Convex Relaxation for Optimal Distributed Control Problem

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***Abstract*—This paper is concerned with the optimal distributed control (ODC) problem for discrete-time deterministic and stochastic systems. The objective is to design a fixed-order distributed controller with a pre-specified structure that is globally optimal with respect to a quadratic cost functional. It is shown that this NP-hard problem has a quadratic formulation, which can be relaxed to a semidefinite program (SDP). If the SDP relaxation has a rank-1 solution, a globally optimal distributed controller can be recovered from this solution. By utilizing the notion of treewidth, it is proved that the nonlinearity of the ODC problem appears in such a sparse way that an SDP relaxation of this problem has a matrix solution with rank at most 3. Since the proposed SDP relaxation is computationally expensive for a large-scale system, a computationally-cheap SDP relaxation is also developed with the property that its objective function indirectly penalizes the rank of the SDP solution. Various techniques are proposed to approximate a low-rank SDP solution with a rank-1 matrix, leading to recovering a near-global controller together with a bound on its optimality degree. The above results are developed for both finite-horizon and infinite horizon ODC problems. While the finite-horizon ODC is investigated using a time-domain formulation, the infinite-horizon ODC problem for both deterministic and stochastic systems is studied via a Lyapunov formulation. The SDP relaxations developed in this work are exact for the design of a centralized controller, hence serving as an alternative for solving Riccati equations. The efficacy of the proposed SDP relaxations is elucidated in numerical examples.**

I. Introduction

The area of decentralized control is created to address the challenges arising in the control of real-world systems with many interconnected subsystems. The objective is to design a structurally constrained controller—a set of partially interacting local controllers—with the aim of reducing the computation or communication complexity of the overall controller. The local controllers of a decentralized controller may not be allowed to exchange information. The term *distributed control* is often used in lieu of decentralized control in the case where there is some information exchange between the local controllers (possibly distributed over a geographical area). It has been long known that the design of a globally optimal decentralized (distributed) controller is a daunting task because it amounts to an NP-hard optimization problem in general [1], [2]. Great effort has been devoted to investigating this highly complex problem for special types of systems, including spatially distributed systems [3]–[7], dynamically decoupled systems [8], [9], weakly coupled systems [10], and strongly connected systems [11]. Another special case that has received considerable attention is the design of an optimal static distributed controller [12], [13]. Early approaches for the optimal decentralized control problem were based on parameterization techniques [14], [15], which were then evolved into matrix optimization methods [16], [17]. In fact, with a structural assumption on the exchange of information between subsystems, the performance offered by linear static controllers may be far less than the optimal performance achievable using a nonlinear time-varying controller [1].

Due to the recent advances in the area of convex optimization, the focus of the existing research efforts has shifted from deriving a closed-form solution for the above control synthesis problem to finding a convex formulation of the problem that can be efficiently solved numerically [18]–[22]. This has been carried out in the seminal work [23] by deriving a sufficient condition named quadratic invariance, which has been specialized in [24] by deploying the concept of partially order sets. These conditions have been further investigated in several other papers [25]–[27]. A different approach is taken in the recent papers [28] and [29], where it has been shown that the distributed control problem can be cast as a convex optimization for positive systems.

There is no surprise that the decentralized control problem is computationally hard to solve. This is a consequence of the fact that several classes of optimization problems, including polynomial optimization and quadratically-constrained quadratic program as a special case, are NP-hard in the worst case. Due to the complexity of such problems, various convex relaxation methods based on linear matrix inequality (LMI), semidefinite programming (SDP), and second-order cone programming (SOCP) have gained popularity [30], [31]. These techniques enlarge the possibly non-convex feasible set into a convex set characterizable via convex functions, and then provide the exact or a lower bound on the optimal objective value. The effectiveness of these techniques has been reported in several papers. For instance, [32] shows how SDP relaxation can be used to find better approximations for maximum cut (MAX CUT) and maximum 2-satisfiability (MAX 2SAT) problems. Another approach is proposed in [33] to solve the max-3-cut problem via a complex SDP. The approaches in [32] and [33] have been generalized in several papers, including [34], [35].

Semidefinite programming relaxation usually converts an optimization with a vector variable to a convex optimization with a matrix variable, via a lifting technique. The exactness of the relaxation can then be interpreted as the existence of a low-rank (e.g., rank-1) solution for SDP relaxation. Several papers have studied the existence of a low-rank solution to matrix optimizations with linear or nonlinear (e.g., LMI) constraints. For instance, the papers [36], [37] provide upper

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bounds on the lowest rank among all solutions of a feasible LMI problem. A rank-1 matrix decomposition technique is developed in [38] to find a rank-1 solution whenever the number of constraints is small. We have shown in [39] and [40] that SDP relaxation is able to solve a large class of non-convex energy-related optimization problems performed over power networks. We related the success of the relaxation to the hidden structure of those optimizations induced by the physics of a power grid. Inspired by this positive result, we developed the notion of "nonlinear optimization over graph" in [41]–[43]. Our technique maps the structure of an abstract nonlinear optimization into a graph from which the exactness of SDP relaxation may be concluded. By adopting the graph technique developed in [41], the objective of the present work is to study the potential of SDP relaxation for the optimal distributed control problem.

In this paper, we cast the optimal distributed control (ODC) problem as a non-convex optimization problem with only quadratic scalar and matrix constraints, from which an SDP relaxation can be obtained. The goal is to show that this relaxation has a low-rank solution whose rank depends on the topology of the controller to be designed. In particular, we prove that the design of a static distributed controller with a pre-specified structure amounts to a sparse SDP relaxation with a solution of rank at most 3. This positive result is a consequence of the fact that the sparsity graph associated with the underlying optimization problem has a small treewidth. The notion of "treewidth" used in this paper could potentially help to understand how much approximation is needed to make the ODC problem tractable. This is due to a recent result stating that a rank-constrained optimization problem has an almost equivalent convex formulation whose size depends on the treewidth of a certain graph [44]. In this work, we also discuss how to round the rank-3 SDP matrix to a rank-1 matrix in order to design a near-global controller.

The results of this work hold true for both a time-domain formulation corresponding to a finite-horizon control problem and a Lyapunov-domain formulation associated with an infinite-horizon deterministic/stochastic control problem. We first investigate the ODC problem for the deterministic systems and then the ODC problem for stochastic systems. Our approach rests on formulating each of these problems as a rank-constrained optimization from which an SDP relaxation can be derived. With no loss of generality, this paper focuses on the design of a static controller. Since the proposed relaxations with guaranteed low-rank solutions are computationally expensive, we also design computationally-cheap SDP relaxations for numerical purposes. Afterwards, we develop some heuristic methods to recover a near-optimal controller from a low-rank SDP solution. Note that the computationally-cheap SDP relaxations associated with the infinite-horizon ODC are exact in both deterministic and stochastic cases for the classical (centralized) LQR and H_2 problems. Although the focus of the paper is static controllers, its results can be naturally generalized to the dynamic case as well.

We conduct case studies on a mass-spring system and 100 random systems to elucidate the efficacy of the proposed relaxations. In particular, the design of many near-optimal structured controllers with global optimality degrees above 99% will be demonstrated. An additional study is conducted on electrical power systems in our supplementary paper [45].

This work is organized as follows. The problem is introduced in Section II, and then the SDP relaxation of a quadratically-constrained quadratic program (QCQP) is studied via a graph-theoretic approach. Three different SDP relaxations of the finite-horizon deterministic ODC problem are presented for the static controller design in Section III. The infinite-horizon deterministic ODC problem is studied in Section IV. The results are generalized to an infinite-horizon stochastic ODC problem in Section V, followed by a brief discussion on dynamic controllers in Section VI. Various experiments and simulations are provided in Section VII. Concluding remarks are drawn in Section VIII.

A. *Notations*

$\mathbb{R}$, $\mathbb{S}_n$ and $\mathbb{S}_n^+$ denote the sets of real numbers, $n \times n$ symmetric matrices and $n \times n$ positive semidefinite matrices, respectively. The m by n rectangular identity matrix whose (i, j) entry is equal to the Kronecker delta δ_{ij} is denoted by $I_{m\times n}$ or alternatively I_n when $m = n$. $\operatorname{rank}\{W\}$ and $\operatorname{trace}\{W\}$ denote the rank and trace of a matrix W. The notation $W \succeq 0$ means that W is symmetric and positive semidefinite. Given a matrix W, its (l, m) entry is denoted as W_{lm}. Given a block matrix $\mathbf{W}$, its (l, m) block is shown as $\mathbf{W}_{lm}$. Given a matrix M, its Moore Penrose pseudoinverse is denoted as M^+. The superscript $(\cdot)^{\text{opt}}$ is used to show a *globally optimal value* of an optimization parameter. The symbols $(\cdot)^T$ and $\|\cdot\|$ denote the transpose and 2-norm operators, respectively. The symbols $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_F$ denote the Frobinous inner product and norm of matrices, respectively. The notation $|.|$ shows the size of a vector, the cardinality of a set or the number of vertices a graph, depending on the context. The expected value of a random variable x is shown as $\mathcal{E}\{x\}$. The submatirx of M formed by rows form the set $\mathcal{S}_1$ and columns from the set $\mathcal{S}_2$ is denoted by $M\{\mathcal{S}_1, \mathcal{S}_2\}$. The notation $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ implies that $\mathcal{G}$ is a graph with the vertex set $\mathcal{V}$ and the edge set $\mathcal{E}$.

II. Preliminaries

In this paper, the Optimal Distributed Control (ODC) problem is studied based on the following steps:

- First, the problem is cast as a non-convex optimization problem with only quadratic scalar and/or matrix constraints.
- Second, the resulting non-convex problem is formulated as a rank-constrained optimization.
- Third, a convex relaxation of the problem is derived by dropping the non-convex rank constraint.
- Last, the rank of the minimum-rank solution of the SDP relaxation is analyzed.

Since there is no unique SDP relaxation for the ODC problem, a major part of this work is devoted to designing a sparse quadratic formulation of the ODC problem with a guaranteed low-rank SDP solution. To achieve this goal, a graph is associated to each SDP, which is then sparsified to contrive a problem with a low-rank solution. Note that this paper significantly improves our recent result in [46].

A. Problem Formulation

The following variations of the Optimal Distributed Control (ODC) problem are studied in this work.

1) Finite-horizon deterministic ODC problem: Consider the discrete-time system

$$x[\tau+1] = Ax[\tau] + Bu[\tau], \qquad \tau = 0,1,\dots,p-1 \tag{1a}$$
$$y[\tau] = Cx[\tau], \qquad \tau = 0,1,\dots,p \tag{1b}$$

with the known matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{r\times n}$, and $x[0] = x_0 \in \mathbb{R}^n$, where p is the terminal time. The goal is to design a distributed static controller $u[\tau] = Ky[\tau]$ minimizing a quadratic cost function under the constraint that the controller gain K must belong to a given linear subspace $\mathcal{K} \subseteq \mathbb{R}^{m\times r}$. The set $\mathcal{K}$ captures the sparsity structure of the unknown constrained controller and, more specifically, it contains all $m \times r$ real-valued matrices with forced zeros in certain entries. The cost function

$$\sum_{\tau=0}^{p} \left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau] \right) + \alpha \|K\|_F^2 \tag{2}$$

is considered in this work, where α is a nonnegative scalar, and Q and R are positive-semidefinite matrices. This problem will be studied in Section III.

Remark 1. *The third term in the objective function of the ODC problem is a soft penalty term aimed at avoiding a high-gain controller. Instead of this soft penalty, we could impose a hard constraint $\|K\|_F \leq \beta$, for a given number β. The method to be developed later can be adopted for the modified case.*

2) Infinite-horizon deterministic ODC problem: The infinite-horizon ODC problem corresponds to the case $p = +\infty$ subject to the additional constraint that the controller must be stabilizing. This problem will be studied through a Lyapunov domain formulation in Section IV.

3) Infinite-horizon stochastic ODC problem: Consider the discrete-time stochastic system

$$x[\tau+1] = Ax[\tau] + Bu[\tau] + Ed[\tau], \qquad \tau = 0,1,\dots \tag{3a}$$
$$y[\tau] = Cx[\tau] + Fv[\tau], \qquad \tau = 0,1,\dots \tag{3b}$$

with the known matrices A, B, C, E, and F, where $d[\tau]$ and $v[\tau]$ denote the input disturbance and measurement noise, which are assumed to be zero-mean white-noise random processes. The ODC problem for the above system will be investigated in Section V.

The extension of the above results to the design of dynamic controllers will be briefly discussed in Section VI.

B. Graph Theory Preliminaries

Definition 1. *For two simple graphs $\mathcal{G}_1 = (\mathcal{V}, \mathcal{E}_1)$ and $\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$ with the same set of vertices, their union is defined as $\mathcal{G}_1 \cup \mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_1 \cup \mathcal{E}_2)$.*

Definition 2. *The representative graph of an $n \times n$ symmetric matrix W, denoted by $\mathcal{G}(W)$, is a simple graph with n vertices whose edges are specified by the locations of the nonzero off-diagonal entries of W. In other words, two disparate vertices i and j are connected if W_{ij} is nonzero.*

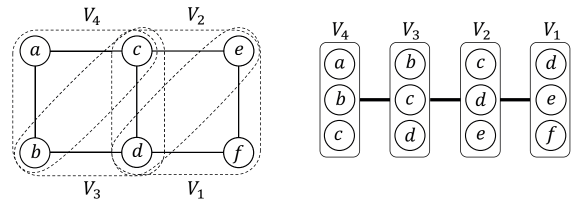


Fig. 1: A minimal tree decomposition for a ladder graph.

Consider a graph $\mathcal{G}$ identified by a set of "vertices" and a set of edges. This graph may have cycles in which case it cannot be a tree. Using the notion to be explained below, we can map $\mathcal{G}$ into a tree $\mathcal{T}$ identified by a set of "nodes" and a set of edges where each node of $\mathcal{T}$ contains a group of vertices of $\mathcal{G}$.

Definition 3 (Treewidth). *Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a tree $\mathcal{T}$ is called* a tree decomposition of $\mathcal{G}$ *if it satisfies the following properties:*

1) *Every node of $\mathcal{T}$ corresponds to and is identified by a subset of $\mathcal{V}$.*
2) *Every vertex of $\mathcal{G}$ is a member of at least one node of $\mathcal{T}$.*
3) *For every edge (i, j) of $\mathcal{G}$, there should be a node in $\mathcal{T}$ containing vertices i and j simultaneously.*
4) *Given an arbitrary vertex k of $\mathcal{G}$, the subgraph induced by all nodes of $\mathcal{T}$ containing vertex k must be connected (more precisely, a tree).*

Each node of $\mathcal{T}$ is a bag (collection) of vertices of $\mathcal{G}$ and hence it is referred to as bag. The width of $\mathcal{T}$ is the cardinality of its biggest bag minus one. The treewidth of $\mathcal{G}$ is the minimum width over all possible tree decompositions of $\mathcal{G}$ and is denoted by $\mathrm{tw}(\mathcal{G})$.

Every graph has a trivial tree decomposition with one single bag consisting of all its vertices. Figure 1 shows a graph $\mathcal{G}$ with 6 vertices named a, b, c, d, e, f, together with its minimal tree decomposition $\mathcal{T}$. Every node of $\mathcal{T}$ is a set containing three members of $\mathcal{V}$. The width of this decomposition is therefore equal to 2. Observe that the edges of the tree decomposition are chosen in such a way that every subgraph induced by all bags containing each member of $\mathcal{V}$ is a tree (as required by Property 4 stated before).

Note that if $\mathcal{G}$ is a tree itself, it has a minimal tree decomposition $\mathcal{T}$ such that: each bag corresponds to two connected vertices of $\mathcal{G}$ and every two adjacent bags in $\mathcal{T}$ share a vertex in common. Therefore, the treewidth of a tree is equal to 1. The reader is referred to [47] for a comprehensive literature review on treewidth.

C. SDP Relaxation

The objective of this subsection is to study SDP relaxation of a quadratically-constrained quadratic program (QCQP) using a graph-theoretic approach. Consider the standard nonconvex QCQP problem

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \qquad f_0(x) \tag{4a}$$
$$\text{subject to} \qquad f_k(x) \leq 0, \qquad k = 1, \dots, q, \tag{4b}$$

where $f_k(x) = x^T A_k x + 2b_k^T x + c_k$ for $k = 0, \dots, q$. Define

$$F_k \triangleq \begin{bmatrix} c_k & b_k^T \\ b_k & A_k \end{bmatrix}. \tag{5}$$

Each f_k has the linear representation $f_k(x) = \langle F_k, W\rangle$ for the following choice of W:

$$W \triangleq [x_0 \quad x^T]^T [x_0 \quad x^T] \tag{6}$$

where x_0 is considered as 1. On the other hand, an arbitrary matrix $W \in \mathbb{S}_{n+1}$ can be factorized as (6) if and only if it satisfies three properties: $W_{11} = 1$, $W \succeq 0$, and $\operatorname{rank}\{W\} = 1$. In this representation of QCQP, the rank constraint carries all the nonconvexity. Neglecting this constraint yields the convex problem

$$\begin{aligned} \underset{W\in\mathbb{S}_{n+1}}{\text{minimize}} \quad & \langle F_0, W\rangle && \text{(7a)}\\ \text{subject to} \quad & \langle F_k, W\rangle \le 0 \qquad k = 1,\dots,q, && \text{(7b)}\\ & W_{11} = 1, && \text{(7c)}\\ & W \succeq 0, && \text{(7d)} \end{aligned}$$

known as a semidefinite programming (SDP) relaxation of the QCQP (4). The existence of a rank-1 solution for an SDP relaxation guarantees the equivalence of the original QCQP and its relaxed problem.

D. Connection Between Rank and Sparsity

To explore the rank of the minimum-rank solution of SDP relaxation, define $\mathcal{G} = \mathcal{G}(F_0) \cup \cdots \cup \mathcal{G}(F_q)$ as the **sparsity graph** associated with the problem (7). The graph $\mathcal{G}$ describes the zero-nonzero pattern of the matrices $F_0, \dots, F_q$, or alternatively captures the sparsity level of the QCQP problem (4). Let $\mathcal{T} = (\mathcal{V}_\mathcal{T}, \mathcal{E}_\mathcal{T})$ be a tree decomposition of $\mathcal{G}$. Denote its width as t and its bags as $\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_{|\mathcal{T}|}$. It is known that given such a decomposition, every solution $W^{\text{ref}} \in \mathbb{S}_{n+1}$ of the SDP problem (7) can be transformed into a solution W^{opt} whose rank is upper bounded by $t + 1$ [37]. To perform this transformation, a suitable polynomial-time recursive algorithm will be proposed below.

Rank reduction algorithm:

1) Set $\mathcal{T}' := \mathcal{T}$ and $W := W^{\text{ref}}$.
2) If $\mathcal{T}'$ has a single node, then consider W^{opt} as W and terminate; otherwise continue to the next step.
3) Choose a pair of bags $\mathcal{B}_i, \mathcal{B}_j$ of $\mathcal{T}'$ such that $\mathcal{B}_i$ is a leaf of $\mathcal{T}'$ and $\mathcal{B}_j$ is its unique neighbor.
4) Using the notation $W\{\cdot,\cdot\}$ introduced in Section I-A, define

$$O \triangleq W\{\mathcal{B}_i \cap \mathcal{B}_j, \mathcal{B}_i \cap \mathcal{B}_j\} \tag{8a}$$
$$V_i \triangleq W\{\mathcal{B}_i \setminus \mathcal{B}_j, \mathcal{B}_i \cap \mathcal{B}_j\} \tag{8b}$$
$$V_j \triangleq W\{\mathcal{B}_j \setminus \mathcal{B}_i, \mathcal{B}_i \cap \mathcal{B}_j\} \tag{8c}$$
$$H_i \triangleq W\{\mathcal{B}_i \setminus \mathcal{B}_j, \mathcal{B}_i \setminus \mathcal{B}_j\} \in \mathbb{R}^{n_i\times n_i} \tag{8d}$$
$$H_j \triangleq W\{\mathcal{B}_j \setminus \mathcal{B}_i, \mathcal{B}_j \setminus \mathcal{B}_i\} \in \mathbb{R}^{n_j\times n_j} \tag{8e}$$
$$S_i \triangleq H_i - V_i O^+ V_i^T = Q_i \Lambda_i Q_i^T \tag{8f}$$
$$S_j \triangleq H_j - V_j O^+ V_j^T = Q_j \Lambda_j Q_j^T \tag{8g}$$

where $Q_i\Lambda_iQ_i^T$ and $Q_j\Lambda_jQ_j^T$ denote the eigenvalue decompositions of S_i and S_j with the diagonals of Λ_i and Λ_j arranged in descending order. Then, update a part of W as follows:

$$\begin{aligned} W\{\mathcal{B}_j \setminus \mathcal{B}_i, \mathcal{B}_i \setminus \mathcal{B}_j\} := {} & V_j O^+ V_i^T \\ & + Q_j \sqrt{\Lambda_j}\ \ I_{n_j\times n_i} \sqrt{\Lambda_i}\ Q_i^T \end{aligned}$$

and update $W\{\mathcal{B}_i \setminus \mathcal{B}_j, \mathcal{B}_j \setminus \mathcal{B}_i\}$ accordingly to preserve the Hermitian property of W.
5) Update $\mathcal{T}'$ by merging $\mathcal{B}_i$ into $\mathcal{B}_j$, i.e., replace $\mathcal{B}_j$ with $\mathcal{B}_i \cup \mathcal{B}_j$ and then remove $\mathcal{B}_i$ from $\mathcal{T}'$.
6) Go back to step 2.

Theorem 1. *The output of the rank reduction algorithm, denoted as W^{opt}, is a solution of the SDP problem* (7) *whose rank is smaller than or equal to $t + 1$.*

Proof. Consider one run of Step 4 of the rank reduction algorithm. Our first objective is to show that $W\{\mathcal{B}_i \cup \mathcal{B}_j, \mathcal{B}_i \cup \mathcal{B}_j\}$ is a positive semidefinite matrix whose rank is upper bounded by the maximum ranks of $W\{\mathcal{B}_i, \mathcal{B}_i\}$ and $W\{\mathcal{B}_j, \mathcal{B}_j\}$. To this end, one can write:

$$W\{\mathcal{B}_i \cup \mathcal{B}_j, \mathcal{B}_i \cup \mathcal{B}_j\} = \begin{bmatrix} O & V_i^T & V_j^T \\ V_i & H_i & Z^T \\ V_j & Z & H_j \end{bmatrix} \tag{9}$$

where $Z \triangleq W\{\mathcal{B}_j \setminus \mathcal{B}_i, \mathcal{B}_i \setminus \mathcal{B}_j\}$. Now, define

$$\begin{aligned} S &\triangleq \begin{bmatrix} H_i & Z^T \\ Z & H_j \end{bmatrix} - \begin{bmatrix} V_i \\ V_j \end{bmatrix} O^+ \begin{bmatrix} V_i^T & V_j^T \end{bmatrix} \\ &= \begin{bmatrix} Q_i & 0 \\ o & Q_j \end{bmatrix} N \begin{bmatrix} Q_i^T & 0 \\ 0 & Q_j^T \end{bmatrix} \end{aligned} \tag{10}$$

where

$$N \triangleq \begin{bmatrix} \Lambda_i & \sqrt{\Lambda_i}\ \ I_{n_i\times n_j} \sqrt{\Lambda_j} \\ \sqrt{\Lambda_j}\ \ I_{n_j\times n_i} \sqrt{\Lambda_i} & \Lambda_j \end{bmatrix} \tag{11}$$

It is straightforward to verify that

$$\operatorname{rank}\{S\} = \operatorname{rank}\{N\} = \max\{\operatorname{rank}\{S_i\}, \operatorname{rank}\{S_j\}\}$$

On the other hand, the Schur complement formula yields:

$$\begin{aligned} &\operatorname{rank}\{W\{\mathcal{B}_i, \mathcal{B}_i\}\} = \operatorname{rank}\{O^+\} + \operatorname{rank}\{S_i\} \\ &\operatorname{rank}\{W\{\mathcal{B}_j, \mathcal{B}_j\}\} = \operatorname{rank}\{O^+\} + \operatorname{rank}\{S_j\} \\ &\operatorname{rank}\{W\{\mathcal{B}_i \cup \mathcal{B}_j, \mathcal{B}_i \cup \mathcal{B}_j\}\} = \operatorname{rank}\{O^+\} + \operatorname{rank}\{S\} \end{aligned}$$

(see [48]). Combining the above equations leads to the conclusion that the rank of $W\{\mathcal{B}_i \cup \mathcal{B}_j, \mathcal{B}_i \cup \mathcal{B}_j\}$ is upper bounded by the maximum ranks of $W\{\mathcal{B}_i, \mathcal{B}_i\}$ and $W\{\mathcal{B}_j, \mathcal{B}_j\}$. On the other hand, since N is positive semidefinite, it follows from (10) that $W\{\mathcal{B}_i \cup \mathcal{B}_j, \mathcal{B}_i \cup \mathcal{B}_j\} \succeq 0$. A simple induction concludes that the output W^{opt} of the matrix completion algorithm is a positive semidefinite matrix whose rank is upper bounded by $t + 1$. The proof is completed by noting that W^{opt} and W^{ref} share the same values on their diagonals and those off-diagonal locations corresponding to the edges of the sparsity graph $\mathcal{G}$. $\square$

III. Finite-horizon Deterministic ODC Problem

The primary objective of the ODC problem is to design a structurally constrained gain K. Assume that the matrix K has l free entries to be designed. Denote these parameters as $h_1, h_2, \ldots, h_l$. To formulate the ODC problem, the space of permissible controllers can be characterized as

$$\mathcal{K} \triangleq \left\{ \sum_{i=1}^{l} h_i N_i \;\middle|\; h \in \mathbb{R}^l \right\}, \tag{12}$$

for some (fixed) 0-1 matrices $N_1, \ldots, N_l \in \mathbb{R}^{m\times r}$. Now, the ODC problem can be stated as follows.

Finite-Horizon ODC Problem: Minimize

$$\sum_{\tau=0}^{p} \left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau] \right) + \alpha \|K\|_F^2 \tag{13a}$$

subject to

$$x[0] = x_0 \tag{13b}$$

$$x[\tau+1] = Ax[\tau] + Bu[\tau] \qquad \tau = 0,1,\ldots,p-1 \tag{13c}$$

$$y[\tau] = Cx[\tau] \qquad \tau = 0,1,\ldots,p \tag{13d}$$

$$u[\tau] = Ky[\tau] \qquad \tau = 0,1,\ldots,p \tag{13e}$$

$$K = h_1 N_1 + \ldots + h_l N_l \tag{13f}$$

over the variables $\{x[\tau] \in \mathbb{R}^n\}_{\tau=0}^p$, $\{y[\tau] \in \mathbb{R}^r\}_{\tau=0}^p$, $\{u[\tau] \in \mathbb{R}^m\}_{\tau=0}^p$, $K \in \mathbb{R}^{m\times r}$ and $h \in \mathbb{R}^l$.

A. Sparsification of ODC Problem

The finite-horizon ODC is naturally a QCQP problem. Consider an arbitrary SDP relaxation of the ODC problem and let $\mathcal{G}$ be the sparsity graph of this relaxation. Due to existence of nonzero off-diagonal elements in Q and R, certain edges (and probably cycles) may exist in the subgraphs of $\mathcal{G}$ associated with the state and input vectors $x[\tau]$ and $u[\tau]$. Under this circumstance, the treewidth of $\mathcal{G}$ could be as high as n. To understand the effect of a non-diagonal controller K, consider the case $m = r = 2$ and assume that the controller K under design has three free elements as follows:

$$K = \begin{bmatrix} K_{11} & K_{12} \\ 0 & K_{22} \end{bmatrix} \tag{14}$$

(i.e., $h_1 = K_{11}$, $h_2 = K_{12}$ and $h_3 = K_{22}$). Figure 2 shows a part of the graph $\mathcal{G}$. It can be observed that this subgraph is acyclic for $K_{12} = 0$ but has a cycle as soon as K_{12} becomes a free parameter. As a result, the treewidth of $\mathcal{G}$ is contingent upon the zero pattern of K. In order to guarantee existence of a low rank solution, we diagonalize Q, R and K through a reformulation of the ODC problem. Note that this transformation is redundant if Q, R and K are all diagonal.

Let $Q_d \in \mathbb{R}^{n\times n}$ and $R_d \in \mathbb{R}^{m\times m}$ be the respective eigenvector matrices of Q and R, i.e.,

$$Q = Q_d^T \Lambda_Q Q_d, , \qquad R = R_d^T \Lambda_R R_d \tag{15}$$

where $\Lambda_Q \in \mathbb{R}^{n\times n}$ and $\Lambda_R \in \mathbb{R}^{m\times m}$ are diagonal matrices. Notice that there exist two constant binary matrices $\Phi_1 \in \mathbb{R}^{m\times l}$ and $\Phi_2 \in \mathbb{R}^{l\times r}$ such that

$$\mathcal{K} = \left\{ \Phi_1 \mathrm{diag}\{h\} \Phi_2 \mid h \in \mathbb{R}^l \right\}, \tag{16}$$

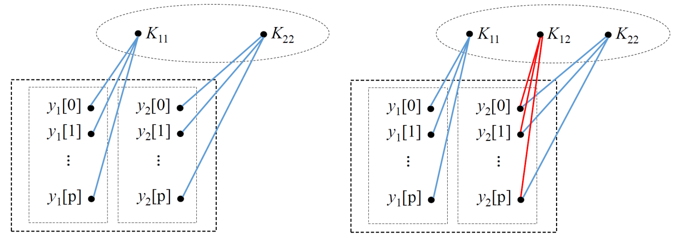


Fig. 2: Effect of a nonzero off-diagonal entry of the controller K on the sparsity graph of the finite-horizon ODC: (a) a subgraph of $\mathcal{G}$ for the case where K_{11} and K_{22} are the only free parameters of the controller K, (b) a subgraph of $\mathcal{G}$ for the case where K_{12} is also a free parameter of the controller.

where $\mathrm{diag}\{h\}$ denotes a diagonal matrix whose diagonal entries are inherited from the vector h [49]. Now, a sparse formulation of the ODC problem can be obtained in terms of the matrices

$$\bar{A} \triangleq Q_d A Q_d^T, \qquad \bar{B} \triangleq Q_d B R_d^T,$$
$$\bar{C} \triangleq \Phi_2 C Q_d^T, \qquad \bar{x}_0 \triangleq Q_d x_0,$$

and the new set of variables $\bar{x}[\tau] \triangleq Q_d x[\tau]$, $\bar{y}[\tau] \triangleq \Phi_2 y[\tau]$ and $\bar{u}[\tau] \triangleq R_d u[\tau]$ for every $\tau = 0,1,\ldots,p$.

Reformulated Finite-Horizon ODC Problem: Minimize

$$\sum_{\tau=0}^{p} \left(\bar{x}[\tau]^T \Lambda_Q \bar{x}[\tau] + \bar{u}[\tau]^T \Lambda_R \bar{u}[\tau] \right) + \alpha \|h\|_2^2 \tag{17a}$$

subject to

$$\bar{x}[0] = \bar{x}_0 \times z^2 \tag{17b}$$

$$\bar{x}[\tau+1] = \bar{A}\bar{x}[\tau] + \bar{B}\bar{u}[\tau] \qquad \tau = 0,1,\ldots,p-1 \tag{17c}$$

$$\bar{y}[\tau] = \bar{C}\bar{x}[\tau] \qquad \tau = 0,1,\ldots,p \tag{17d}$$

$$\bar{u}[\tau] = R_d \Phi_1 \mathrm{diag}\{h\} \bar{y}[\tau] \qquad \tau = 0,1,\ldots,p \tag{17e}$$

$$z^2 = 1 \tag{17f}$$

over the variables $\{\bar{x}[\tau] \in \mathbb{R}^n\}_{\tau=0}^p$, $\{\bar{y}[\tau] \in \mathbb{R}^l\}_{\tau=0}^p$, $\{\bar{u}[\tau] \in \mathbb{R}^m\}_{\tau=0}^p$, $h \in \mathbb{R}^l$ and $z \in \mathbb{R}$.

To cast the reformulated finite-horizon ODC as a quadratic optimization, define

$$w \triangleq \begin{bmatrix} z \ h^T \ \bar{x}^T \ \bar{u}^T \ \bar{y}^T \end{bmatrix}^T \in \mathbb{R}^{n_w} \tag{18}$$

where $\bar{x} \triangleq \begin{bmatrix} \bar{x}[0]^T \cdots \bar{x}[p]^T \end{bmatrix}^T$, $\bar{u} \triangleq \begin{bmatrix} \bar{u}[0]^T \cdots \bar{u}[p]^T \end{bmatrix}^T$, $\bar{y} \triangleq \begin{bmatrix} \bar{y}[0]^T \cdots \bar{y}[p]^T \end{bmatrix}^T$ and $n_w \triangleq 1+l+(p+1)(n+l+m)$. The scalar auxiliary variable z plays the role of number 1 (it suffices to impose the additional quadratic constraint (17f) as opposed to $z = 1$ without affecting the solution).

B. SDP Relaxations of ODC Problem

In this subsection, two SDP relaxations are proposed for the reformulated finite-horizon ODC problem given in (17). For the first relaxation, there is a guarantee on the rank of the solution. In contrast, the second relaxation offers a tighter lower bound on the optimal cost of the ODC problem, but its solution might be high rank and therefore its rounding to a rank-1 solution could be more challenging.

1) Sparse SDP relaxation: Let $e_1, \ldots, e_{n_w}$ denote the standard basis for $\mathbb{R}^{n_w}$. The ODC problem consists of $n_l \triangleq (p+1)(n+l)$ linear constraints given in (17b), (17c) and (17d), which can be formulated as

$$D^T w = 0 \tag{19}$$

for some matrix $D \in \mathbb{R}^{n_w \times n_l}$. Moreover, the objective function (17a) and the constraints in (17e) and (17f) are all quadratic and can be expressed in terms of some matrices $M \in \mathbb{S}_{n_w}$, $\{M_i[\tau] \in \mathbb{S}_{n_w}\}_{i=1,\ldots,m;\, \tau=0,1,\ldots,p}$ and $E \triangleq e_1 e_1^T$. This leads to the following formulation of (17).

Sparse Formulation of ODC Problem: Minimize

$$\langle M, ww^T \rangle \tag{20a}$$

subject to

$$D^T w = 0 \tag{20b}$$
$$\langle M_i[\tau], ww^T \rangle = 0 \quad i = 1, \ldots, m, \quad \tau = 0, 1, \ldots, p \tag{20c}$$
$$\langle E, ww^T \rangle = 1 \tag{20d}$$

with the variable $w \in \mathbb{R}^{n_w}$.

For every $j = 1, \ldots, n_l$, define

$$D_j = D_{:,j} e_j^T + e_j D_{:,j}^T \tag{21}$$

where $D_{:,j}$ denotes the j-th column of D. An SDP relaxation of (20) will be obtained below.

Sparse Relaxation of Finite-Horizon ODC: Minimize

$$\langle M, W \rangle \tag{22a}$$

subject to

$$\langle D_j, W \rangle = 0 \quad j = 1, \ldots, n_l \tag{22b}$$
$$\langle M_i[\tau], W \rangle = 0 \quad i = 1, \ldots, m, \quad \tau = 0, 1, \ldots, p \tag{22c}$$
$$\langle E, W \rangle = 1 \tag{22d}$$
$$W \succeq 0 \tag{22e}$$

with the variable $W \in \mathbb{S}_{n_w}$.

The problem (22) is a convex relaxation of the QCQP problem (20). The sparsity graph of this problem is equal to

$$\mathcal{G} = \mathcal{G}(D_1) \cup \ldots \cup \mathcal{G}(D_{n_l}) \cup \mathcal{G}(M_1[0]) \cup \ldots \cup \mathcal{G}(M_m[0]) \\ \cup \ldots \cup \mathcal{G}(M_1[p]) \cup \ldots \cup \mathcal{G}(M_m[p])$$

where the vertices of $\mathcal{G}$ correspond to the entries of w. In particular, the vertex set of $\mathcal{G}$ can be partitioned into five vertex subsets, where subset 1 consists of a single vertex associated with the variable z and subsets 2-5 correspond to the vectors $\bar{x}$, $\bar{u}$, $\bar{y}$ and h, respectively. The underlying sparsity graph $\mathcal{G}$ for the sparse formulation of the ODC problem is drawn in Figure 3, where each vertex of the graph is labeled by its corresponding variable. To maintain the readability of the graph, some edges of vertex z are not shown in the picture. Indeed, z is connected to all vertices corresponding to the elements of $\bar{x}$, $\bar{u}$ and $\bar{y}$ due to the linear terms in (20b).

Theorem 2. *The sparsity graph of the sparse relaxation of the finite-horizon ODC problem has treewidth 2.*

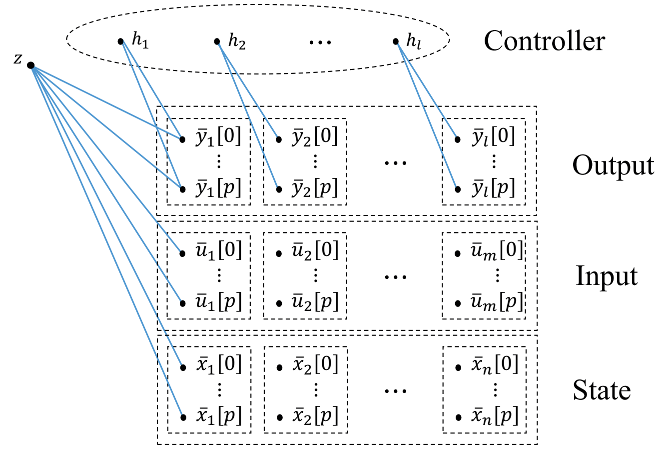


Fig. 3: Sparsity graph of the problem (22) (some edges of vertex z are not shown to improve the legibility of the graph).

Proof. It follows from the graph drawn in Figure 3 that removing vertex z from the sparsity graph $\mathcal{G}$ makes the remaining subgraph acyclic. This implies that the treewidth of $\mathcal{G}$ is at most 2. On the other hand, the treewidth cannot be 1 in light of the cycles of the graph. □

Consider the variable W of the SDP relaxation (22). The exactness of this relaxation is tantamount to the existence of an optimal rank-1 solution W^{opt} for (22). In this case, an optimal vector w^{opt} for the ODC problem can be recovered by decomposing W^{opt} as $(w^{\text{opt}})(w^{\text{opt}})^T$ (note that w has been defined in (18)). The following observation can be made.

Corollary 1. *The sparse relaxation of the finite-horizon ODC problem has a matrix solution with rank at most 3.*

Proof. This corollary is an immediate consequence of Theorems 1 and 2. □

Remark 2. *Since the treewidth of the sparse relaxation of the finite-horizon ODC problem (22) is equal to 2, it is possible to significantly reduce its computational complexity. More precisely, the complicating constraint $W \succeq 0$ can be replaced by positive semidefinite constraints on certain 3×3 submatrices of W, as given below:*

$$W\{\mathcal{B}_i, \mathcal{B}_i\} \succeq 0, \quad k = 1, \ldots, |\mathcal{T}| \tag{23}$$

where $\mathcal{T}$ is an optimal tree decomposition of the sparsity graph $\mathcal{G}$, and $\mathcal{B}_1, \ldots, \mathcal{B}_{|\mathcal{T}|}$ denote its bags. After this simplification of the hard constraint $W \succeq 0$, a quadratic number of entries of W turn out to be redundant (not appearing in any constraint) and can be removed from the optimization [37], [50].

2) Dense SDP relaxation: Define $D^\perp \in \mathbb{R}^{n_w \times (n_w - n_l)}$ as an arbitrary full row rank matrix satisfying the relation $D^T D^\perp = 0$. It follows from (20b) that every feasible vector w satisfies the equation $w = D^\perp \tilde{w}$, for a vector $\tilde{w} \in \mathbb{R}^{(n_w - n_l)}$. Define

$$\tilde{M} = (D^\perp)^T M D^\perp \tag{24a}$$
$$\tilde{M}_i[\tau] = (D^\perp)^T M_i[\tau] D^\perp \tag{24b}$$
$$\tilde{E} = (D^\perp)^T e_1 e_1^T D^\perp. \tag{24c}$$

The problem (20) can be cast in terms of $\tilde{w}$ as shown below.

Dense Formulation of ODC Problem: Minimize

$$\langle \tilde{M}, \tilde{w}\tilde{w}^T \rangle \tag{25a}$$

subject to

$$\langle \tilde{M}_i[\tau], \tilde{w}\tilde{w}^T \rangle = 0 \quad i = 1, \ldots, m; \quad \tau = 0, 1, \ldots, p \tag{25b}$$

$$\langle \tilde{E}, \tilde{w}\tilde{w}^T \rangle = 1 \tag{25c}$$

over $\tilde{w} \in \mathbb{R}^{(n_w - n_l)}$.

The SDP relaxation of the above formulation is provided next.

Dense Relaxation of Finite-Horizon ODC: Minimize

$$\langle \tilde{M}, \tilde{W} \rangle \tag{26a}$$

subject to

$$\langle \tilde{M}_i[\tau], \tilde{W} \rangle = 0 \quad i = 1, \ldots, m; \quad \tau = 0, 1, \ldots, p \tag{26b}$$

$$\langle \tilde{E}, \tilde{W} \rangle = 1 \tag{26c}$$

$$\tilde{W} \succeq 0 \tag{26d}$$

over $\tilde{W} \in \mathbb{S}_{(n_w - n_l)}$.

Remark 3. *Let $\mathcal{F}_s$ and $\mathcal{F}_d$ denote the feasible sets for the sparse and dense SDP relaxation problems in* (22) *and* (26)*, respectively. It can be easily seen that*

$$\{D^{\perp}\tilde{W}(D^{\perp})^T \mid \tilde{W} \in \mathcal{F}_d\} \subseteq \mathcal{F}_s \tag{27}$$

Therefore, the lower bound provided by the dense SDP relaxation problem (26) *is equal to or tighter than that of the sparse SDP relaxation* (22)*. However, the rank of the SDP solution of the dense relaxation may be high, which complicates its rounding to a rank-1 matrix. Hence, the sparse relaxation may be useful for recovering a near-global controller, while the dense relaxation may be used to bound the global optimality degree of the recovered controller.*

C. Rounding of SDP Solution to Rank-1 Matrix

Let W^{opt} either denote a low-rank solution for the sparse relaxation (22) or be equal to $D^{\perp}\tilde{W}^{\text{opt}}(D^{\perp})^T$ for a low-rank solution $\tilde{W}^{\text{opt}}$ (if any) of the dense relaxation (26). If the rank of W^{opt} is 1, then W^{opt} can be mapped back into a globally optimal controller for the ODC problem through an eigenvalue decomposition $W^{\text{opt}} = w^{\text{opt}}(w^{\text{opt}})^T$. Assume that W^{opt} does not have rank 1. There are multiple heuristic methods to recover a near-global controller, some of which are delineated below.

Direct Recovery Method: If W^{opt} had rank 1, then the $(2,1), (3,1), \ldots, (|h|+1, 1)$ entries of W^{opt} would have corresponded to the vector h^{opt} containing the free entries of K^{opt}. Inspired by this observation, if W^{opt} has rank greater than 1, then a near-global controller may still be recovered from the first column of W^{opt}. We refer to this approach as *Direct Recovery Method*.

Penalized SDP Relaxation: Recall that an SDP relaxation can be obtained by eliminating a rank constraint. In the case where this removal changes the solution, one strategy is to compensate for the rank constraint by incorporating an additive penalty function, denoted as $\Psi(W)$, into the objective of SDP relaxation. A common penalty function $\Psi(\cdot)$ is $\varepsilon \times \text{trace}\{W\}$, where ε is a design parameter. This problem is referred to as *Penalized SDP Relaxation* throughout this paper.

Indirect Recovery Method: Define x as the aggregate state vector obtained by stacking $x[0], ..., x[p]$. The objective function of every proposed SDP relaxation depends strongly on x and only weakly on k if α is small. In particular, if $\alpha = 0$, then the SDP objective function is not in terms of K. This implies that the relaxation may have two feasible matrix solutions both leading to the same optimal cost such that their first columns overlap on the part corresponding to x and not the part corresponding to h. Hence, unlike the direct method that recovers h from the first column of W^{opt}, it may be advantageous to first recover x and then solve a second convex optimization to generate a structured controller that is able to generate state values as closely to the recovered aggregate state vector as possible. More precisely, given an SDP solution W^{opt}, define $\hat{x} \in \mathbb{R}^{n(p+1)}$ as a vector containing the entries $(|h|+2, 1), (|h|+3, 1), \ldots, (1+|h|+n(p+1), 1)$ of W^{opt}. Define the indirect recovery method as the convex optimization problem

$$\text{minimize} \quad \sum_{\tau=0}^{p} \|\hat{x}[\tau+1] - (A + BKC)\hat{x}[\tau]\|^2 \tag{28a}$$

$$\text{subject to} \quad K = h_1 M_1 + \ldots + h_l M_l \tag{28b}$$

with the variables $K \in \mathbb{R}^{m \times r}$ and $h \in \mathbb{R}^l$. Let $\hat{K}$ denote a solution of the above problem. In the case where W^{opt} has rank 1 or the state part of the matrix W^{opt} corresponds to the true solution of the ODC problem, $\hat{x}$ is the same as x^{opt} and $\hat{K}$ is an optimal controller. Otherwise, $\hat{K}$ is a feasible controller that aims to make the closed-loop system follow the near-optimal state trajectory vector $\hat{x}$. As tested in [45], the above controller recovery method exhibits a remarkable performance on power systems.

D. Computationally-Cheap SDP Relaxation

Two SDP relaxations have been proposed earlier. Although these problems are convex, it may be difficult to solve them efficiently for a large-scale system. This is due to the fact that the size of each SDP matrix depends on the number of scalar variables at all times from 0 to p. There is an efficient approach to derive a computationally-cheap SDP relaxation. This will be explained below for the case where Q and R are non-singular and $r, m \leq n$.

Notation 1. *For every matrix $M \in \mathbb{R}^{n_1 \times n_2}$, define the sparsity pattern of M as follows*

$$\mathcal{S}(M) \triangleq \{S \in \mathbb{R}^{n_1 \times n_2} \mid \forall (i,j) \; M_{ij} = 0 \Rightarrow S_{ij} = 0\} \tag{29}$$

With no loss of generality, we assume that C has full row rank. There exists an invertible matrix $\Phi \in \mathbb{R}^{n \times n}$ such that $C\Phi = \begin{bmatrix} I_r & 0 \end{bmatrix}$. Define also

$$\mathcal{K}^2 \triangleq \{\Phi_1 S \Phi_1^T \mid S \in \mathcal{S}(\Phi_2 \Phi_2^T)\}. \tag{30}$$

Indeed, $\mathcal{K}^2$ captures the sparsity pattern of the matrix KK^T. For example, if $\mathcal{K}$ consists of block-diagonal (rectangular) matrix, $\mathcal{K}^2$ will also include block-diagonal (square) matrices. Let $\mu \in \mathbb{R}$ be a positive number such that $Q \succ \mu \times \Phi^{-T}\Phi^{-1}$, where Φ^{-T} denotes the transpose of the inverse of Φ. Define

$$\widehat{Q} := Q - \mu \times \Phi^{-T}\Phi^{-1}. \tag{31}$$

Using the slack matrix variables

$$X \triangleq [x[0] \ x[1] \ \dots \ x[p]], \tag{32a}$$

$$U \triangleq [u[0] \ u[1] \ \dots \ u[p]], \tag{32b}$$

an efficient relaxation of the ODC problem can be obtained.

Computationally-Cheap Relaxation of Finite-Horizon ODC: Minimize

$$\text{trace}\{X^T\widehat{Q}X + \mu \ \mathbf{W}_{22} + U^TRU\} + \alpha \ \text{trace}\{\mathbf{W}_{33}\} \tag{33a}$$

subject to

$$x[\tau+1] = Ax[\tau] + Bu[\tau], \quad \tau = 0, 1, \dots, p-1, \tag{33b}$$

$$x[0] = x_0, \tag{33c}$$

$$\mathbf{W} = \left[\begin{array}{c|c|c} I_n & \Phi^{-1}X & [K \quad 0]^T \\ \hline X^T\Phi^{-T} & \mathbf{W}_{22} & U^T \\ \hline [K \quad 0] & U & \mathbf{W}_{33} \end{array}\right], \tag{33d}$$

$$K \in \mathcal{K}, \tag{33e}$$

$$\mathbf{W}_{33} \in \mathcal{K}^2, \tag{33f}$$

$$\mathbf{W} \succeq 0, \tag{33g}$$

over $K \in \mathbb{R}^{m\times r}$, $X \in \mathbb{R}^{n\times(p+1)}$, $U \in \mathbb{R}^{m\times(p+1)}$ and $\mathbf{W} \in \mathbb{S}_{n+m+p+1}$ (note that $\mathbf{W}_{22}$ and $\mathbf{W}_{33}$ are two blocks of the variable $\mathbf{W}$).

Note that the above relaxation can be naturally cast as an SDP problem by replacing each quadratic term in its objective with a new variable and then using the Schur complement. We refer to the SDP formulation of this problem as **computationally-cheap SDP relaxation**.

Theorem 3. *The problem* (33) *is a convex relaxation of the ODC problem. Furthermore, the relaxation is exact if and only if it possesses a solution* $(K^{opt}, X^{opt}, U^{opt}, \mathbf{W}^{opt})$ *such that* $\text{rank}\{\mathbf{W}^{opt}\} = n$.

Proof. It is evident that the problem (33) is a convex program. To prove the remaining parts of the theorem, it suffices to show that the ODC problem is equivalent to (33) subject to the additional constraint $\text{rank}\{\mathbf{W}\} = n$. To this end, consider a feasible solution $(K, X, U, \mathbf{W})$ such that $\text{rank}\{\mathbf{W}\} = n$. Since I_n has rank n, taking the Schur complement of the blocks $(1,1)$, $(1,2)$, $(2,1)$ and $(2,2)$ of $\mathbf{W}$ yields that

$$0 = \mathbf{W}_{22} - X^T\Phi^{-T}(I_n)^{-1}\Phi^{-1}X \tag{34}$$

Likewise,

$$0 = \mathbf{W}_{33} - KK^T \tag{35}$$

On the other hand,

$$\sum_{\tau=0}^{p} \left(x[\tau]^TQx[\tau] + u[\tau]^TRu[\tau]\right) = \text{trace}\{X^TQX + U^TRU\} \tag{36}$$

It follows from (34), (35) and (36) that the ODC problem and its computationally cheap relaxation lead to the same objective at the respective points (K, X, U) and $(K, X, U, \mathbf{W})$. In addition, it can be concluded from the Schur complement of the blocks $(1,1)$, $(1,2)$, $(3,1)$ and $(3,2)$ of $\mathbf{W}$ that

$$U = [K \quad 0]\Phi^{-1}X = KCX \tag{37}$$

or equivalently

$$u[\tau] = KCx[\tau] \qquad \text{for} \quad \tau = 0, 1, \dots, p \tag{38}$$

This implies that (K, X, U) is a feasible solution of the ODC problem. Hence, the optimal objective value of the ODC problem is a lower bound on that of the computationally-cheap relaxation under the additional constraint $\text{rank}\{\mathbf{W}\} = n$.

Now, consider a feasible solution (K, X, U) of the ODC problem. Define $\mathbf{W}_{22} = X^T\Phi^{-T}\Phi^{-1}X$ and $K_2 = KK^T$. Observe that $\mathbf{W}$ can be written as the rank-n matrix $W_rW_r^T$, where

$$W_r = \begin{bmatrix} I_n & \Phi^{-1}X & [K \quad 0]^T \end{bmatrix}^T \tag{39}$$

Thus, $(K, X, U, \mathbf{W})$ is a feasible solution of the computationally-cheap SDP relaxation. This implies that the optimal objective value of the ODC problem is an upper bound on that of the computationally-cheap SDP relaxation under the additional constraint $\text{rank}\{\mathbf{W}\} = n$. The proof is completed by combining this property with its opposite statement proved earlier. $\square$

The sparse and dense SDP relaxations were both obtained by defining a matrix W as the product of two *vectors*. However, the computationally-cheap relaxation of the finite-horizon ODC Problem is obtained by defining $\mathbf{W}$ as the product of two *matrices*. This significantly reduces the computational complexity. To shed light on this fact, notice that the numbers of rows for the matrix variables of sparse and dense SDP relaxations are on the order of np, whereas the number of rows for the computationally-cheap SDP solution is on the order of $n + p$.

Remark 4. *The computationally-cheap relaxation of the finite-horizon ODC Problem automatically acts as a* <u>*penalized*</u> *SDP relaxation. To explain this remarkable feature of the proposed relaxation, notice that the terms* $\text{trace}\{\mathbf{W}_{22}\}$ *and* $\text{trace}\{\mathbf{W}_{33}\}$ *in the objective function of the relaxation inherently penalize the trace of* $\mathbf{W}$. *This automatic penalization helps significantly with the reduction of the rank of* $\mathbf{W}$ *at optimality. As a result, it is expected that the quality of the relaxation will be better for higher values of* α *and* μ.

Remark 5. *Consider the extreme case where* $r = n$, $C = I_n$, $\alpha = 0$, $p = \infty$, *and the unknown controller* K *is unstructured. This amounts to the famous LQR problem and the optimal controller can be found using the Riccati equation. It is straightforward to verify that the computationally-cheap relaxation of the ODC problem is always exact in this case even though it is infinite-dimensional. The proof is based on the following facts:*

- *When* K *is unstructured, the constraint* (33e) *and* (33f) *can be omitted. Therefore, there is no structural constraint on* $\mathbf{W}_{33}$ *and* $\mathbf{W}_{31}$ *(i.e., the* $(3,1)$ *block entry).*

- *Then, the constraint* (33d) *reduces to* $\mathbf{W}_{22} = X^T\Phi^{-T}\Phi^{-1}X$ *due to the term* $\text{trace}\{\mathbf{W}_{22}\}$ *in the objective function. Consequently, the objective function can be rearranged as* $\sum_{\tau=0}^{\infty}\left(x[\tau]^TQx[\tau]+u[\tau]^TRu[\tau]\right)$.
- *The only remaining constraints are the state evolution equation and* $x[0]=x_0$. *It is known that the remaining feed-forward problem has a solution* (X^{opt}, U^{opt}) *such that* $U^{opt}=K^{opt}X^{opt}$ *for some matrix* K^{opt}.

E. Stability Enforcement

The finite-horizon ODC problem studied before had no stability conditions. It is verified in some simulations in [45] that the closed-loop stability may be automatically guaranteed for physical systems, provided p is large enough. In this subsection, we aim to directly enforce stability by imposing additional constraints on the proposed SDP relaxations.

Theorem 4. *There exists a controller* $u[\tau]=Ky[\tau]$ *with the structure* $K\in\mathcal{K}$ *to stabilize the system* (1) *if and only if there exist a (Lyapunov) matrix* $P\in\mathbb{S}_n$, *a matrix* $K\in\mathbb{R}^{m\times r}$, *and auxiliary variables* $L\in\mathbb{R}^{m\times n}$ *and* $\mathbf{G}\in\mathbb{S}_{2n+m}$ *such that*

$$\begin{bmatrix} P-I_n & AP+B\mathbf{G}_{32} \\ PA^T+\mathbf{G}_{23}B^T & P \end{bmatrix} \succeq 0, \tag{40a}$$

$$K\in\mathcal{K}, \tag{40b}$$

$$\mathbf{G}\succeq 0, \tag{40c}$$

$$\mathbf{G}_{33}\in\mathcal{K}^2, \tag{40d}$$

$$\text{rank}\{\mathbf{G}\}=n, \tag{40e}$$

where

$$\mathbf{G}\triangleq\left[\begin{array}{c|c|c} I_n & \Phi^{-1}P & [K\ \ 0]^T \\ \hline P\Phi^{-T} & \mathbf{G}_{22} & \mathbf{G}_{23} \\ \hline [K\ \ 0] & \mathbf{G}_{32} & \mathbf{G}_{33} \end{array}\right] \tag{41}$$

Proof. It is well-known that the system (1) is stable under a controller $u[\tau]=Ky[\tau]$ if and only if there exists a positive-definite matrix $P\in\mathbb{S}_n$ to satisfy the Lyapunov inequality:

$$(A+BKC)^TP(A+BKC)-P+I_n\preceq 0 \tag{42}$$

or equivalently

$$\begin{bmatrix} P-I_n & AP+BKCP \\ PA^T+PK^TC^TB^T & P \end{bmatrix}\succeq 0 \tag{43}$$

Due to the analogy between $\mathbf{W}$ and $\mathbf{G}$, the argument made in the proof of Theorem 3 can be adopted to complete the proof of this theorem (note that $\mathbf{G}_{32}$ plays the role of KCP). □

Theorem 4 translates the stability of the closed-loop system into a rank-n condition. Consider one of the aforementioned SDP relaxations of the ODC problem. To enforce stability, it results from Theorem 4 that two actions can be taken: (i) addition of the convex constraints (40a)-(40d) to SDP relaxations, (ii) compensation for the rank-n condition through an appropriate convex penalization of $\mathbf{G}$ in the objective function of SDP relaxations. Note that the penalization is vital because otherwise $\mathbf{G}_{22}$ and $\mathbf{G}_{33}$ would grow unboundedly to satisfy the condition $\mathbf{G}\succeq 0$.

IV. Infinite-horizon Deterministic ODC Problem

In this section, we study the infinite-horizon ODC problem, corresponding to $p=+\infty$ and subject to a stability condition.

A. Lyapunov Formulation

The finite-horizon ODC was investigated through a time-domain formulation. However, to deal with the infinite dimension of the infinite-horizon ODC and its hard stability constraint, a Lyapunov approach will be taken here. Consider the following optimization problem.

Lyapunov Formulation of ODC: Minimize

$$x_0^TPx_0+\alpha\|K\|_F^2 \tag{44a}$$

subject to

$$\begin{bmatrix} G & G & (AG+BL)^T & L^T \\ G & Q^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix}\succeq 0, \tag{44b}$$

$$\begin{bmatrix} P & I_n \\ I_n & G \end{bmatrix}\succeq 0, \tag{44c}$$

$$K\in\mathcal{K}, \tag{44d}$$

$$L=KCG, \tag{44e}$$

over $K\in\mathbb{R}^{m\times r}$, $L\in\mathbb{R}^{m\times n}$, $P\in\mathbb{S}_n$ and $G\in\mathbb{S}_n$.

It will be shown in the next theorem that the above formulation is tantamount to the infinite-horizon ODC problem.

Theorem 5. *The infinite-horizon deterministic ODC problem is equivalent to finding a controller* $K\in\mathcal{K}$, *a symmetric Lyapunov matrix* $P\in\mathbb{S}_n$, *an auxiliary symmetric matrix* $G\in\mathbb{S}_n$ *and an auxiliary matrix* $L\in\mathbb{R}^{m\times n}$ *to solve the optimization problem* (44).

Proof. Given an arbitrary control gain K, we have:

$$\sum_{\tau=0}^{\infty}\left(x[\tau]^TQx[\tau]+u[\tau]^TRu[\tau]\right)=x[0]^TPx[0] \tag{45}$$

where

$$P=(A+BKC)^TP(A+BKC)+Q+(KC)^TR(KC) \tag{46a}$$

$$P\succeq 0 \tag{46b}$$

On the other hand, it is well-known that replacing the equality sign "=" in (46a) with the inequality sign "$\succeq$" does not affect the solution of the optimization problem [31]. After pre- and post-multiplying the Lyapunov inequality obtained from (46a) with P^{-1} and using the Schur complement formula, the constraints (46a) and (46b) can be combined as

$$\begin{bmatrix} P^{-1} & P^{-1} & S^T & P^{-1}(KC)^T \\ P^{-1} & Q^{-1} & 0 & 0 \\ S & 0 & P^{-1} & 0 \\ (KC)P^{-1} & 0 & 0 & R^{-1} \end{bmatrix}\succeq 0 \tag{47}$$

where $S=(A+BKC)P^{-1}$. By replacing P^{-1} with a new variable G in the above matrix and defining L as KCG, the constraints (44b) and (44e) will be obtained. On the other hand, (44c) implies that $G\succ 0$ and $P\succeq G^{-1}$. Therefore, the minimization of $x_0^TPx_0$ subject to the constraint (44c) ensures that $P=G^{-1}$ is satisfied for at least one optimal solution of the optimization problem. □

Theorem 6. *Consider the special case where $r = n$, $C = I_n$, $\alpha = 0$ and $\mathcal{K}$ contains the set of all unstructured controllers. Then, the infinite-horizon deterministic ODC problem has the same solution as the convex optimization problem obtained from the nonlinear optimization* (44) *by removing its non-convex constraint* (44e).

Proof. It is easy to verify that a solution $(K^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}}, L^{\text{opt}})$ of the convex problem stated in the theorem can be mapped to the solution $(L^{\text{opt}}(G^{\text{opt}})^{-1}, P^{\text{opt}}, G^{\text{opt}}, L^{\text{opt}})$ of the non-convex problem (44) and vice versa (recall that $C = I_n$ by assumption). This completes the proof. □

B. SDP Relaxation

Theorem 6 states that a classical optimal control problem can be precisely solved via a convex relaxation of the nonlinear optimization (44) by eliminating its constraint (44e). However, this simple convex relaxation does not work satisfactorily for a general control structure $K = \Phi_1 \text{diag}\{h\}\Phi_2$. To design a better relaxation, define

$$w = \begin{bmatrix} 1 & h^T & \text{vec}\{\Phi_2 CG\}^T \end{bmatrix}^T \tag{48}$$

where $\text{vec}\{\Phi_2 CG\}$ is an $nl \times 1$ column vector obtained by stacking the columns of $\Phi_2 CG$. It is possible to write every entry of the bilinear matrix term KCG as a linear function of the entries of the parametric matrix ww^T. Hence, by introducing a new matrix variable $\mathbf{W}$ playing the role of ww^T, the nonlinear constraint (44e) can be rewritten as a linear constraint in term of $\mathbf{W}$.

Notation 2. *Define the sampling operator* $\text{samp} : \mathbb{R}^{l\times nl} \to \mathbb{R}^{l\times n}$ *as follows:*

$$\text{samp}\{X\} = \begin{bmatrix} X_{i,(n-1)j+i} \end{bmatrix}_{i=1,\ldots,l;\ j=1,\ldots,n}. \tag{49}$$

Now, one can relax the non-convex mapping constraint $\mathbf{W} = ww^T$ to $\mathbf{W} \succeq 0$ and another constraint stating that the first column of $\mathbf{W}$ is equal to w. This yields the following convex relaxation of problem (44).

SDP Relaxation of Infinite-Horizon Deterministic ODC: Minimize

$$x_0^T P x_0 + \alpha \text{ trace}\{\mathbf{W}_{33}\} \tag{50a}$$

subject to

$$\begin{bmatrix} G & G & (AG+BL)^T & L^T \\ G & Q^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \tag{50b}$$

$$\begin{bmatrix} P & I_n \\ I_n & G \end{bmatrix} \succeq 0, \tag{50c}$$

$$L = \Phi_1 \times \text{samp}\{\mathbf{W}_{32}\}, \tag{50d}$$

$$\mathbf{W} = \left[\begin{array}{c|c|c} 1 & \text{vec}\{\Phi_2 CG\}^T & h^T \\ \hline \text{vec}\{\Phi_2 CG\} & \mathbf{W}_{22} & \mathbf{W}_{23} \\ \hline h & \mathbf{W}_{32} & \mathbf{W}_{33} \end{array}\right], \tag{50e}$$

$$\mathbf{W} \succeq 0, \tag{50f}$$

over $h \in \mathbb{R}^l$, $L \in \mathbb{R}^{m\times n}$, $P \in \mathbb{S}_n$, $G \in \mathbb{S}_n$ and $\mathbf{W} \in \mathbb{S}_{1+l(n+1)}$.

If the relaxed problem (50) has the same solution as the infinite-horizon ODC in (44), the relaxation is exact.

Theorem 7. *The following statements hold regarding the relaxation of the infinite-horizon deterministic ODC in* (50)*:*

i) *The relaxation is exact if it has a solution* $(h^{opt}, P^{opt}, G^{opt}, L^{opt}, \mathbf{W}^{opt})$ *such that* $\text{rank}\{\mathbf{W}^{opt}\} = 1$.
ii) *The relaxation always has a solution* $(h^{opt}, P^{opt}, G^{opt}, L^{opt}, \mathbf{W}^{opt})$ *such that* $\text{rank}\{\mathbf{W}^{opt}\} \leq 3$.

Proof. Consider a sparsity graph $\mathcal{G}$ of (50), constructed as follows. The graph $\mathcal{G}$ has $1 + l(n+1)$ vertices corresponding to the rows of $\mathbf{W}$. Two arbitrary disparate vertices $i, j \in \{1, 2, \ldots, 1+l(n+1)\}$ are adjacent in $\mathcal{G}$ if $\mathbf{W}_{ij}$ appears in at least one of the constraints of the problem (50) excluding the global constraint $\mathbf{W} \succeq 0$. For example, vertex 1 is connected to all remaining vertices of $\mathcal{G}$. The graph $\mathcal{G}$ with its vertex 1 removed is depicted in Figure 4. This graph is acyclic and therefore the treewidth of $\mathcal{G}$ is at most 2. Hence, it follows from Theorem 1 that (50) has a matrix solution with rank at most 3. □

Theorem 7 states that the SDP relaxation of the infinite-horizon ODC problem has a low-rank solution. However, it does not imply that every solution of the relaxation is low-rank. Theorem 1 provides a procedure for converting a high-rank solution of the SDP relaxation into a low-rank one.

C. Computationally-Cheap Relaxation

The aforementioned SDP relaxation has a high dimension for a large-scale system, which makes it less interesting for computational purposes. Moreover, the quality of its optimal objective value can be improved using some indirect penalty technique. The objective of this subsection is to offer a computationally-cheap SDP relaxation for the ODC problem, whose solution outperforms that of the previous SDP relaxation. For this purpose, consider again a scalar μ such that $Q \succ \mu \times \Phi^{-T}\Phi^{-1}$ and define $\hat{Q}$ according to (31).

Computationally-Cheap Relaxation of Infinite-horizon Deterministic ODC: Minimize

$$x_0^T P x_0 + \alpha \text{ trace}\{\mathbf{W}_{33}\} \tag{51a}$$

subject to

$$\begin{bmatrix} G - \mu\mathbf{W}_{22} & G & (AG+BL)^T & L^T \\ G & \hat{Q}^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \tag{51b}$$

$$\begin{bmatrix} P & I_n \\ I_n & G \end{bmatrix} \succeq 0, \tag{51c}$$

$$\mathbf{W} = \left[\begin{array}{c|c|c} I_n & \Phi^{-1}G & [K \quad 0]^T \\ \hline G\Phi^{-T} & \mathbf{W}_{22} & L^T \\ \hline [K \quad 0] & L & \mathbf{W}_{33} \end{array}\right], \tag{51d}$$

$$K \in \mathcal{K}, \tag{51e}$$

$$\mathbf{W}_{33} \in \mathcal{K}^2, \tag{51f}$$

$$\mathbf{W} \succeq 0, \tag{51g}$$

over $K \in \mathbb{R}^{m\times r}$, $L \in \mathbb{R}^{m\times n}$, $P \in \mathbb{S}_n$, $G \in \mathbb{S}_n$ and $\mathbf{W} \in \mathbb{S}_{2n+m}$.

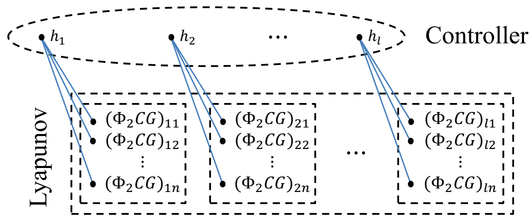


Fig. 4: The sparsity graph for the infinite-horizon deterministic ODC in the case where $\mathcal{K}$ consists of diagonal matrices (the central vertex corresponding to the constant 1 is removed for simplicity).

The following remarks can be made regarding (51):

- The constraint (51b) corresponds to the Lyapunov inequality associated with (46a), where $\mathbf{W}_{22}$ in its first block aims to play the role of $P^{-1}\Phi^{-T}\Phi^{-1}P^{-1}$.
- The constraint (51c) ensures that the relation $P = G^{-1}$ occurs at optimality (at least for one of the solution of the problem).
- The constraint (51d) is a surrogate for the only complicating constraint of the ODC problem, i.e., $L = KCG$.
- Since no non-convex rank constraint is imposed on the problem to maintain the convexity of the relaxation, the rank constraint is compensated in various ways. More precisely, the entries of $\mathbf{W}$ are constrained in the objective function (51a) through the term α trace$\{\mathbf{W}_{33}\}$, in the first block of the constraint (51b) through the term $G - \mu\mathbf{W}_{22}$, and also via the constraint (51e) and (51f). These terms aim to automatically penalize the rank of $\mathbf{W}$ indirectly.
- The proposed relaxation takes advantage of the sparsity of not only K, but also KK^T (through the constraint (51f)).

Theorem 8. *The problem* (51) *is a convex relaxation of the infinite-horizon ODC problem. Furthermore, the relaxation is exact if and only if it possesses a solution* $(K^{opt}, L^{opt}, P^{opt}, G^{opt}, \mathbf{W}^{opt})$ *such that rank*$\{\mathbf{W}^{opt}\} = n$.

Proof. The objective function and constraints of the problem (51) are all linear functions of the tuple $(K, L, P, G, \mathbf{W})$. Hence, this relaxation is indeed convex. To study the relationship between this optimization problem and the infinite-horizon ODC, consider a feasible point (K, L, P, G) of the ODC formulation (44). It can be deduced from the relation $L = KCG$ that $(K, L, P, G, \mathbf{W})$ is a feasible solution of the problem (51) if the free blocks of $\mathbf{W}$ are considered as

$$\mathbf{W}_{22} = G\Phi^{-T}\Phi^{-1}G, \qquad \mathbf{W}_{33} = KK^T \tag{52}$$

(note that (44b) and (51b) are equivalent for this choice of $\mathbf{W}$). This implies that the problem (51) is a convex relaxation of the infinite-horizon ODC problem.

Consider now a solution $(K^{\text{opt}}, L^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}}, \mathbf{W}^{\text{opt}})$ of the computationally-cheap SDP relaxation such that rank$\{\mathbf{W}^{\text{opt}}\} = n$. Since the rank of the first block of $\mathbf{W}^{\text{opt}}$ (i.e., I_n) is already n, a Schur complement argument on the blocks $(1,1)$, $(1,3)$, $(2,1)$ and $(2,3)$ of $\mathbf{W}^{\text{opt}}$ yields that

$$0 = L^{\text{opt}} - [K^{\text{opt}} \quad 0](I_n)^{-1}\Phi^{-1}G^{\text{opt}} \tag{53}$$

or equivalently $L^{\text{opt}} = K^{\text{opt}}CG^{\text{opt}}$, which is tantamount to the constraint (44e). This implies that $(K^{\text{opt}}, L^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}})$ is a solution of the infinite-horizon ODC problem (44) and hence the relaxation is exact. So far, we have shown that the existence of a rank-n solution $\mathbf{W}^{\text{opt}}$ guarantees the exactness of the relaxation. The converse of this statement can also be proved similarly. □

The variable $\mathbf{W}$ in the first SDP relaxation (50) had $1+l(n+1)$ rows. In contrast, this number reduces to $2n+m$ for the matrix $\mathbf{W}$ in the computationally-cheap relaxation (51). This significantly reduces the computation time of the relaxation.

Corollary 2. *Consider the special case where $r = n$, $C = I_n$, $\alpha = 0$ and $\mathcal{K}$ contains the set of all unstructured controllers. Then, the computationally-cheap relaxation problem* (51) *is exact for the infinite-horizon ODC problem.*

Proof. The proof follows from that of Theorem 6. □

D. Controller Recovery

In this subsection, two controller recovery methods will be described. With no loss of generality, our focus will be on the computationally-cheap relaxation problem (51).

Direct Recovery Method for Infinite-Horizon ODC: A near-optimal controller K for the infinite-horizon ODC problem is chosen to be equal to the optimal matrix K^{opt} obtained from the computationally-cheap relaxation problem (51).

Indirect Recovery Method for Infinite-Horizon ODC: Let $(K^{\text{opt}}, L^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}}, \mathbf{W}^{\text{opt}})$ denote a solution of the computationally-cheap relaxation problem (51). Given a pre-specified nonnegative number ε, a near-optimal controller $\hat{K}$ for the infinite-horizon ODC problem is recovered by minimizing

$$\varepsilon \times \gamma + \alpha\|K\|_F^2 \tag{54a}$$

subject to

$$\begin{bmatrix} (G^{\text{opt}})^{-1} - Q + \gamma I_n & (A+BKC)^T & (KC)^T \\ (A+BKC) & G^{\text{opt}} & 0 \\ (KC) & 0 & R^{-1} \end{bmatrix} \succ 0 \tag{54b}$$

$$K = h_1 N_1 + \ldots + h_l N_l. \tag{54c}$$

over $K \in \mathbb{R}^{m\times r}$, $h \in \mathbb{R}^l$ and $\gamma \in \mathbb{R}$. Note that this is a convex program. The direct recovery method assumes that the controller K^{opt} obtained from the computationally-cheap relaxation problem (51) is near-optimal, whereas the indirect method assumes that the controller K^{opt} might be unacceptably imprecise while the inverse of the Lyapunov matrix is near-optimal. The indirect method is built on SDP relaxation by fixing G at its optimal value and then perturbing Q as $Q - \gamma I_n$ to facilitate the recovery of a stabilizing controller. The underlying idea is that the SDP relaxation depends strongly on G and weakly on P (note that G appears 9 times in the formulation, while P appears only twice to indirectly account for the inverse of G). In other words, there might be two feasible solutions with similar costs for the SDP relaxation whose G parts are identical while their P parts are very different. Hence, the indirect method focuses on G.

V. Infinite-Horizon Stochastic ODC Problem

This section is mainly concerned with the stochastic optimal distributed control (SODC) problem, which aims to design a stabilizing static controller $u[\tau] = Ky[\tau]$ to minimize the cost function

$$\lim_{\tau\to+\infty} \mathcal{E}\left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau]\right) + \alpha \|K\|_F^2 \tag{55}$$

subject to the system dynamics (3) and the controller requirement $K \in \mathcal{K}$, for a nonnegative scalar α and positive-definite matrices Q and R. Define two covariance matrices as

$$\Sigma_d = \mathcal{E}\{Ed[0]d[0]^T E^T\} \quad \Sigma_v = \mathcal{E}\{Fv[0]v[0]^T F^T\} \tag{56}$$

Consider the following optimization problem.

Lyapunov Formulation of SODC: Minimize

$$\langle P, \Sigma_d\rangle + \langle M + K^T RK, \Sigma_v\rangle + \alpha\|K\|_F^2 \tag{57a}$$

subject to

$$\begin{bmatrix} G & G & (AG+BL)^T & L^T \\ G & Q^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \tag{57b}$$

$$\begin{bmatrix} P & I_n \\ I_n & G \end{bmatrix} \succeq 0, \tag{57c}$$

$$\begin{bmatrix} M & (BK)^T \\ BK & G \end{bmatrix} \succeq 0, \tag{57d}$$

$$K \in \mathcal{K} \tag{57e}$$

$$L = KCG \tag{57f}$$

over the controller $K \in \mathbb{R}^{m\times r}$, Lyapunov matrix $P \in \mathbb{S}_n$ and auxiliary matrices $G \in \mathbb{S}_n$, $L \in \mathbb{R}^{m\times n}$ and $M \in \mathbb{S}_r$.

Theorem 9. *The infinite-horizon SODC problem adopts the non-convex formulation* (57).

Proof. It is straightforward to verify that

$$\begin{aligned} x[\tau] &= (A+BKC)^\tau x[0] \\ &\quad + \sum_{t=0}^{\tau-1} (A+BKC)^{\tau-t-1}(Ed[t] + BKFv[t]) \end{aligned} \tag{58}$$

for $\tau = 1, 2, \ldots, \infty$. On the other hand, since the controller under design must be stabilizing, $(A+BKC)^\tau$ approaches zero as τ goes to $+\infty$. In light of the above equation, it can be verified that

$$\begin{aligned} &\mathcal{E}\left\{\lim_{\tau\to+\infty}\left(x[\tau]^T Q x[\tau] + u[\tau]^T R u[\tau]\right)\right\} \\ &= \mathcal{E}\left\{\lim_{\tau\to+\infty} x[\tau]^T\left(Q + C^T K^T RKC\right)x[\tau]\right\} \\ &\quad + \mathcal{E}\left\{\lim_{\tau\to+\infty} v[\tau]^T F^T K^T RKFv[\tau]\right\} \\ &= \langle P, \Sigma_d\rangle + \langle (BK)^T P(BK) + K^T RK, \Sigma_v\rangle \end{aligned} \tag{59}$$

where

$$P = \sum_{t=0}^{\infty}\left((A+BKC)^t\right)^T (Q + C^T K^T RKC)(A+BKC)^t$$

Similar to the proof of Theorem 5, the above infinite series can be replaced by the expanded Lyapunov inequality (47): After replacing P^{-1} and KCP^{-1} in (47) with new variables G and L, it can be concluded that:

- The condition (47) is identical to the set of constraints (57b) and (57f).
- The cost function (59) can be expressed as
$$\langle P, \Sigma_d\rangle + \langle (BK)^T G^{-1}(BK) + K^T RK, \Sigma_v\rangle + \alpha\|K\|_F^2$$
- Since P appears only once in the constraints of the optimization problem (57) (i.e., the condition (57c)) and the objective function of this optimization includes the term $\langle P, \Sigma_d\rangle$, an optimal value of P is equal to G^{-1} (Notice that $\Sigma_d \succeq 0$).
- Similarly, the optimal value of M is equal to $(BK)^T G^{-1}(BK)$.

The proof follows from the above observations. □

The traditional H_2 optimal control problem (i.e., in the centralized case) can be solved using Riccati equations. It will be shown in the next proposition that dropping the nonconvex constraint (57f) results in a convex optimization that correctly solves the centralized H_2 optimal control problem.

Proposition 1. *Consider the special case where $r = n$, $C = I_n$, $\alpha = 0$, $\Sigma_v = 0$, and $\mathcal{K}$ contains the set of all unstructured controllers. Then, the SODC problem has the same solution as the convex optimization problem obtained from the nonlinear optimization* (57a)-(57) *by removing its non-convex constraint* (57f).

Proof. It is similar to the proof of Theorem 6. □

Consider the vector w defined in (48). Similar to the infinite-horizon ODC case, the bilinear matrix term KCG can be represented as a linear function of the entries of the parametric matrix $\mathbf{W}$ defined as ww^T. Now, a convex relaxation can be attained by relaxing the constraint $\mathbf{W} = ww^T$ to $\mathbf{W} \succeq 0$ and adding another constraint stating that the first column of $\mathbf{W}$ is equal to w.

Relaxation of Infinite-Horizon SODC: Minimize

$$\langle P, \Sigma_d\rangle + \langle M + K^T RK, \Sigma_v\rangle + \alpha\ \mathrm{trace}\{\mathbf{W}_{33}\} \tag{60a}$$

subject to

$$\begin{bmatrix} G & G & (AG+BL)^T & L^T \\ G & Q^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1} \end{bmatrix} \succeq 0, \tag{60b}$$

$$\begin{bmatrix} P & I_n \\ I_n & G \end{bmatrix} \succeq 0, \tag{60c}$$

$$K = \Phi_1 \mathrm{diag}\{h\}\Phi_2, \tag{60d}$$

$$\begin{bmatrix} M & (BK)^T \\ BK & G \end{bmatrix} \succeq 0, \tag{60e}$$

$$L = \Phi_1 \mathrm{samp}\{\mathbf{W}_{32}\}, \tag{60f}$$

$$\mathbf{W} = \left[\begin{array}{c|c|c} 1 & \mathrm{vec}\{\Phi_2 CG\}^T & h^T \\ \hline \mathrm{vec}\{\Phi_2 CG\} & \mathbf{W}_{22} & \mathbf{W}_{23} \\ \hline h & \mathbf{W}_{32} & \mathbf{W}_{33} \end{array}\right], \tag{60g}$$

$$\mathbf{W} \succeq 0, \tag{60h}$$

over the controller $K \in \mathbb{R}^{m\times r}$, Lyapunov matrix $P \in \mathbb{S}_n$ and auxiliary matrices $G \in \mathbb{S}_n$, $L \in \mathbb{R}^{m\times n}$, $M \in \mathbb{S}_r$, $h \in \mathbb{R}^l$ and $\mathbf{W} \in \mathbb{S}_{1+l(n+1)}$.

Theorem 10. *The following statements hold regarding the convex relaxation of the infinite-horizon SODC problem:*

i) *The relaxation is exact if it has a solution* $(h^{opt}, K^{opt}, P^{opt}, G^{opt}, L^{opt}, M^{opt}, \mathbf{W}^{opt})$ *such that* $\text{rank}\{W^{opt}\} = 1$.

ii) *The relaxation always has a solution* $(h^{opt}, K^{opt}, P^{opt}, G^{opt}, L^{opt}, M^{opt}, \mathbf{W}^{opt})$ *such that* $\text{rank}\{W^{opt}\} \leq 3$.

Proof. The proof is omitted (see Theorems 7 and 9). □

As before, it can be deduced from Theorem 10 that the infinite-horizon SODC problem has a convex relaxation with the property that its exactness amounts to the existence of a rank-1 matrix solution $\mathbf{W}^{\text{opt}}$. Moreover, it is always guaranteed that this relaxation has a solution such that $\text{rank}\{\mathbf{W}^{\text{opt}}\} \leq 3$.

A computationally-cheap SDP relaxation for the SODC problem will be derived below. Let μ_1 and μ_2 be two nonnegative numbers such that

$$Q \succ \mu_1 \times \Phi^{-T}\Phi^{-1}, \quad \Sigma_v \succeq \mu_2 \times I_r \tag{61}$$

Define $\widehat{Q} := Q - \mu_1 \times \Phi^{-T}\Phi^{-1}$ and $\widehat{\Sigma}_v := \Sigma_v - \mu_2 \times I_r$.

Computationally-Cheap Relaxation of Infinite-Horizon SODC: Minimize

$$\langle P, \Sigma_d\rangle + \langle M, \Sigma_v\rangle + \langle K^T R K, \widehat{\Sigma}_v\rangle + \langle \mu_2 R + \alpha I_m, \mathbf{W}_{33}\rangle \tag{62a}$$

subject to

$$\begin{bmatrix} G-\mu_1\mathbf{W}_{22} & G & (AG+BL)^T & L^T \\ G & \widehat{Q}^{-1} & 0 & 0 \\ AG+BL & 0 & G & 0 \\ L & 0 & 0 & R^{-1}\end{bmatrix} \succeq 0, \tag{62b}$$

$$\begin{bmatrix} P & I_n \\ I_n & G \end{bmatrix} \succeq 0, \tag{62c}$$

$$\begin{bmatrix} M & (BK)^T \\ BK & G \end{bmatrix} \succeq 0, \tag{62d}$$

$$\mathbf{W} = \left[\begin{array}{c|c|c} I_n & \Phi^{-1}G & [K \quad 0]^T \\ \hline G\Phi^{-T} & \mathbf{W}_{22} & L^T \\ \hline [K \quad 0] & L & \mathbf{W}_{33} \end{array}\right], \tag{62e}$$

$$K \in \mathcal{K}, \tag{62f}$$

$$\mathbf{W}_{33} \in \mathcal{K}^2, \tag{62g}$$

$$\mathbf{W} \succeq 0, \tag{62h}$$

over $K \in \mathbb{R}^{m\times r}$, $P \in \mathbb{S}_n$, $G \in \mathbb{S}_n$, $L \in \mathbb{R}^{m\times n}$, $M \in \mathbb{S}_r$ and $\mathbf{W} \in \mathbb{S}_{2n+m}$.

It should be noted that the constraint (62d) ensures that the relation $M = (BK)^T G^{-1}(BK)$ occurs at optimality.

Theorem 11. *The problem* (62) *is a convex relaxation of the SODC problem. Furthermore, the relaxation is exact if and only if it possesses a solution* $(K^{opt}, L^{opt}, P^{opt}, G^{opt}, M^{opt}, \mathbf{W}^{opt})$ *such that rank*$\{\mathbf{W}^{opt}\} = n$.

Proof. Since the proof is similar to that of the infinite-horizon case presented earlier, it is omitted here. □

For the retrieval of a near-optimal controller, the direct recovery method delineated for the infinite-horizon ODC problem can be readily deployed. However, the indirect recovery method requires some modifications, which will be explained below. Let $(K^{\text{opt}}, L^{\text{opt}}, P^{\text{opt}}, G^{\text{opt}}, M^{\text{opt}}, \mathbf{W}^{\text{opt}})$ denote a solution of the computationally-cheap relaxation of SODC. A near-optimal controller K for SODC may be recovered by minimizing

$$\langle K^T(B^T(G^{\text{opt}})^{-1}B + R)K, \Sigma_v\rangle + \alpha\|K\|_F^2 + \varepsilon \times \gamma \tag{63a}$$

subject to

$$\begin{bmatrix} (G^{\text{opt}})^{-1} - Q + \gamma I_n & (A+BKC)^T & (KC)^T \\ (A+BKC) & G^{\text{opt}} & 0 \\ (KC) & 0 & R^{-1}\end{bmatrix} \succ 0 \tag{63b}$$

$$K \in h_1 N_1 + \ldots + h_l N_l. \tag{63c}$$

over $K \in \mathbb{R}^{m\times r}$, $h \in \mathbb{R}^l$ and $\gamma \in \mathbb{R}$, where ε is a pre-specified nonnegative number. This is a convex program.

VI. Extension to Dynamic Controllers

Consider the problem of finding an optimal fixed-order dynamic controller with a pre-specified structure. To formulate the problem, denote the unknown controller as

$$\begin{cases} z_c[\tau+1] = A_c z_c[\tau] + B_c y[\tau] \\ u[\tau] = C_c z_c[\tau] + D_c y[\tau] \end{cases} \tag{64}$$

where $z_c[\tau] \in \mathbb{R}^{n_c}$ represents the state of the controller, and n_c denotes its known degree. The unknown quadruple (A_c, B_c, C_c, D_c) must belong to a given polytope $\mathcal{K}$. More precisely, A_c, B_c, C_c, and D_c are often required to be block matrices with certain forced zero blocks. It is shown in [51] how the design of a fixed-order distributed controller for an interconnected system adopts the above formulation. The augmentation of the system (1) with the above unknown controller leads to the closed-loop system $\tilde{x}[\tau+1] = \tilde{A}\tilde{x}[\tau]$, where $\tilde{x}[\tau] = \left[\; x[\tau+1]^T \quad z_c[\tau+1]^T \;\right]^T$ and

$$\tilde{A} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix} \tag{65}$$

Note that this closed-loop system reduces to $x[\tau+1] = (A+BKC)x[\tau]$ in the static case. Since $\tilde{A}$ is a linear structured matrix with respect to (A_c, B_c, C_c, D_c), the state evolution equation $\tilde{x}[\tau+1] = \tilde{A}\tilde{x}[\tau]$ is bilinear, similar to its static counterpart $x[\tau+1] = (A+BKC)x[\tau]$. Hence, the parameterized matrix $\tilde{A}$ plays the role of $A+BKC$, which makes it possible to naturally generalize all results of this work to the dynamic case in both finite- and infinite-horizon cases. Note that the existence of a Lyapunov matrix guarantees the stability of $\tilde{A}$ or the internal stability of the system.

VII. Numerical Examples

In what follows, we offer multiple experiments on random systems and mass-spring systems. More simulations are provided in [45].

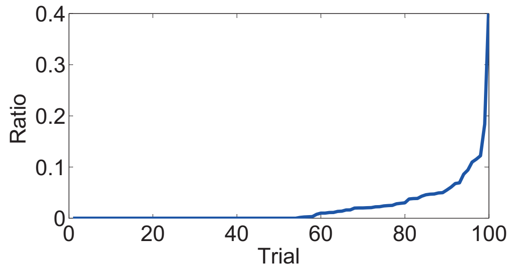


Fig. 5: The ratio $\frac{\lambda_2}{\lambda_1}$ obtained from the dense SDP relaxation of the finite-horizon ODC Problem (26) for 100 random systems.

A. Random Systems

Consider the system (1) with $n = 5$ and $m = r = 3$. The goal is to design a decentralized static controller $u[\tau] = Ky[\tau]$ (i.e., a diagonal matrix K) minimizing the cost function

$$\left(\sum_{\tau=0}^{20} x[\tau]^T x[\tau] + u[\tau]^T u[\tau]\right) + 10^{-3}\|K\|_F \tag{66}$$

This function accounts for the state regulation, input energy, and controller gain. The SDP relaxation problems (22), (26) and (33) have a 235×235, 168×168 and 29×29 matrix variables, respectively. According to Corollary 1, it is guaranteed that the sparse SDP relaxation problem (22) has a solution W^{opt} with rank at most 3 (i.e., at least 233 eigenvalues of this solution must be zero), independent of the values of the matrices A, B, C, and $x[0]$. Note that this result does not imply that all solutions of problem (22) have rank at most 3, but Theorem 1 can be used to find such a low-rank solution.

Since real-world systems are normally highly structured in many ways, we consider some structure for the system under study by assuming that B can be expressed as $[b \quad b \quad b]$ for some vector $b \in \mathbb{R}^5$. Assume that A, b, and $x[0]$ are normal random variables with the standard deviations 0.2, 1, and 1, respectively, while C is equal to $[I_3 \quad 0_{3\times 2}]$. We generated 100 random systems according to the above probability distributions for the parameters of the system and checked the rank of a low-rank solution of the sparse, dense, and computationally-cheap SDP relaxation problems for every trial. Let λ_1 and λ_2 denote the largest and the second largest eigenvalues of W^{opt} associated with the dense relaxation. We arranged the obtained 100 ratios $\frac{\lambda_2}{\lambda_1}$ in ascending order and subsequently labeled their corresponding trials as $1, 2, \ldots, 100$. Figure 5 plots the ratio $\frac{\lambda_2}{\lambda_1}$ for the ordered trials. It can be observed that this ratio is equal to 0 for 53 trials, implying that the dense SDP relaxation has found the solution of the ODC problem for 53 samples of the system. In addition, $\frac{\lambda_2}{\lambda_1}$ is less than 0.1 in 95 trials. Also, three near-global solutions of the ODC problem were found using different relaxations in all 100 cases. Figure 6 (a) depicts the (global) optimality degrees of these solutions after re-arranging the trials based on their associated optimality degrees for the dense SDP relaxation problem. Optimality degree is defined as

$$\text{Optimality degree (\%)} = 100 - \frac{\text{upper bound - lower bound}}{\text{upper bound}} \times 100$$

where "upper bound" and 'lower bound" denote the cost of the near-global controller recovered using the direct method and the optimal SDP cost, respectively. The optimality degree is an upper bound on the closeness of the cost of the near-optimal controller to the minimum cost, which is expressed in percentage. Notice that the employed optimality measure evaluates the global performance within the specified set of controllers. For example, the optimality degree of 100% means that a globally optimal controller is found among all linear static structured controllers.

As an alternative, we solved a penalized SDP relaxation with the penalty term $\Psi(W) = 0.5 \ \text{trace}\{W\}$ added to the objective of the SDP relaxation. Interestingly, the matrix $\tilde{W}^{\text{opt}}$ became rank 1 for all of the 100 trials. Figure 6 (b) depicts the optimality degrees associated with the penalized dense SDP relaxation problem of the 100 random systems. It can be seen that the optimality degree is greater than 99.8% for 69 trials and is never less than 98.2%.

B. Mass-Spring Systems

In this subsection, the aim is to evaluate the performance of the developed controller design techniques in Lyapunov domain on the *Mass-Spring* system, as a classical physical system. Consider a mass-spring system consisting of N masses. This system is exemplified in Figure 7 for $N = 2$. The system can be modeled in the continuous-time domain as

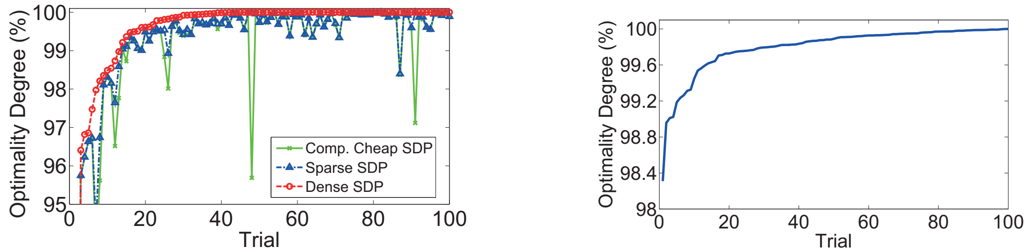
$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t) \tag{67}$$

where the state vector $x_c(t)$ can be partitioned as $[o_1(t)^T \quad o_2(t)^T]$ with $o_1(t) \in \mathbb{R}^n$ equal to the vector of positions and $o_2(t) \in \mathbb{R}^n$ equal to the vector of velocities of the N masses. We assume that $N = 10$ and adopt the values of A_c and B_c from [52]. The goal is to design a static sampled-data controller with a pre-specified structure (i.e., the controller is composed of a sampler, a static discrete-time structured controller and a zero-order holder). Consider two different control structures shown in Figure 8. The free parameters of each controller are colored in red in this figure. Notice that Structure (a) corresponds to a fully decentralized controller, where each local controller has access to the position and velocity of its associated mass. In contrast, Structure (b) allows limited communications between neighboring local controllers. Two ODC problems will be solved for these structures below.

Infinite-Horizon Deterministic ODC: In this experiment, we first discretize the system with the sampling time of 0.1 second and denote the obtained system as

$$x[\tau + 1] = Ax[\tau] + Bu[\tau], \qquad \tau = 0, 1, \ldots \tag{68}$$

It is aimed to design a constrained controller $u[\tau] = Kx[\tau]$ to minimize the cost function $\sum_{\tau=0}^{\infty} \left(x[\tau]^T x[\tau] + u[\tau]^T u[\tau]\right)$. Consider 100 randomly-generated initial states $x[0]$ with entries drawn from a normal distribution. We solved the computationally-cheap SDP relaxation of the infinite-horizon ODC problem combined with the direct recovery method to design a controller of Structure (a) minimizing the above cost function. The optimality degrees of the controllers designed for these 100 random trials are depicted in Figure 9. As can be seen, the optimality degree is better than 95% for



(a) (b)

Fig. 6: Optimal degrees of different relaxations for 100 random systems.

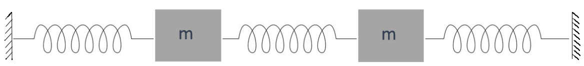
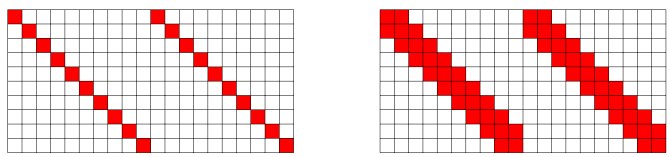


Fig. 7: Mass-spring system with two masses



(a) Decentralized (b) Distributed

Fig. 8: Two different structures (decentralized and distributed) for the controller K: the free parameters are colored in red (uncolored entries are set to zero).

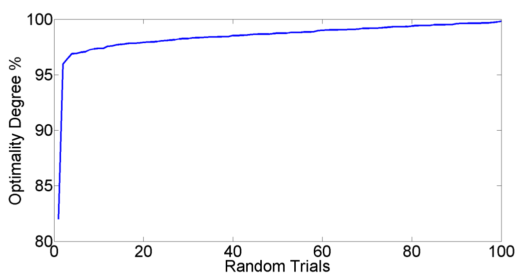


Fig. 9: Optimality degree (%) of the decentralized controller $\hat{K}$ for a mass-spring system under 100 random initial states.

more than 98 trials. It should be mentioned that all of these controllers stabilize the system.

Infinite-Horizon Stochastic ODC: Assume that the system is subject to both input disturbance and measurement noise. Consider the case $\Sigma_d = I_n$ and $\Sigma_v = \sigma I_r$, where σ varies from 0 to 5. Using the computationally-cheap relaxation problem (62) in conjunction with the indirect recovery method, a near-optimal controller is designed for each of the aforementioned control structures under various noise levels. The results are reported in Figure 10. The designed structured controllers are all stable with optimality degrees higher than 95% in the worst case and close to 99% in many cases.

VIII. Conclusions

This paper studies the optimal distributed control (ODC) problem for discrete-time deterministic and stochastic systems. The objective is to design a fixed-order distributed controller with a pre-determined structure to minimize a quadratic cost functional. Both time domain and Lyapunov domain formulations of the ODC problem are cast as rank-constrained optimization problems with only one non-convex constraint requiring the rank of a variable matrix to be 1. We propose semidefinite programming (SDP) relaxations of these problems. The notion of tree decomposition is exploited to prove the existence of a low-rank solution for the SDP relaxation problems with rank at most 3. This result can be a basis for a better understanding of the complexity of the ODC problem because it states that almost all eigenvalues of the SDP solution are zero. Moreover, multiple recovery methods are proposed to round the rank-3 solution to rank 1, from which a near-global controller may be retrieved. Computationally-cheap relaxations are also developed for finite-horizon, infinite-horizon, and stochastic ODC problems. These relaxations are guaranteed to exactly solve the LQR and H_2 problems for the classical centralized control problem. The results are tested on multiple examples. In our supplementary paper [45], we have conducted a case study on electrical power systems to further evaluate the performance of the methods proposed in this paper.

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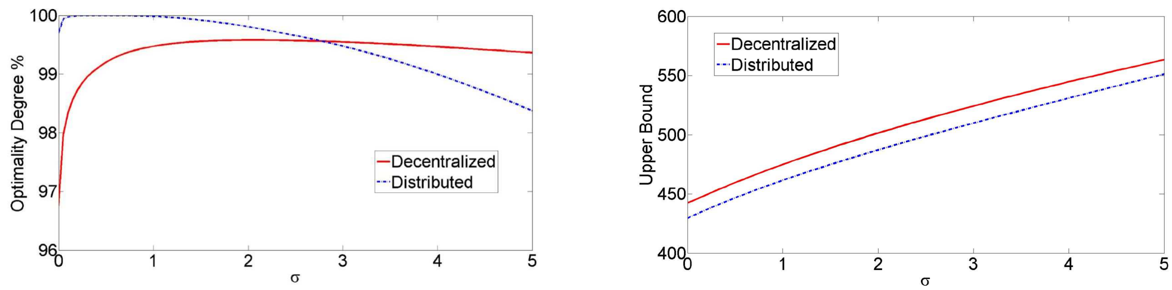

(a) Optimality degree

(b) Cost of near-optimal controller

Fig. 10: The optimality degree and optimal cost of the near-optimal controller designed for the mass-spring system for two different control structures. The noise covariance matrix Σ_v is assumed to be equal to σI_r, where σ varies over a wide range.

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