A Note on Torsion in Multiply-connected Sections

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In "*Variational methods for the solution of problems of equilibrium and vibrations*[1] R. Courant developed and presented the most profound way to constructing numerical solutions in mathematical physics. A theoretical analysis of torsion for the multiply connected domains appeared in pages 6 and 7 with a numerical example in the Appendix, in pages 20 and 21. A derivation of $\int\limits_{C_i} \frac{\partial u}{\partial n} ds$ is presented [2] here for the following sketch:

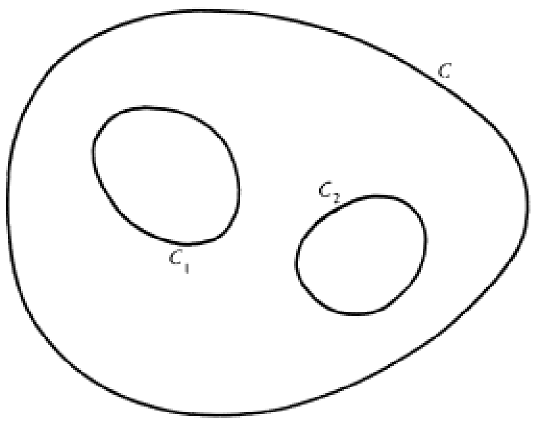

FIG. 1

The St. Venant warping function $\psi(x, y)$ is the axial displacement profile for unit twist per axis length. Around a cavity, ψ should be continuous, so also Prandtl $\varphi(x, y)$, that is defined from $\psi + \sqrt{-1}\varphi$ to be analytic in the entire region[3] A^*, where:

$$\Delta u(x,y) = 1, \ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{and } u\Big|_{(x,y)\in C_i} = c_i \text{ with } u\Big|_{(x,y)\in C} = 0 \tag{1}$$

and φ on C is proportional to $\frac{1}{2}(x^2 + y^2)$. Select the scaling factor λ so that:

$$u = \frac{(x^2+y^2)}{4} + \lambda \, \varphi; \text{ and } u\Big|_C = 0; \text{ recall: } \Delta\psi = \Delta\varphi = 0 \text{ and } \frac{\partial \psi}{\partial s} = -\frac{\partial \varphi}{\partial n}; \tag{2}$$

$$\int\limits_{C_i} \frac{\partial u}{\partial n} ds = \lambda \int\limits_{C_i} \frac{\partial \varphi}{\partial n} ds + \int\limits_{C_i} \frac{\partial}{\partial n}\left(\frac{x^2+y^2}{4}\right) ds = \int\limits_{C_i} \vec{n}.\nabla\left(\frac{x^2+y^2}{4}\right) ds - \lambda \int\limits_{C_i} \frac{\partial \psi}{\partial s} ds$$

$$= \int\limits_{A_i} \nabla.\nabla\left(\frac{x^2+y^2}{4}\right) dA + 0 \text{ since } w, \psi \text{ are continuos on } C_i \tag{3}$$

$$= \int\limits_{A_i} \Delta\left(\frac{x^2+y^2}{4}\right) dA = \int\limits_{A_i} dA = A_i; \text{ hence: } \int\limits_{C_i} \frac{\partial u}{\partial n} ds - A_i = 0 \quad \blacksquare \tag{4}$$

[1] Source: Bull. Amer. Math. Soc. Volume 49, Number 1 (1943), 1-23

[2] the equation differs from Cournat's expression on page 7 that was indicated as $\int\limits_{C_i} \frac{\partial u}{\partial n} ds + c_i \ A_i = 0$

[3] a cavity C_i encloses an area A_i the area enclosed by C is A and, A^* is the complement of $\cup_i A_i$ in A