{\rho_1}) - \nabla \psi_{\rho_1}(1 + \psi_{\rho_2})\Big). \tag{32}$$

We may then define

$$\check{\beta} := \frac{1}{2|\rho_1|}\check{\beta}_d = \mu^2\frac{\rho_1-\rho_2}{2|\rho_1|} + \mu^2\hat{h}, \qquad \check{\gamma} := \frac{1}{2|\rho_1|}\check{\gamma}_d, \tag{33}$$

where $\hat{h}$ is defined as in (28). Then, we deduce from Theorem 3.4 that $|\rho_1|\mu^2\hat{h}$ is bounded uniformly in $C^k(\bar{X};\mathbb{C})$. When $|\rho_1|$ is sufficiently large, then $\check{\beta}$ is close to $\mu^2\frac{\rho_1-\rho_2}{2|\rho_1|}$, which is a non-vanishing vector when $\rho_1 \neq \rho_2$. Provided that the real part or the imaginary part of $\check{\beta}$ does not vanish, then (30) gives an equation for μ that can be uniquely solved since μ is known on ∂X.

Note that the data d_k are complex valued. The only possibility to construct two different complex valued data with two real valued data is to assume that $d_2 = \bar{d}_1$, the complex conjugate of d_1. For the construction of CGOs, this implies that we choose $\rho_2 = \overline{\rho_1}$. Indeed, we verify that $\overline{u_\rho} = u_{\bar{\rho}}$ since for $\rho = \mathfrak{k} + i\mathfrak{k}^\perp$ with $|\mathfrak{k}| = |\mathfrak{k}^\perp|$ and $\mathfrak{k}\cdot\mathfrak{k}^\perp = 0$, we have $\overline{e^{\rho\cdot x}} = e^{\bar{\rho}\cdot x}$ and $\overline{\psi_\rho} = \psi_{\bar{\rho}}$ by uniqueness of the solution to the equation satisfied by ψ_ρ. This implies then that $\check{\beta}$ defined in (33) with $\rho_2 = \bar{\rho}_1$ is given by

$$\check{\beta}_\rho = i\mu^2\mathfrak{k}^\perp + \mu^2\hat{h}. \tag{34}$$

As soon as $|\rho| > C_0$ so that $\|\hat{h}\|_{C^0(\bar{X})} < 1$, we obtain that any point in X is connected to a point in ∂X by an integral curve of $\beta_\rho := \Im\check{\beta}_\rho$.

Note that u_ρ solves (29) with the unknown boundary condition $u_{\rho|\partial X} \in C^{k,\alpha}(\partial X;\mathbb{C})$ for some $\alpha > \frac{1}{2}$ since u_ρ is known to be a little more regular than being of class $C^{k+1}(\bar{X};\mathbb{C})$ by construction (since $\varepsilon > 0$).

Let us now define boundary conditions $g \in C^{k,\alpha}(\partial X;\mathbb{C})$ such that

$$\|g - u_{\rho|\partial X}\|_{C^{k,\alpha}(\partial X;\mathbb{C})} \leq \epsilon, \tag{35}$$

for some $\epsilon > 0$ sufficiently small. Let u be the solution of (29) with g as in (35). By elliptic regularity, we thus have

$$\|u - u_\rho\|_{C^{k+1}(\bar{X};\mathbb{C})} \leq C\epsilon, \tag{36}$$

for some positive constant C. Define the complex valued internal data $d = \mu u$. Since $\mu \in C^{k+1}(\bar{X})$, we deduce that

$$\|d - d_\rho\|_{C^{k+1}(\bar{X};\mathbb{C})} \leq C_0\epsilon, \tag{37}$$

for $C_0 > 0$. Once d is constructed, define $d_1 = d$ and $d_2 = \bar{d}$ and define $\check{\beta}_d$ and $\check{\mu}_d$ as in (31) with $\chi(x) = e^{-2\mathfrak{k}\cdot x}$ and the normalized quantities $\check{\beta}$ and $\check{\gamma}$ as in (33). Note that $\chi(x)$ is positive and bounded on $\bar{X}$.

Let us define

$$\beta := \Im\check{\beta} = \frac{1}{2|\mathfrak{k}|}\Im\check{\beta}_d, \qquad \gamma := \Im\check{\gamma} = \frac{1}{2|\mathfrak{k}|}\Im\check{\gamma}_d. \tag{38}$$

Thanks to (37) and (34), we obtain the error estimate

$$\|\beta - \mu^2\hat{\mathfrak{k}}\|_{C^k(\bar{X})} \le C\frac{1+\epsilon}{|\mathfrak{k}|}. \tag{39}$$

As a consequence, as soon as $|\mathfrak{k}|$ is sufficiently large and ϵ sufficiently small, we obtain that $\beta\cdot\hat{\mathfrak{k}} \ge \zeta > 0$ so that any point $x \in X$ is mapped to a point in ∂X in a time less than $|\zeta|^{-1}\mathrm{diam}(X)$ by an integral curve of β.

Moreover, we have the equation with real-valued coefficients:

$$\beta\cdot\nabla\mu + \gamma\mu = 0. \tag{40}$$

Since $\mu = d/g$ is known on ∂X, this equation provides a unique reconstruction for μ.

Let us define the set of parameters

$$\begin{aligned}\mathcal{P} \;=\; \Big\{&(\mu,q)\in C^{k+1}(\bar{X})\times H^{\frac{n}{2}+k+\varepsilon}(X);\ 0 \text{ not an eigenvalue of } \Delta+q,\\ &\|\mu\|_{C^{k+1}(\bar{X})} + \|q\|_{H^{\frac{n}{2}+k+\varepsilon}(X)} \le P < \infty\Big\}.\end{aligned} \tag{41}$$

The above construction of the vector field allows us to obtain the following uniqueness result.

Theorem 3.5 *Let X be a bounded, open subset of $\mathbb{R}^n$ with boundary ∂X of class C^2. Let (μ,q) and $(\tilde\mu,\tilde q)$ be two elements in $\mathcal{P}$. Let $\mathfrak{k}\in\mathbb{R}^n$ with $|\mathfrak{k}|\ge|\mathfrak{k}_0|$ and $|\mathfrak{k}_0|$ sufficiently large and define $\rho=\mathfrak{k}+i\mathfrak{k}^\perp$ so that $\rho\cdot\rho=0$. Let u_ρ be the corresponding CGO for q and u constructed as above with $d=\mu u$ and with ϵ sufficiently small. Let $\tilde d$ be constructed similarly with the parameters $(\tilde\mu,\tilde q)$.*

Then $d=\tilde d$ implies that $(\mu,q)=(\tilde\mu,\tilde q)$.

Proof. Since the two measurements $d=\tilde d$, we have that μ and $\tilde\mu$ solve the same equation (40). Since $\mu=\tilde\mu=d/g$ on ∂X, we deduce that $\mu=\tilde\mu$ since the integral curves of β map any point $x\in X$ to the boundary ∂X. More precisely, consider the flow $\varphi_x(t)$ associated to β, i.e., the solution to

$$\dot\varphi_x(t) = \beta(\varphi_x(t)), \qquad \varphi_x(0) = x\in\bar{X}. \tag{42}$$

By the Picard-Lindelöf theorem, the above equations admit unique solutions since β is of class C^1. And by hypothesis on β since $\beta\cdot\hat{\mathfrak{k}}\ge\zeta>0$ for $|\mathfrak{k}|$ sufficiently large, any point x is mapped to two points (for positive and negative values of t) on ∂X by the flow $\varphi_x(t)$ in a time less than $\zeta^{-1}\mathrm{diam}(X)$. For $x\in X$, let us define $x_\pm(x)\in\partial X$ and $\pm t_\pm(x)>0$ such that

$$\varphi_x(t_\pm(x)) = x_\pm(x)\in\partial X. \tag{43}$$

Then by the method of characteristics, $\mu(x)$ solution of (40) is given by

$$\mu(x) = \mu_0(x_\pm(x))e^{-\int_0^{t_\pm(x)}\gamma(\varphi_x(s))ds}. \tag{44}$$

The solution $\tilde{\mu}(x)$ is given by the same formula since $\varphi_x(t) = \tilde{\varphi}_x(t)$ so that $\tilde{\mu} = \mu$. This implies that $u = \tilde{u}$ since $d = \tilde{d}$. It remains to use the equation for u to deduce that $q = \tilde{q}$ on the domain where $u \neq 0$. By unique continuation, u cannot vanish on an open set in X different from the empty set for otherwise u vanishes everywhere and this is impossible to satisfy the boundary conditions. This shows that the set $F \subset X$ where $|u| > 0$ is open and $\bar{F} = \bar{X}$ since the complement of $\bar{F}$ has to be empty. By continuity, this shows that q is known on $\bar{X}$. □

The above result shows that there exists an open set of boundary conditions g close to $u_{\rho|\partial X}$ so that data $d_1 = d$ and $d_2 = \bar{d}$ obtained from one complex-valued solution u or equivalently from two real valued solutions $\Re u$ and $\Im u$, uniquely determine the parameters (μ, q). A more explicit characterization of the open set of illuminations is lacking. However, we observe that larger values of q require larger values of $|\mathfrak{k}|$ in order to straighten the vector field β. Although (35) seems to be independent of $\mathfrak{k}$ and ρ, in fact u_ρ itself grows exponentially with $|\mathfrak{k}| = |\rho|$ so that g has to be in the ϵ vicinity of an exponentially growing quantity. This means that $|\mathfrak{k}|$ has to be sufficiently large that the field β is sufficiently flat while at the same time not so large that the imposed illuminations become physically infeasible.

The above uniqueness result does not guaranty stability in the reconstruction. We easily verify that the construction provides stability of the reconstruction of μ in most of the domain X. However, small changes in the data may generate small changes in the field β. This in turn may significantly modify the value of the reconstructed function μ at points where β is "almost" tangent to the boundary ∂X. We will see below that under some geometric constraints of sufficient convexity of X, the above procedure provides a stable reconstruction of the parameters (μ, q). When such conditions are not met, we can still obtain stability by acquiring more measurements. Indeed, if a sufficient number of vector fields β can be constructed at every point so that the span of these vector fields is exactly $\mathbb{R}^n$, then we face a significantly more favorable situation. We now consider such a case where $2n$ real-valued measurements are available. Later, we will derive stability results in the two-measurement setting under additional geometric constraints.

3.3. ISID with $2n$ real-valued internal data

Let us consider first the setting in which we can access $2n$ real-valued internal data viewed as n complex-valued internal data (since the measurements are linear in u, we can measure the real and imaginary parts separately).

Let us define $\mathfrak{k}_j = |\mathfrak{k}| e_j$ where $(e_1, \ldots, e_n)$ is an orthonormal basis. We define the complex vectors

$$\rho_j = \mathfrak{k}_1 + i\mathfrak{k}_j, \quad 2 \leq j \neq n, \qquad \rho_1 = -\mathfrak{k}_1 - i\mathfrak{k}_2 = -\rho_2. \tag{45}$$

Let u_{ρ_j} be the corresponding CGOs. We choose boundary conditions g_j such that

$$\|g_j - u_{\rho_j|\partial X}\|_{C^{k,\alpha}(\partial X;\mathbb{C})} \leq \epsilon, \tag{46}$$

for ϵ sufficiently small. We define u_j as the solutions to (29) with boundary conditions g_j. These are n complex-valued solutions whose real and imaginary parts consist of $2n$ real-valued solutions. For $1 \leq j \leq n$, we define $d_j = \mu u_j$. We now construct the n vector field β_j. For $2 \leq j \leq n$, the real-valued vector fields and scalar terms are constructed as in the preceding section; for $j = 1$, the vector field is constructed by using $\rho_2 = -\rho_1$:

$$\begin{aligned} \beta_1 &= \frac{1}{2|\mathfrak{k}|}\Re\big(d_2\nabla d_1 - d_1\nabla d_2\big), & \gamma_1 &= \frac{1}{4|\mathfrak{k}|}\Re\big(d_1\Delta d_2 - d_2\Delta d_1\big), \\ \beta_j &= \frac{e^{-2\mathfrak{k}_1\cdot x}}{2|\mathfrak{k}|}\Im\big(d_j\nabla \bar d_j - \bar d_j\nabla d_j\big), & \gamma_j &= \frac{e^{-2\mathfrak{k}_1\cdot x}}{4|\mathfrak{k}|}\Im\big(\bar d_j\Delta d_j - d_j\Delta \bar d_j\big), \end{aligned} \tag{47}$$

for $2 \leq j \leq n$. As in the preceding section, we verify that

$$\|\beta_j - \mu^2\hat{\mathfrak{k}}_j\|_{C^k(\bar X)} \leq C\frac{1+\epsilon}{|\mathfrak{k}|}. \tag{48}$$

For $|\mathfrak{k}|$ sufficiently large, and thanks to the bound $\mu_0^{-1} \leq \mu \leq \mu_0$, we obtain that at each point $x \in X$, the vectors $\beta_j(x)$ form a basis. Moreover, the matrix a_{ij} such that $\beta_j = \sum a_{jk}e_k$ is an invertible matrix with inverse of class $C^k(\bar X)$. In other words, we have constructed a vector-valued function $\Gamma(x) \in (C^k(\bar X))^n$ such that (40) may be recast as

$$\nabla\mu + \Gamma(x)\mu = 0. \tag{49}$$

Finally, the construction of Γ is stable under small perturbations in the data d_j. Indeed, invertibility of a_{jk} is ensured for vector fields close to β_j. Let Γ and $\tilde\Gamma$ be two vector fields constructed from knowledge of two sets of internal data $d = \{d_j, 1 \leq j \leq n\}$ and $\tilde d = \{\tilde d_j, 1 \leq j \leq n\}$. Then we find that

$$\|\Gamma - \tilde\Gamma\|_{(C^k(\bar X))^n} \leq C\|d - \tilde d\|_{(C^{k+1}(\bar X;\mathbb{C}))^n}, \tag{50}$$

provided the right-hand side is sufficiently small.

Let us now assume that X is connected (otherwise, the method applies to each connected component) and μ is known and equal to $\mu_0 = d/g$ for some point $x_0 \in \partial X$. In other words, we want to solve the over-determined problem

$$\nabla\mu + \Gamma(x)\mu = 0, \quad X, \qquad \mu(x_0) = \mu_0(x_0), \quad x_0 \in \partial X. \tag{51}$$

Let $x \in X$ be an arbitrary point and assume that X is bounded and connected and ∂X is smooth. Then we find a smooth curve that links x to the point $x_0 \in \partial X$. Restricted to this curve, (51) becomes a stable ordinary differential equation. The solution of the ordinary differential equation is then stable with respect to modifications in Γ (the curve between x and x_0 is kept constant). The solution μ then clearly inherits the smoothness of $\Gamma(x)$ directly from (51). Moreover, since $\mu_0(x_0) - \tilde\mu_0(x_0)$ (with $\tilde\mu_0 = \tilde d/g$ on ∂X) is small and equation (50) is stable with respect to changes in the value of $\mu_0(x_0)$, we deduce that the reconstruction of μ is stable with respect to perturbations in d.

We may thus state the main result of this section:

Theorem 3.6 *Let $k \geq 1$. We assume that we have access to n well-chosen complex-valued measurements and that (μ, q) and $(\tilde{\mu}, \tilde{q})$ are elements in $\mathcal{P}$. Under the hypotheses outlined above, and provided that $\|u_{\rho_j|\partial X} - g_j\|_{C^{k,\alpha}(\bar{X};\mathbb{C})}$ is sufficiently small, then we have the following stability result:*

$$\|\mu - \tilde{\mu}\|_{C^k(\bar{X})} + \|q - \tilde{q}\|_{C^{k-2}(\bar{X})} \leq C\|d - \tilde{d}\|_{(C^{k+1}(\bar{X}))^{2n}}, \tag{52}$$

Proof. The inequality for $\mu - \tilde{\mu}$ is a direct consequence of the results proved above. This provides a stability result for $\nu = \mu^{-1}$ and for $u_j = \nu d_j$ from the data d_j. We thus have a stability result for $\Delta u_j = -u_j q$ and hence the above stability result for $u_j(q - \tilde{q})$ since $(u_j - \tilde{u}_j)\tilde{q}$ is small.

Now, $u_\rho = e^{\rho\cdot x}(1+\psi_\rho)$ does not vanish on X when $|\rho|$ is sufficiently large since $|\rho|\psi_\rho$ is bounded. When the boundary condition $g_j - u_{\rho_j|\partial X}$ is small, then by the maximum principle, u_j does not vanish on X either. This means that either its real part or its imaginary part does not vanish everywhere in X. This provides control of $q - \tilde{q}$ in X as given in (52). □

3.4. Vector fields and stability of solutions

The above construction allows one to stably reconstruct the two functions μ and q provided that we have constructed $J = 2n$ well-chosen real-valued boundary conditions and collected $2n$ corresponding internal data. We now return to the reconstruction of μ and q in the presence of $J = 2$ well-chosen real-valued internal data. Such internal data are obtained as Theorem 3.5.

We recall that $\mathfrak{k}$ is fixed and $\rho = \mathfrak{k} + i\mathfrak{k}^\perp$. We define u as the solution to (29) with g close to $u_{\rho|\partial X}$. The complex-valued internal data are then $d = \mu u$. The vector field β and the scalar γ are then given by

$$\beta \;=\; \frac{e^{-2\mathfrak{k}\cdot x}}{2|\mathfrak{k}|}\Im\big(d\nabla\bar{d} - \bar{d}\nabla d\big), \quad \gamma \;=\; \frac{e^{-2\mathfrak{k}\cdot x}}{4|\mathfrak{k}|}\Im\big(\bar{d}\Delta d - d\Delta\bar{d}\big), \tag{53}$$

and we verify that

$$\beta\cdot\nabla\mu + \gamma\mu = 0 \qquad X. \tag{54}$$

As earlier, we verify that

$$\|\beta - \mu^2\hat{\mathfrak{k}}^\perp\|_{C^k(\bar{X})} \leq C\frac{1+\epsilon}{|\mathfrak{k}|}. \tag{55}$$

As a consequence, the integral curves of β given by $\varphi_x(t)$ map any point $x \in X$ to two points on ∂X when $|\mathfrak{k}|$ is sufficiently large as was mentioned earlier.

However, the stability of equation (54) with respect to changes in β and γ is not as good as in the presence of n complex internal data. The stability of the reconstruction degrades for points x close to $x_0 \in \partial X$ where $\beta_\mu(x_0)\cdot n(x_0)$ is close to 0. The stability of the reconstruction of μ will however be good when X is a convex domain with "sufficient" convexity as established in Hypothesis 2.3. We prove the following result:

Proposition 3.7 *Let $k \geq 1$. Let μ and $\tilde{\mu}$ be solutions of (54) corresponding to coefficients (β,γ) and $(\tilde{\beta},\tilde{\gamma})$, respectively, where*

$$\beta = \mu^2\hat{\mathfrak{k}}^\perp + \frac{1}{|\mathfrak{k}|}h, \qquad \tilde{\beta} = \mu^2\hat{\mathfrak{k}}^\perp + \frac{1}{|\mathfrak{k}|}\tilde{h},$$

for h, γ, $\tilde{h}$, and $\tilde{\gamma}$ bounded in $C^k(\bar{X})$.

Let us assume that $\mu_{|\partial X} = \mu_0$ and $\tilde{\mu}_{|\partial X} = \tilde{\mu}_0$ on ∂X for some functions $\mu_0, \tilde{\mu}_0 \in C^k(\partial X)$. Let us assume that X is sufficiently convex so that Hypothesis 2.3 holds for some $R < \infty$. We also assume that $|\mathfrak{k}| \geq \mathfrak{k}_0$ is sufficiently large. Then there is a constant C such that

$$\begin{aligned} \|\mu - \tilde{\mu}\|_{C^{k-1}(\bar{X})} &\leq C\|\mu_0\|_{C^k(\partial X)}\big(\|\beta-\tilde{\beta}\|_{C^{k-1}(\bar{X})} + \|\gamma-\tilde{\gamma}\|_{C^{k-1}(\bar{X})}\big) \\ &+ C\|\mu_0 - \tilde{\mu}_0\|_{C^k(\partial X)}. \end{aligned} \tag{56}$$

Proof of Proposition 3.7. We recall that $\varphi_x(t)$ is the flow defined in (42) and that $x_\pm(x)$ and $t_\pm(x)$ are defined in (43). By the method of characteristics, $\mu(x)$ solution of (54) is given by

$$\mu(x) = \mu_0(x_\pm(x))e^{-\int_0^{t_\pm(x)}\gamma(\varphi_x(s))ds}. \tag{57}$$

The solution $\tilde{\mu}(x)$ is given similarly. We first assume that $\tilde{\mu}_0 = \mu_0$.

From the equality

$$\varphi_x(t) - \tilde{\varphi}_x(t) = \int_0^t [\beta(\varphi_x(s)) - \tilde{\beta}(\tilde{\varphi}_x(s))]ds,$$

and using the Lipschitz continuity of β and Gronwall's lemma, we thus deduce the existence of a constant C such that

$$|\varphi_x(t) - \tilde{\varphi}_x(t)| \leq Ct\|\beta - \tilde{\beta}\|_{C^0(X)}$$

uniformly in t knowing that all characteristics exit X in finite time and provided that $\varphi_x(t)$ and $\tilde{\varphi}_x(t)$ are in $\bar{X}$.

Such estimates are stable with respect to modifications in the initial conditions. Let us define $W(t) = D_x\varphi_x(t)$. Then classically, W solves the equation $\dot{W} = D_x\beta(\varphi_x)W$ with $W(0) = I$ and by using Gronwall's lemma once more, we deduce that

$$|W - \tilde{W}|(t) \leq Ct\|D_x\beta - D_x\tilde{\beta}\|_{C^0(\bar{X})},$$

for all times provided that $\varphi_x(t)$ and $\tilde{\varphi}_x(t)$ are in $\bar{X}$. As a consequence, since β and $\tilde{\beta}$ are of class $C^k(\bar{X})$, then we obtain similarly that:

$$|D_x^{k-1}\varphi_x(t) - D_x^{k-1}\tilde{\varphi}_x(t)| \leq Ct\|\beta - \tilde{\beta}\|_{C^{k-1}(X)},$$

and this again for all times provided that $\varphi_x(t)$ and $\tilde{\varphi}_x(t)$ are in $\bar{X}$.

However, this does not imply that $x_+(x)$ is close to $\tilde{x}_+(x)$. When ∂X is flat for instance, we may very well have that $x_+(x)$ is such that $n(x_+(x)) \cdot \dot{\varphi}_x(t_+(x))$ is very small and that $x_+(x) - \tilde{x}_+(x)$ is arbitrarily large if $\tilde{\beta}$ is parallel to the surface ∂X for instance. This behavior, however, cannot occur when both β and $\tilde{\beta}$ are sufficiently flat, which is the case when $|\mathfrak{k}|$ is sufficiently large, and when ∂X is sufficiently curved, which is obtained from the existence of $R < \infty$ in Hypothesis 2.3. In such a setting, we can obtain the following result:

Lemma 3.8 *Let $k \geq 1$ and assume that β and $\tilde{\beta}$ are $C^k(\bar{X})$ vector fields that are sufficiently flat, i.e., that $|\mathfrak{k}|$ is sufficiently large. Let us assume that ∂X is sufficiently convex so that Hypothesis 2.3 holds for some $R < \infty$. Then we have that*

$$\|x_+ - \tilde{x}_+\|_{C^{k-1}(\bar{X})} + \|t_+ - \tilde{t}_+\|_{C^{k-1}(\bar{X})} \leq C\|\beta - \tilde{\beta}\|_{C^{k-1}(\bar{X})}, \tag{58}$$

where C is a constant that depends on $|\mathfrak{k}|$ and R.

The above lemma is mostly a consequence of the following result:

Lemma 3.9 *Let C_0 be the constant defined such that*

$$|\ddot{\varphi}_x(t)| = |\nabla\beta(\varphi_x(t))\beta(\varphi_x(t))| \leq \frac{C_0}{|\mathfrak{k}|}.$$

Let t_M be the maximal time spent by any trajectory in X, which we know is bounded. Assume that $|\mathfrak{k}|$ is sufficiently large that for all $x \in X$,

$$\Big(\frac{C_0 R}{|\mathfrak{k}|} + \frac{C_0^2 t_M^2}{4|\mathfrak{k}|^2}\Big)\frac{1}{|\beta(x_+(x))|^2} = \rho < 1.$$

Then we have that for all $x \in X$,

$$t_+(x) \leq \frac{2R}{|\beta(x_+(x))|^2(1-\rho)} n(x_+(x)) \cdot \beta(x_+(x)). \tag{59}$$

In other words, the vector field $\beta(x_+(x))$ is close to being tangent to ∂X only when the time spent in X is small.

Let now $x_0 \in X$ and $x \in \partial X$ and define $v_0 = \beta(x_0)$. Assume moreover that

$$|x - x_0| \leq C_1\delta^2, \qquad |n(x) \cdot v_0| \leq C_2\delta. \tag{60}$$

Then we have

$$t_+(x_0) \leq C_3\delta, \tag{61}$$

for some constant $C_3 > 0$ independent of x_0. In other words, a trajectory close to ∂X and almost tangent to ∂X exits X in a short time.

We postpone the proof of the above two lemmas to the end of the section. Let us conclude the proof of the proposition. We recall that

$$\mu(x) = \mu_0(x_+(x))e^{-\int_0^{t_+(x)} \gamma(\varphi_x(s))ds},$$

with a similar expression for $\tilde{\mu}$. Since $x \to e^{-x}$ is smooth, by the Leibniz rule it is sufficient to prove the stability result for $\mu_0(x_+(x))$ and for $\int_0^{t_+(x)} \gamma(\varphi_x(s))ds$. It is clear from the above lemmas that

$$\|\mu_0(x_+(x)) - \mu_0(\tilde{x}_+(x))\|_{C^{k-1}(\bar{X})} \le \|\mu_0\|_{C^k(\partial X)} \|x_+ - \tilde{x}_+\|_{C^{k-1}(\bar{X})}$$
$$\le C\|\mu_0\|_{C^k(\partial X)} \|\beta - \tilde{\beta}\|_{C^{k-1}(\bar{X})}.$$

Let us now assume without loss of generality that $\tilde{t}_+(x) \ge t_+(x)$. Then we have

$$\int_0^{t_+(x)} \big(\gamma(\varphi_x(s)) - \tilde{\gamma}(\tilde{\varphi}_x(s))\big)ds = \int_0^{t_+(x)} \big(\gamma(\varphi_x(s)) - \gamma(\tilde{\varphi}_x(s)) + (\gamma - \tilde{\gamma})(\varphi_x(s))\big)ds.$$

We verify that the above expression has $k-1$ derivatives uniformly bounded since (i) $x \to t_+(x)$ is $C^{k-1}(\bar{X})$; (ii) γ has C^k derivatives bounded on $\bar{X}$; (iii) $(\varphi_x - \tilde{\varphi}_x)(s)$ has $k-1$ derivatives bounded by $\|\beta - \tilde{\beta}\|_{C^{k-1}(\bar{X})}$.

It thus remains to handle the term

$$\upsilon(x) = \int_{t_+(x)}^{\tilde{t}_+(x)} \tilde{\gamma}(\tilde{\varphi}_x(s))ds.$$

The function $x \to \tilde{\gamma}(\tilde{\varphi}_x(s))$ is of class $C^{k-1}(\bar{X})$ by regularity of the flow and because $\tilde{\gamma}$ is of class $C^k(\bar{X})$. Derivatives of order $k-1$ of $\upsilon(x)$ thus involve terms of size $\tilde{t}_+(x) - t(x)$ and terms of the form

$$D_x^m \Big(\tilde{t}_+ D_x^{k-1-m} \tilde{\gamma}(\tilde{\varphi}_x(\tilde{t}_+)) - t_+ D_x^{k-1-m} \tilde{\gamma}(\tilde{\varphi}_x(t_+))\Big), \quad 0 \le m \le k-1.$$

Because $\tilde{\beta}$ is of class $C^k(\bar{X})$, then so is $x \to \tilde{\gamma}(\tilde{\varphi}_x(s))$. Since the latter function has $k-1$ derivatives that are Lipschitz continuous, we thus find that

$$|D_x^{k-1} \upsilon(x)| \le C\|\tilde{t}_+ - t_+\|_{C^{k-1}(\bar{X})}.$$

This concludes the proof of the proposition when $\tilde{\mu}_0 = \mu_0$. Applying Lemma 3.8 as before, we verify that

$$\big|D_x^{k-1}\big[(\tilde{\mu}_0(x_\pm(x)) - \mu_0(x_\pm(x)))e^{-\int_0^{t_\pm(x)} \gamma(\varphi_x(s))ds}\big]\big| \le C\|\mu_0 - \tilde{\mu}_0\|_{C^k(\bar{X})}.$$

By the triangle inequality, we deduce the error estimate on $\mu - \tilde{\mu}$ described in the proposition. □

Proof of Lemma 3.8. Let us assume without loss of generality that $t_+(x) \le \tilde{t}_+(x)$. We have seen that

$$|\varphi_x(t_+(x)) - \tilde{\varphi}_x(t_+(x))| \le Ct_+(x)\delta, \qquad \delta = \|\beta - \tilde{\beta}\|_{C^0(\bar{X})}.$$

We also have that $|\beta(\varphi_x(t_+)) - \tilde{\beta}(\tilde{\varphi}_x(t_+))| \le C_2\delta$. From Lemma 3.9, we know that $\beta(x_+(x)) \cdot n(x_+(x)) \ge 2C_1 t_+(x)$ for some constant $C_1 > 0$.

Let us assume first that $C_1 t_+ \ge C_2\delta$ so that $v_0 \cdot e_1 \ge C_1 t_+(x)$ where $x_0 = \tilde{\varphi}_x(t_+)$, $v_0 = \tilde{\beta}(x_0)$, and $e_1 = n(x_+(x))$. We want to show that the integral curve of $\tilde{\beta}$ starting at x_0 at time 0 with velocity v_0 exits X in a time of order δ so that $x_+(x) - \tilde{x}_+(x)$ is of order δ.

In an appropriate system of coordinates, we have $x_0 = (-y, 0)$ and $x_+ = (0, c)$ where $0 < y \leq C\delta t_+$. We verify that

$$\tilde{\varphi}_{x_0}(t) - x_0 - tv_0 = \int_0^t \int_0^s \partial_t^2 \tilde{\varphi}_x(u) du ds.$$

$\tilde{\varphi}_{x_0}(t)$ will be outside of the convex domain X as soon as its first component becomes non-negative, which implies that

$$-y + tv \cdot e_1 + e_1 \cdot \int_0^t \int_0^s \partial_t^2 \tilde{\varphi}_x(u) du ds \leq 0,$$

or equivalently

$$t_+ t \leq C_3 t_+ \delta + C_4 t^2.$$

Now take $t = 2C_3\delta$ so that the above constraint becomes

$$t_+ \leq 4C_4C_3\delta.$$

It remains to choose $t_+ > 4C_4C_3\delta$ to obtain the existence of a time t so that $\tilde{\varphi}_{x_0}(2C_3\delta)$ is outside of X. This shows that the distance traveled by $\tilde{\varphi}_{x_0}$ is of size δ so that $|x_+(x) - \tilde{x}_+(x)| \leq C\delta$.

We have treated the case $t_+(x) > \alpha\delta$ for some constant α sufficiently large. It remains to address the case $t_+(x) < \alpha\delta$. For this, a sufficiently positive curvature of ∂X is necessary and we use Lemma 3.9. Indeed, we know that

$$|\varphi_x(t_+(x)) - \tilde{\varphi}_x(t_+(x))| \leq C\alpha\delta^2.$$

We also know that $\beta(x_+(x)) \cdot n(x_+(x)) \geq 2\alpha C_1 \delta$. We can then invoke the second result of Lemma 3.9 and obtain that $|x_+(x) - \tilde{x}_+(x)| \leq C\delta$.

At this stage, we have thus proved that independent of $t_+(x)$, $|x_+(x) - \tilde{x}_+(x)| \leq C\delta$ with C a constant independent of δ. The proof of the above result shows that $|t_+(x) - \tilde{t}_+(x)| \leq C\delta$ as well. Higher order derivatives are now treated in a similar fashion. We have seen that

$$|W(t_+) - \tilde{W}(t_+)| \leq Ct_+ \|D_x\beta - D_x\tilde{\beta}\|_{C^0(\bar{X})}.$$

Since $\tilde{W}(t)$ is of class C^1 and $|t_+(x) - \tilde{t}_+(x)| \leq C\delta$, we deduce that

$$|W(t_+) - \tilde{W}(\tilde{t}_+)| \leq C\|\beta - \tilde{\beta}\|_{C^1(\bar{X})}$$

which is equivalent to

$$\|D_x x_+ - D_x \tilde{x}_+\|_{C^0(\bar{X})} \leq C\|\beta - \tilde{\beta}\|_{C^1(\bar{X})}.$$

Higher-order derivatives are treated in exactly the same manner providing a bound for $k-1$ derivatives of $x_+ - \tilde{x}_+$ in the uniform norm.

The error on t_+ is obtained as follows. We note that

$$\varphi_x(t_+) - \tilde{\varphi}_x(\tilde{t}_+) = (x_+ - \tilde{x}_+)(x).$$

After differentiation in space, we obtain

$$W(t_+(x))D_x t_+(x) - \tilde{W}(\tilde{t}_+(x))D_x \tilde{t}_+(x) = D_x(x_+ - \tilde{x}_+)(x).$$

Since W is Lipschitz, and $t_+ - \tilde{t}_+$ is small, this implies that

$$W(t_+(x))(D_x t_+(x) - D_x \tilde{t}_+(x)) = O(\delta).$$

We have that $\dot{W}$ is of order $|\mathfrak{k}|^{-1}$ and that $W(0) = I$ so that for $|\mathfrak{k}|$ sufficiently large, $W(t_+(x))$ is invertible. This implies

$$(D_x t_+(x) - D_x \tilde{t}_+(x)) = O(\delta).$$

Higher-order derivative are treated in the same manner by using the Leibniz product rule and the invertibility of $W(t_+(x))$. This concludes the proof of the lemma. □

Proof of Lemma 3.9. Instead of running the characteristics forward from x to $x_+(x)$, we run the characteristics backwards from $x_+(x)$ to $x_-(x)$ and show that the time spent in X is controlled by the angle the trajectories makes with the normal to X at $x_+(x) \in \partial X$. More precisely, we set $y = x_+(x)$ and $v = \dot{\varphi}_x(t_+(x)) = \beta(x_+(x))$ and run characteristics backwards.

From the equality

$$\dot{\varphi}_x(-t) = v - \int_0^t \ddot{\varphi}_x(-s)ds,$$

we deduce that

$$|\dot{\varphi}_x(-t) - v| \leq \frac{C_0 t}{|\mathfrak{k}|},$$

and hence

$$\big|\varphi_x(-t) - \big(y - tv\big)\big| \leq \frac{C_0 t^2}{2|\mathfrak{k}|}.$$

Let t_m the time it takes from $x_+(x)$ to $x_-(x)$. We obviously have that $t_+(x) \leq t_m$. Let $B_y(R)$ the (unique) ball of radius R tangent to ∂X at $y \in \partial X$ and such that $X \subset B_y(R)$.

In a system of coordinates with $B_y(R)$ centered at 0 and $v = e_1$, we find that $|y - tv|^2 = R^2 + |v|^2 t^2 - 2rtv \cdot n(y)$. We want $\varphi_x(-t) \in X \subset B_y(R)$. This imposes that

$$|y - tv| \leq R + \frac{C_0}{2|\mathfrak{k}|} t^2.$$

Let us define t_M as the maximal time a trajectory spends in X, which is a bounded quantity. Then the above imposes that

$$|v|^2 t^2 - 2Rtv \cdot n(y) \leq \frac{RC_0}{|\mathfrak{k}|} t^2 + \frac{C_0^2 t_M^2}{4|\mathfrak{k}|^2} t^2 \leq |v|^2 \rho t^2,$$

and in other words that t_m is bounded by $\frac{2R}{|v|^2(1-\rho)}v\cdot n(y)$.

For the second result, we define $B_x(R)$ as the ball of radius R tangent to ∂X at x and such that $X\subset B_x(R)$. We again have that $|\varphi_{x_0}(t)-(x_0+v_0t)|$ is bounded by $\rho|v_0|^2t^2$. In a system of coordinates where $n(x)=e_2$, $x=Re_2$, and $v_0=(v_0\cdot e_1)e_1+(v_0\cdot e_2)e_2$, we obtain that $\varphi_{x_0}(t)\in X\subset B_x(R)$ implies that

$$|x+tv_0|\le|x-x_0|+R+\frac{C_0}{2|\mathfrak{k}|}t^2.$$

This is equivalent to

$$2tRv_0\cdot e_2+t^2|v_0|^2\le|x-x_0|^2+\frac{C_0^2t_M^2}{4|\mathfrak{k}|^2}t^2+2|x-x_0|R+2|x-x_0|\frac{C_0t_M^2}{2|\mathfrak{k}|}+\frac{RC_0}{|\mathfrak{k}|}t^2.$$

This implies that

$$(1-\rho)|v|^2t^2\le C|x-x_0|+2tR|v\cdot e_2|\le C(\delta^2+t\delta),$$

for some constants C that can be made explicit. Solving this quadratic inequality yields that t_m is bounded as prescribed. □

3.5. ISID with two real-valued measurements

We are now in a position to state our main stability result in the presence of one well chosen complex-valued internal data. We fix $\mathfrak{k}$ and let $\rho=\mathfrak{k}+i\mathfrak{k}^\perp$. We define u as the solution to (29) with the complex-valued illumination g close to $u_{\rho|\partial X}$. The complex-valued internal data are then $d=\mu u$. As before, we assume that $k\ge1$ and $\mu\in C^{k+1}(\bar X)$ so that $d\in C^{k+1}(\bar X;\mathbb{C})$, which implies that β and γ defined in (53) are of class $C^k(\bar X)$.

The results of Proposition 3.7 yield the following result:

Theorem 3.10 *Let us assume that (μ,q) and $(\tilde\mu,\tilde q)$ are elements in $\mathcal{P}$ and that $\|g-u_{\rho|\partial X}\|_{C^0(\bar X)}$ is sufficiently small so that u does not vanish on X. Under the hypotheses of Proposition 3.7 and assuming that $|\mathfrak{k}|\ge|\mathfrak{k}_0|$ with $|\mathfrak{k}_0|$ sufficiently large, we have that*

$$\|\mu-\tilde\mu\|_{C^{k-1}(\bar X)}\le C\|d-\tilde d\|_{(C^k(\bar X;\mathbb{C}))}. \tag{62}$$

Moreover, we have the following stability result provided that $k\ge3$:

$$\|q-\tilde q\|_{C^{k-3}(\bar X)}\le C\|d-\tilde d\|_{(C^k(\bar X;\mathbb{C}))}. \tag{63}$$

Proof. Let us define $\mu_0=d_{|\partial X}/g$ and $\tilde\mu=\tilde d_{|\partial X}/g$. By assumption, g does not vanish on ∂X. We thus deduce that $\|\mu_0-\tilde\mu_0\|_{C^k(\partial X)}$ is controlled by $\|d-\tilde d\|_{(C^k(\bar X;\mathbb{C}))}$. The rest of the proof of the theorem is a direct consequence of the results obtained in the preceding section and of the proof of Theorem 3.6. □

We now make a few comments on the differences between the $2n-$ and $2-$ internal data settings. In the presence of n fields, the geometry of X is allowed to be rather general. In contrast, the $2-$ data setting requires much stronger convexity assumptions on X to avoid that integral curves of the vector field be too close to the boundary ∂X for too long, which would result in a severe lack of stability. Since the integral curves of the vector fields β and $\tilde{\beta}$ are not known a priori, more (Lipschitz) regularity is required on the vector fields to ensure that information propagates along near-by trajectories. This is the reason for the replacement of k in Theorem 3.6 by "$k-1$" in Theorem 3.10.

4. Inverse Diffusion with Internal Data (IDID)

We now return to the diffusion equation with unknown diffusion coefficient D and unknown absorption coefficient σ_a:

$$-\nabla \cdot D\nabla u + \sigma_a u = 0, \quad X, \qquad u = g, \quad \partial X.$$

The theorems stated in section 2 are straightforward consequences of the results presented in this section in a slightly more general setting.

Using the standard Liouville change of variables, $v = \sqrt{D}u$ solves

$$\Delta v + qv = 0,$$

with

$$q = -\frac{\Delta\sqrt{D}}{\sqrt{D}} - \frac{\sigma_a}{D}.$$

The internal data in photoacoustics are given by

$$d = \sigma_a u = \frac{\sigma_a}{\sqrt{D}} v = \mu v, \qquad \mu := \frac{\sigma_a}{\sqrt{D}}.$$

We assume we know $\sqrt{D}$ on ∂X. This allows us to prescribe v on ∂X and thus to reconstruct μ and q as in the preceding section. Then we find that

$$-\Delta\sqrt{D} - q\sqrt{D} = \mu, \tag{64}$$

so we can solve for $\sqrt{D}$ and then get $\sigma_a = \mu\sqrt{D}$.

Let us recall that $\sqrt{D} \in Y = H^{\frac{n}{2}+k+2+\varepsilon}(X) \subset C^{k+1}(\bar{X})$. We assume that $(D, \sigma_a) \in \mathcal{M}$ for some M. This implies that $(q, \mu) \in \mathcal{P}$ for some P. Indeed, that $\sqrt{D}$ solves (64) implies that $\Delta + q$ whose inverse is compact does not have 0 as an eigenvalue. Here, as always, $k \geq 1$.

The above calculations show the unique reconstruction of (D, σ_a) from internal data for well-chosen boundary distributions as stated in Theorem 2.1. From Theorem 3.6 in the $2n-$internal data setting, we get the following result.

Theorem 4.1 *Let $k \geq 2$ and assume that (D, σ_a) and $(\tilde{D}, \tilde{\sigma}_a)$ are in $\mathcal{M}$ with $D_{|\partial X} = \tilde{D}_{|\partial X}$ on ∂X. Then there is an open set of $2n$ real valued boundary values g in $C^{k,\alpha}(\partial X)$ for $\alpha > \frac{1}{2}$ such that we have the stability estimate*

$$\|D - \tilde{D}\|_{C^k(X)} + \|\sigma_a - \tilde{\sigma}_a\|_{C^k(X)} \leq C(M, k)\|d - \tilde{d}\|_{(C^{k+1}(X))^{2n}}. \tag{65}$$

Proof. The main result consists of getting the stability on D mentioned above. Since $k \geq 2$, we have stability of the reconstruction of q in $C^{k-2}(\bar{X})$ and of μ in $C^k(\bar{X})$ provided that the boundary conditions are well-chosen. Thus we have

$$-(\Delta + q)(\sqrt{D} - \sqrt{\tilde{D}}) = \mu - \tilde{\mu} + (q - \tilde{q})\sqrt{\tilde{D}}.$$

By elliptic regularity, we deduce that $(\sqrt{D} - \sqrt{\tilde{D}})$ is bounded in $C^k(\bar{X})$, and hence the result. □

Finally, we deduce from Theorem 3.10 the following result

Theorem 4.2 *Under the hypotheses of Theorem 3.10 and those of Theorem 4.1, we obtain in the 2-internal data setting the following result. Let* $k \geq 3$ *and* (D, σ_a) *and* $(\tilde{D}, \tilde{\sigma}_a)$ *be in* $\mathcal{M}$*. Let us assume that* $D_{|\partial X} = \tilde{D}_{|\partial X}$*.*

Then there is an open set of 2 *real-valued boundary conditions* g *in* $C^{k,\alpha}(\partial X)$ *for* $\alpha > \frac{1}{2}$ *such that we have the stability estimate*

$$\|D - \tilde{D}\|_{C^{k-1}(X)} + \|\sigma_a - \tilde{\sigma}_a\|_{C^{k-1}(X)} \leq C(M, k)\|d - \tilde{d}\|_{C^k(X;\mathbb{C})}. \tag{66}$$

The proof of the above theorem is the same as that of Theorem 4.1.

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