Martingale measures on trees: a convex programming approach to derivative pricing

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Abstract

In this note we show that the martingale measures that have a ubiquitous presence in modern mathematical finance arise very naturally as dual feasible variables for a linear program. Their existence and uniqueness follow very naturally from some simple assumptions on the market and general properties of linear programs. We extend this idea to a market with prportional transaction costs. This extension reveals some interesting connections between stochastic volatility and transaction costs.

1 Problem formulation

In this work we consider a discrete-time and finite-state market with n assets. We will assume that the probability structure is given by the tree $\mathcal{F}$ (see Figure 1). We will denote a generic state at time $t = 0, 1, \ldots, T$ by a ω_t and the set of states at time t by $\mathcal{F}_t$, i.e., at time t the market can be in any of the states in $\mathcal{F}_t$. The "parent" node at time $t-1$ of the node $\omega_t \in \mathcal{F}_t$ is denoted by $\omega_{t-1} \in \mathcal{F}_{t-1}$, i.e., the branch $(\omega_{t-1}, \omega_t) \in \mathcal{F}$. In other words, if the market is in state ω_t at time t then it must have been in state w_{t-1} at time $t-1$. We call a state $\omega_{t+1} \in \mathcal{F}_{t+1}$ a "child" of the state $\omega_t \in \mathcal{F}_t$ if it is a possible state for time $t+1$ given that the market is in state ω_t at time t. $\mathcal{C}(\omega_t)$ is the set of all "children" nodes of the node ω_t. Without loss of generality, we assume that $\mid \mathcal{F}_0 \mid = 1$ and $\mathcal{F}_T = \Omega$, the sample space.

In this section the market is assumed frictionless, i.e., there are no transaction costs. The case of proportional transaction costs will be studied in a subsequent section. The price process of the n assets in the market $\mathbf{S}_t$ is a positive process which is adapted to the tree or equivalently the filtration $\{\mathcal{F}_t\}_{(0 \leq t \leq T)}$.

A Trading strategy in the market is a collection

$$\boldsymbol{\Theta} = \{\boldsymbol{\theta}(\omega_t) \in \mathbf{R}^n, \ \forall \omega_t \in \mathcal{F}_t, \ \forall t \in [0, \ldots, T]\}, \tag{1}$$

where $\boldsymbol{\theta}(\omega_t)$ is the number of shares of the various assets held by the investor in state ω_t at time t. We allow only those trading strategies that are self-financing, i.e. the trading strategy does not call for any influx of fresh capital for times $t \geq 1$. Mathematically this condition is

$$\boldsymbol{\theta}(\omega_t) \cdot \mathbf{S}(\omega_t) \leq \boldsymbol{\theta}(\omega_{t-1}) \cdot \mathbf{S}(\omega_t), \ \forall \omega_t \in \mathcal{F}_t, \ \forall t \in [1, \ldots, T]. \tag{2}$$

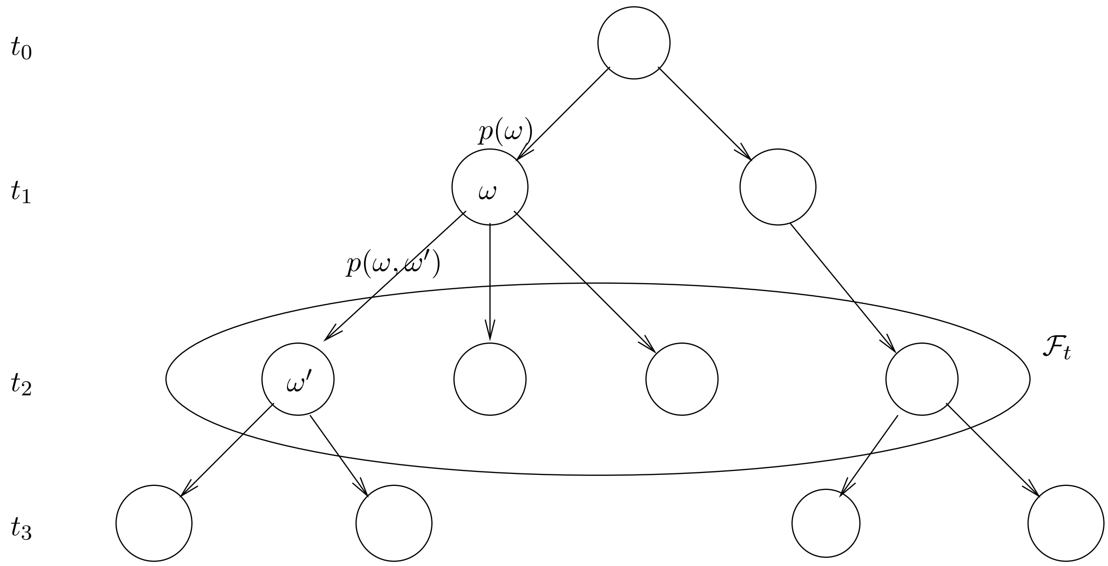

Figure 1: The event tree $\mathcal{F}$.

Since $\boldsymbol{\theta}(\omega_{t-1}) \cdot \mathbf{S}(\omega_t)$ is the total worth of the investor coming in to state $\omega_t \in \mathcal{F}_t$, (2) constrains the investor to move to only those positions $\boldsymbol{\theta}(\omega_t)$ that can be held without any input of fresh capital.

There is a European option B in the market that we wish to price. The option B assigns a certain pay-off for each state $\omega_T \in \mathcal{F}_T$. B is therefore a map from $\mathcal{F}_T \to \mathbf{R}$. In the absence of any other instruments we will attempt to price the option in terms of the prices of the primary assets. Since we are not guaranteed that the market is complete we will attempt to "super-replicate" the option and establish conditions when the super-replication is, in fact, a replication. We shall call a self-financing trading strategy $\boldsymbol{\Theta}$ super-replicating iff

$$\boldsymbol{\theta}(\omega_T) \cdot \mathbf{S}(\omega_T) \geq B(\omega_T), \ \forall \omega_T \in \mathcal{F}_T, \tag{3}$$

i.e., the pay-off the trading strategy is at least as large as the pay-off the option in every state of the world at time T.

The super-replication price B_u is then defined as the solution of the following linear program

$$\begin{array}{rl} \min & \boldsymbol{\theta}(\omega_0) \cdot \mathbf{S}(\omega_0), \\ \text{Subject to} & \\ \text{(a)} & \boldsymbol{\theta}(\omega_t) \cdot \mathbf{S}(\omega_T) \geq B(\omega_T), \quad \forall \omega_T \in \mathcal{F}_T, \\ \text{(b)} & \boldsymbol{\theta}(\omega_{t-1}) \cdot \mathbf{S}(\omega_t) \geq \boldsymbol{\theta}(\omega_t) \cdot \mathbf{S}(\omega_t), \quad \forall \omega_t \in \mathcal{F}_t, \ \forall t \in [1, \ldots, T]. \end{array} \tag{4}$$

Since $\boldsymbol{\theta}$ is self-financing, $\boldsymbol{\theta}(\omega_0) \cdot \mathbf{S}(\omega_0)$ is the initial outlay of the investor at time 0. The constraints in the linear program ensure that the trading strategy is self-financing and super-replicating. The objective then defines the price to be the minimum outlay among all the self-financing super-replicating strategies.

The rest of this section, and indeed the rest of this note, is concerned with manipulating this linear program and its dual to infer various properties of the replicating strategies and the market itself.

The Lagrangian function $L(\boldsymbol{\rho}, \boldsymbol{\pi})$ associated with the linear program (4) is

$$\begin{aligned} L(\boldsymbol{\rho}, \boldsymbol{\pi}) &= \inf_{\boldsymbol{\Theta}} \left\{ \boldsymbol{\theta}(\omega_0) \cdot \mathbf{S}(\omega_0) - \sum_{\omega \in \mathcal{F}_T} \boldsymbol{\rho}(\omega_T) \left[\boldsymbol{\theta}(\omega_t) \cdot \mathbf{S}(\omega_T) - B(\omega_T) \right] \right. \\ &\qquad \left. - \sum_{t=1}^{T} \sum_{\omega_t \in \mathcal{F}_t} \boldsymbol{\pi}(\omega_t) \left[\boldsymbol{\theta}(\omega_{t-1}) \cdot \mathbf{S}(\omega_t) \geq \boldsymbol{\theta}(\omega_t) \cdot \mathbf{S}(\omega_t) \right] \right\}, \qquad (5) \\ &= \sum_{\omega_T \in \mathcal{F}_T} \boldsymbol{\rho}(\omega_T) B(\omega_T) + \inf_{\boldsymbol{\Theta}} \left\{ \sum_{t=1}^{T-1} \sum_{\omega_t \in \mathcal{F}_t} \boldsymbol{\theta}(\omega_t) \cdot \left[\boldsymbol{\pi}(\omega_t) \mathbf{S}(\omega_t) - \sum_{\omega \in \mathcal{C}(\omega_t)} \boldsymbol{\pi}(\omega) \mathbf{S}(\omega) \right] \right. \\ &\qquad \left. + \left[\mathbf{S}(\omega_0) - \sum_{\omega \in \mathcal{C}(\omega_0)} \boldsymbol{\pi}(\omega) \mathbf{S}(\omega) \right] + \sum_{\omega_T \in \mathcal{F}_T} \boldsymbol{\theta}(\omega_t) \cdot \left[\left(\boldsymbol{\pi}(\omega_T) - \boldsymbol{\rho}(\omega_T) \right) \mathbf{S}(\omega_T) \right] \right\}, \qquad (6) \end{aligned}$$

where $\boldsymbol{\rho} \geq \mathbf{0}$ and $\boldsymbol{\pi} \geq \mathbf{0}$.

Since $(\boldsymbol{\rho}, \boldsymbol{\pi}) \geq \mathbf{0}$ and the set of primal feasible variables $\boldsymbol{\Theta}$ satisfies conditions (a) and (b) in (4), therefore it follows from (5) that

$$B_u \geq L(\boldsymbol{\rho}, \boldsymbol{\pi}). \qquad (7)$$

Since the left side of the equation does not contain the variables $(\boldsymbol{\rho}, \boldsymbol{\pi})$ it immediately follows that

$$B_u \geq \max_{(\boldsymbol{\rho}, \boldsymbol{\pi})} L(\boldsymbol{\rho}, \boldsymbol{\pi}). \qquad (8)$$

Without loss of generality we will restrict ourselves to $(\boldsymbol{\rho}, \boldsymbol{\pi})$ such that $L > -\infty$. Since the portfolio of holdings $\boldsymbol{\theta}(\omega_t)$ for the state ω_t at time t is unrestricted, it immediately follows that for $L > -\infty$ we must have that all the terms multiplying the portfolios $\boldsymbol{\theta}(\omega)$ in (6) are identically zero, i.e.,

$$\begin{array}{lll} \text{(a)} & \boldsymbol{\pi}(\omega_t) \mathbf{S}(\omega_t) - \sum_{\omega \in \mathcal{C}(\omega_t)} \boldsymbol{\pi}(\omega) \mathbf{S}(\omega) = 0 & \forall \omega \in \mathcal{F}_t, \ \forall t = 1, \ldots, T-1, \\ \text{(b)} & \mathbf{S}(\omega_0) - \sum_{\omega \in \mathcal{C}(\omega_0)} \boldsymbol{\pi}(\omega) \mathbf{S}(\omega) = 0 & \omega_0 \in \mathcal{F}_0, \\ \text{(c)} & \boldsymbol{\pi}(\omega_T) - \boldsymbol{\rho}(\omega_T) = 0 & \forall \omega_t \in \mathcal{F}_T. \end{array} \qquad (9)$$

Thus, the infimum in (6) is zero. Also,

$$L(\boldsymbol{\rho}, \boldsymbol{\pi}) = \sum_{\omega_T \in \mathcal{F}_t} \boldsymbol{\pi}(\omega_T) B(\omega_T). \qquad (10)$$

The dual program of the linear program (4) is, therefore,

$$\begin{array}{rlll} \max & \sum_{\omega_T \in \mathcal{F}_t} \boldsymbol{\pi}(\omega_T) B(\omega_T) & & \\ \text{Subject to} & & & \\ \text{(a)} & \boldsymbol{\pi}(\omega_t) \mathbf{S}(\omega_t) = \sum_{\omega \in \mathcal{C}(\omega_t)} \boldsymbol{\pi}(\omega) \mathbf{S}(\omega) & \forall \omega_t \in \mathcal{F}_t, \ \forall t = 0, \ldots, T-1, & \\ \text{(b)} & \boldsymbol{\pi}(\omega_0) = 1, \quad \boldsymbol{\pi}(\omega_t) \geq 0 & \forall \omega_t \in \mathcal{F}_t, \ \forall t = 1, \ldots, T, & \end{array} \qquad (11)$$

where we have combined (9)(a) and (9)(b) by defining $\boldsymbol{\pi}(\omega_0) = 1$. Since it is trivially obvious that the primal program has a strictly feasible point it follows from the Slater conditions that this problem has strong duality, i.e., the max in (11) is equal to the min in (4). Thus, one could equivalently solve the dual program.

Assume, without loss of generality, that $\mathbf{S}(\omega_0)_1 = 1$, i.e. asset 1 is the numeraire for the market. Then the first row of condition (11)(a) for $t = 0$ states that

$$\begin{aligned} \boldsymbol{\pi}(\omega_0)\mathbf{S}(\omega_0)_1 &= \sum_{\omega \in \mathcal{C}(\omega_0)} \boldsymbol{\pi}(\omega)\mathbf{S}(\omega)_1, \\ 1 &= \sum_{\omega \in \mathcal{C}(\omega_0)} \boldsymbol{\pi}(\omega)\mathbf{S}(\omega)_1. \end{aligned} \tag{12}$$

In (12) we have used the fact that $\boldsymbol{\pi}(\omega_0)$ was defined to be 1. Similarly for $\omega_t \in \mathcal{F}_t,\ t = 1, \ldots, T-1$ we get,

$$\boldsymbol{\pi}(\omega_t)\mathbf{S}(\omega_t)_1 = \sum_{\omega \in \mathcal{C}(\omega_t)} \boldsymbol{\pi}(\omega)\mathbf{S}(\omega)_1. \tag{13}$$

(12) and (13) together imply that $p : \mathcal{F} \to \mathbf{R}_+$ defined as

$$p(\omega_t) = \boldsymbol{\pi}(\omega_t)\mathbf{S}(\omega_t)_1,\ \forall \omega_t \in \mathcal{F}_t,\ \forall t = 0, \ldots, T, \tag{14}$$

is a probability measure on the tree $\mathcal{F}$.

The other rows of condition (b) can be rewritten as follows:

$$\boldsymbol{\pi}(\omega_t)\mathbf{S}(\omega_t)_k = \sum_{\omega \in \mathcal{C}(\omega_t)} \boldsymbol{\pi}(\omega)\mathbf{S}(\omega_t)_k, \tag{15}$$

$$[\boldsymbol{\pi}(\omega_t)\mathbf{S}(\omega_t)_1] \frac{\mathbf{S}(\omega_t)_k}{\mathbf{S}(\omega_t)_1} = \sum_{\omega \in \mathcal{C}(\omega_t)} [\boldsymbol{\pi}(\omega)\mathbf{S}(\omega)_1] \frac{\mathbf{S}(\omega)_k}{\mathbf{S}(\omega)_1}, \tag{16}$$

$$p(\omega_t)\overline{\mathbf{S}}(\omega_t)_k = \sum_{\omega \in cC(\omega_t)} p(\omega)\overline{\mathbf{S}}(\omega)_k, \tag{17}$$

where $\overline{\mathbf{S}}(\omega)_k = \frac{\mathbf{S}(\omega)_k}{\mathbf{S}(\omega)_1},\ 1 \leq k \leq n$. From (17) it immediately follows that p is a *martingale measure* for the *deflated* price process $\{\overline{\mathbf{S}}_t\}$.

One can rewrite the dual linear program as follows:

$$\begin{array}{rl} \max & \sum_{\omega_T \in \mathcal{F}_T} p(\omega_T)\overline{B}(\omega_T) \\ \text{Subject to} & \\ & p \text{ is a martingale measure for } \{\overline{\mathbf{S}}_t\}, \end{array} \tag{18}$$

where $\overline{B}(\omega_T) = \frac{B(\omega_T)}{\mathbf{S}(\omega_T)_1}$. Thus, the ubiquitous *martingale measures* are merely the *dual feasible variables* for the linear program (4).

Define

$$\mathbf{\Pi} = \{p \mid p \text{ is a martingale measure for } \overline{\mathbf{S}}_t\}. \tag{19}$$

The super-replication price B_u of the option B is given by the solution of the convex program

$$\begin{array}{rl} \max & \sum_{\omega_T \in \mathcal{F}_T} p(\omega_T)\overline{B}(\omega_T) \\ \text{Subject to} & \\ & p \in \ \mathbf{\Pi}, \end{array} \tag{20}$$

This reformulation of the pricing problem leads to some interesting consequences collected in the following two lemmas.

Lemma 1 *If* $\forall p \in \mathbf{\Pi}$, $p(\omega_T) > 0$, $\forall \omega_T \in \mathcal{F}_T$ *then the market is complete.*

Proof: Let p^* be an optimal solution of the dual problem (18). Therefore, $\boldsymbol{\pi}^*(\omega_t) = \frac{p^*(\omega_t)}{\mathbf{S}(\omega_t)_1}$ is an optimal solution of the dual problem (11).

$$\begin{aligned} p^*(\omega_T) > 0 \quad &\Rightarrow \quad \boldsymbol{\pi}^*(\omega_T) > 0, \\ &\Rightarrow \quad \boldsymbol{\rho}^*(\omega_t) > 0. \end{aligned}$$

By complementary slackness it then follows that condition (b) in the primal program (4) must be tight, i.e., for any optimal trading strategy $\mathbf{\Theta}^*$,

$$\boldsymbol{\theta}^*(\omega_T) \cdot \mathbf{S}(\omega_T) = B(\omega_T), \;\; \forall \omega_T \in \mathcal{F}_T. \tag{21}$$

This in turn implies that the strategy $\boldsymbol{\theta}^*$ *replicates* the option B. Since B is arbitrary, it follows that the market is complete. ■

THus, completeness essentially captures the fact that the martingale measures assign strictly positive probability to all the states of nature at the terminal time T.

Lemma 2 *If the market is complete and there are no arbitrage opportunities in the market, then* $\mathbf{\Pi} = \{p^*\}$*, i.e. there is unique martingale measure for the market.*

Proof: Suppose not. First notice that

$$p_1(\omega_T) = p_2(\omega_T), \;\; \forall \omega_T \in \mathcal{F}_T \Leftrightarrow p_1(\omega) = p_2(\omega), \;\; \forall \omega \in \mathcal{F}. \tag{22}$$

This follows from the fact that the probability assignment of the leaf nodes completely determines the probability assignment of every node in the tree. Therefore, if one shows that all the martingale measures agree on $\mathcal{F}_T$ it follows that the martingale measures are identical.

Consider the option $\overline{B} : \mathcal{F}_T \to \mathbf{R}$

$$\overline{B}(\omega) = \chi_{\omega_T}(\omega) = \begin{cases} 1 & \omega = \omega_T \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

Suppose p_1 is the corresponding optimal martingale measure. The price of $\overline{B}$ is

$$\overline{B}_u = \max_{p \in \mathbf{\Pi}} p(\omega_T) = p_1(\omega_T). \tag{24}$$

Since the market is complete there exists a self-financing strategy $\boldsymbol{\theta}^*$ such that

$$\boldsymbol{\theta}^*(\omega_T) \cdot \overline{\mathbf{S}}(\omega_T) \;\; = \;\; \overline{B}(\omega_T), \;\; \forall \omega_T \in \mathcal{F}_t, \tag{25}$$

$$\boldsymbol{\theta}^*(\omega_t) \cdot \overline{\mathbf{S}}(\omega_t) \;\; = \;\; \boldsymbol{\theta}^*(\omega_{t-1}) \cdot \overline{\mathbf{S}}(\omega_{t-1}). \tag{26}$$

Now consider the option $-\overline{B}$. Since $-\mathbf{\Theta}^*$ is clearly a replicating strategy for the option $-\overline{B}$ and there is no arbitrage in the market the price

$$(-\overline{B})_u = -\overline{B}_u = -p_1(\omega_T). \tag{27}$$

On the other hand from the dual formulation

$$(-\overline{B})_u = \max_{p\in\mathbf{\Pi}} -p(\omega_T) = -\min_{p\in\mathbf{\Pi}} p(\omega_T). \tag{28}$$

Therefore

$$-p_1(\omega_T) = -\max_{p\in\mathbf{\Pi}} p(\omega_T) = -\min_{p\in\mathbf{\Pi}} p(\omega_T), \tag{29}$$

or

$$p(\omega_T) = \text{ constant }, \ \forall p \in \mathbf{\Pi}. \tag{30}$$

Since ω_T was chosen arbitrarily, it follows that $\mathbf{\Pi}$ is a singleton. ■

Remark 1 *Notice that we need both completeness and no-arbitrage conditions to conclude the result in the last claim.*

Note that the uniqueness of the martingale measure follows from the fact that a probability measure on the leaves implies a unique measure on the tree. In continuous time-space models this may not be true. It is possible viewing martingale measures as dual variables might help clarify the "bubble" phenomenon.

Let $\mathbf{\Theta}$ be any self-financing strategy and $\boldsymbol{\pi} \in \mathbf{\Pi}$ be any martingale measure for the deflated price process $\{\overline{\mathbf{S}}_t\}$. Define,

$$\begin{aligned} \overline{\mathbf{W}}(\omega_t) &= \boldsymbol{\theta}(\omega_t) \cdot \overline{\mathbf{S}}(\omega_t), \\ &= \frac{\boldsymbol{\theta}(\omega_t) \cdot \mathbf{S}(\omega_t)}{\mathbf{S}(\omega_t)_1}. \end{aligned} \tag{31}$$

$$\tag{32}$$

$\overline{\mathbf{W}}(\omega_t)$ is, therefore, the *deflated* net worth of the investor in state $\omega_t \in \mathcal{F}_t$.

Lemma 3 *The deflated wealth process $\{\overline{\mathbf{W}}_t\}$ is a super martingale under every measure $p \in \mathbf{\Pi}$.*

Proof: The fact that $\{\overline{\mathbf{W}}_t\}$ is adapted to $\{\mathcal{F}_t\}_{(0\leq t\leq T)}$ is obvious. Now for the expected value,

$$\begin{aligned} \mathbf{E}^p\big(\overline{\mathbf{W}}_t \mid \mathcal{F}_{t-1}\big)(\omega_{t-1}) &= \sum_{\omega\in\mathcal{C}(\omega_{t-1})} \frac{p(\omega)}{\boldsymbol{\pi}(\omega_{t-1})} \overline{\mathbf{W}}(\omega), \\ &= \sum_{\omega\in\mathcal{C}(\omega_t)} \frac{p(\omega)}{\boldsymbol{\pi}(\omega_{t-1})} \left[\boldsymbol{\theta}(\omega) \cdot \overline{\mathbf{S}}(\omega)\right], \\ &\leq \sum_{\omega\in\mathcal{C}(\omega_t)} \frac{p(\omega)}{\boldsymbol{\pi}(\omega_{t-1})} \left[\boldsymbol{\theta}(\omega_{t-1}) \cdot \overline{\mathbf{S}}(\omega)\right], \end{aligned}$$

where the last equation follows from $\mathbf{\Theta}$ being self-financing.

Continuing,

$$\begin{aligned}\mathbf{E}^p\big(\overline{\mathbf{W}}_t \mid \mathcal{F}_{t-1}\big)(\omega_{t-1}) &\leq \boldsymbol{\theta}(\omega_{t-1}) \sum_{\omega\in\mathcal{C}(\omega_t)} \cdot \left[\frac{p(\omega)}{\boldsymbol{\pi}(\omega_{t-1})}\overline{\mathbf{S}}(\omega)\right], \\ &= \boldsymbol{\theta}(\omega_{t-1}) \cdot \overline{\mathbf{S}}(\omega_{t-1}). \end{aligned} \tag{33}$$

Equation (33) follows from the fact that p is a martingale measure for $\{\overline{\mathbf{S}}_t\}$. Therefore,

$$\mathbf{E}^p\big(\overline{\mathbf{W}}_t \mid \mathcal{F}_{t-1}\big) \leq \overline{\mathbf{W}}_{t-1}, \tag{34}$$

i.e., $\{\overline{\mathbf{W}}_t\}$ is a supermartingale. ■

2 Markets with transaction costs

In this section we extend the analysis of the last section to markets with transaction costs. We assume that asset 1 is a riskless asset and that it is the numeraire for the market. Since

$$\text{Asset 1 riskless} \Leftrightarrow \mathbf{S}(\omega_t)_1 = \text{ constant }, \ \forall \omega_t \in \mathcal{F}_t, \tag{35}$$

and the relevant quantity of interest is the *deflated* price process, $\{\frac{\mathbf{S}(\omega_t)}{\mathbf{S}(\omega_t)_1}\}$, therefore we can without loss of generality assume that $\mathbf{S}(\omega)_1 \equiv 1$. We assume that proportional transaction costs at rate λ levied on all risky assets. As before, we have a European option B maturing at time T that we wish to price.

Since there is an asymmetry between the assets because of the transaction costs, we alter the definition of the trading strategy to

$$\boldsymbol{\Theta} = \{(\beta(\omega_t), \boldsymbol{\theta}(\omega_t)), \ \forall \omega_t \in \mathcal{F}_t, \ \forall t \in [0, \dots, T]\}, \tag{36}$$

where $\beta(\omega_t)$ is the number of shares of the riskless asset held in state ω_t and $\boldsymbol{\theta}(\omega_t) \in \mathbf{R}^{n-1}$ is the position of the investor in the rest of the assets. The pricing problem is as follows:

$$\begin{array}{rl} \min & W \\ \text{Subject to} & \\ \text{(a)} & \beta(\omega_T) + \boldsymbol{\theta}(\omega_T)\cdot\mathbf{S}(\omega_T) - \lambda \mid \boldsymbol{\theta}(\omega_T) \mid \cdot\mathbf{S}(\omega_T) \geq B(\omega_T), \ \forall \omega_T \in \mathcal{F}_T, \\ \text{(b)} & \beta(\omega_{t-1}) + \boldsymbol{\theta}(\omega_{t-1})\cdot\mathbf{S}(\omega_t) \geq \beta(\omega_t) + \boldsymbol{\theta}(\omega_t)\cdot\mathbf{S}(\omega_t) \\ & \qquad +\lambda \mid \boldsymbol{\theta}(\omega_{t-1}) - \boldsymbol{\theta}(\omega_t) \mid \cdot\mathbf{S}(\omega_t), \ \forall \omega_t \in \mathcal{F}_t, \ \forall t \geq 1, \\ \text{(c)} & W \geq \beta(\omega_0) + \boldsymbol{\theta}(\omega_0)\cdot\mathbf{S}(\omega_0) + \lambda \mid \boldsymbol{\theta}(\omega_0) \mid \cdot\mathbf{S}(\omega_0), \end{array} \tag{37}$$

where (a) ensures that after the investor has liquidated his position in the market he still beats the payoff of the option; (b) is the modified condition of self-financing and (c) ensures that the investor has enough initial capital to buy into the position $(\beta(\omega_0), \boldsymbol{\theta}(\omega_0))$.

We introduce some new variables to convert the convex program (37) into a linear program,

$$\mathbf{c}(\omega_t) \geq \mid \boldsymbol{\theta}(\omega_t) - \boldsymbol{\theta}(o_{t-1}) \mid, \ \forall \omega_t \in \mathcal{F}_t, \ \forall t \geq 1, \tag{38}$$

$$\mathbf{c}(\omega_0) \geq \mid \boldsymbol{\theta}(\omega_0) \mid, \tag{39}$$

$$\boldsymbol{\alpha}(\omega_T) \geq \mid \boldsymbol{\theta}(\omega_T) \mid, \ \forall \omega_T \in \mathcal{F}_T. \tag{40}$$

Substituting the new variables in the convex program (37) we get an equivalent linear program,

$$
\begin{array}{rl}
& \min W \\
\text{Subject to} & \\
\text{(a)} & \beta(\omega_T) + \boldsymbol{\theta}(\omega_T)\cdot \mathbf{S}(\omega_T) - \lambda \boldsymbol{\alpha}(\omega_T)\cdot \mathbf{S}(\omega_T) \geq B(\omega_T),\ \forall \omega_T \in \mathcal{F}_T, \\
\text{(b)} & \beta(\omega_{t-1}) + \boldsymbol{\theta}(\omega_{t-1})\cdot \mathbf{S}(\omega_t) \geq \beta(\omega_t) + \boldsymbol{\theta}(\omega_t)\cdot \mathbf{S}(\omega_t) \\
& \qquad\qquad +\lambda \mathbf{c}(\omega_t)\cdot \mathbf{S}(\omega_t),\ \forall \omega_t \in \mathcal{F}_t,\ \forall t \geq 1, \\
\text{(c)} & W \geq \beta(\omega_0) + \boldsymbol{\theta}(\omega_0)\cdot \mathbf{S}(\omega_0) + \lambda \mathbf{c}(\omega_0)\cdot \mathbf{S}(\omega_0), \\
\text{(d)} & -\boldsymbol{\theta}(\omega_0) \leq \mathbf{c}(\omega_0 \leq \boldsymbol{\theta}(\omega_0), \\
\text{(e)} & \mathbf{c}(\omega_t) \geq \boldsymbol{\theta}(\omega_t) - \boldsymbol{\theta}(\omega_{t-1}),\ \forall \omega_t \in \mathcal{F}_t,\ \forall t \geq 1, \\
\text{(f)} & \mathbf{c}(\omega_t) \geq -\boldsymbol{\theta}(\omega_t) + \boldsymbol{\theta}(\omega_{t-1}),\ \forall t \geq 1,\ \forall t \geq 1, \\
\text{(g)} & -\boldsymbol{\theta}(\omega_T) \leq \boldsymbol{\alpha}(\omega_T) \leq \boldsymbol{\theta}(\omega_T),\ \forall \omega_T \in \mathcal{F}_T.
\end{array}
\tag{41}
$$

The associated Lagrangian function $L(\boldsymbol{\rho}, \boldsymbol{\pi}, \boldsymbol{\nu}, \boldsymbol{\mu}, \boldsymbol{\delta})$ is:

$$
\begin{aligned}
L \;=\; & \inf_{\boldsymbol{\Theta}} \Big\{ W - \sum_{\omega_T \in \mathcal{F}_T} \boldsymbol{\rho}(\omega_T) \left[\beta(\omega_T) + \boldsymbol{\theta}(\omega_T)\cdot \mathbf{S}(\omega_T) - \lambda \boldsymbol{\alpha}(\omega_T)\cdot \mathbf{S}(\omega_T) - B(\omega_T)\right] \\
& - \sum_{t=1}^{T} \sum_{\omega_t \in \mathcal{F}_t} \Big(\boldsymbol{\pi}(\omega_t) \left[\beta(\omega_{t-1}) + \boldsymbol{\theta}(\omega_{t-1})\cdot \mathbf{S}(\omega_t) - \beta(\omega_t) + \boldsymbol{\theta}(\omega_t)\cdot \mathbf{S}(\omega_t) + \lambda \mathbf{c}(\omega_t)\cdot \mathbf{S}(\omega_t)\right] \Big) \\
& - \boldsymbol{\nu}(\omega_0)\cdot [\mathbf{c}(\omega_0) - \boldsymbol{\theta}(\omega_0)] - \boldsymbol{\mu}(\omega_0)\cdot [c(\omega_0) + \boldsymbol{\theta}(\omega_0)] \\
& - \sum_{t=1}^{T} \sum_{\omega_t \in \mathcal{F}_t} \Big(\boldsymbol{\nu}(\omega_t)\cdot [\mathbf{c}(\omega_t) - \boldsymbol{\theta}(\omega_t) + \boldsymbol{\theta}(\omega_{t-1})] + \boldsymbol{\mu}(\omega_t)\cdot [\mathbf{c}(\omega_t) + \boldsymbol{\theta}(\omega_t) - \boldsymbol{\theta}(\omega_{t-1})] \Big) \\
& - \sum_{\omega_T \in \mathcal{F}_T} \Big(\boldsymbol{\gamma}(\omega_T)\cdot [\boldsymbol{\alpha}(\omega_T) - \boldsymbol{\theta}(\omega_T)] + \boldsymbol{\mu}(\omega_T)[\boldsymbol{\alpha}(\omega_T) + \boldsymbol{\theta}(\omega_T)] \Big) \\
& - \boldsymbol{\rho}(\omega_0)[W - \beta(\omega_0) + \boldsymbol{\theta}(\omega_0)\cdot \mathbf{S}(\omega_0) + \lambda \mathbf{c}(\omega_0)\cdot \mathbf{S}(\omega_0)] \Big\}.
\end{aligned}
\tag{42}
$$

Regrouping terms one gets,

$$
\begin{aligned}
L \;=\; & \sum_{\omega_T \in \mathcal{F}_T} \boldsymbol{\rho}(\omega_T) B(\omega_T) + \inf_{\boldsymbol{\Theta}} \Big\{ W(1 - \boldsymbol{\rho}(\omega_0)) + \beta(\omega_0) \Big[\boldsymbol{\rho}(\omega_0) - \sum_{\omega \in \mathcal{C}(\omega_o)} \boldsymbol{\pi}(\omega) \Big] \\
& + \sum_{t=1}^{T-1} \sum_{\omega_t \in \mathcal{F}_t} \beta(\omega_t) \Big[\boldsymbol{\pi}(\omega_t) - \sum_{\omega \in \mathcal{C}(\omega_t)} \boldsymbol{\pi}(\omega) \Big] \\
& + \sum_{\omega_T \in \mathcal{F}_T} \beta(\omega_T) \left[\boldsymbol{\rho}(\omega_T) - \boldsymbol{\pi}(\omega_T)\right] \\
& + \boldsymbol{\theta}(\omega_0)\cdot \Big[\boldsymbol{\rho}(\omega_0)\mathbf{S}(\omega_0) + \boldsymbol{\nu}(\omega_0) - \boldsymbol{\mu}(\omega_0) - \sum_{\omega \in \mathcal{C}(\omega_0)} (\boldsymbol{\pi}(\omega)\mathbf{S}(\omega) + \boldsymbol{\nu}(\omega) - \boldsymbol{\mu}(\omega)) \Big] \\
& + \sum_{t=1}^{T-1} \sum_{\omega_t \in \mathcal{F}_t} \Big\{ \boldsymbol{\theta}(\omega_t)\cdot \Big[\boldsymbol{\rho}(\omega_t)\mathbf{S}(\omega_t) + \boldsymbol{\nu}(\omega_t) - \boldsymbol{\mu}(\omega_t) - \sum_{\omega \in \mathcal{C}(\omega_t)} (\boldsymbol{\pi}(\omega)\mathbf{S}(\omega) + \boldsymbol{\nu}(\omega) - \boldsymbol{\mu}(\omega)) \Big] \Big\} \\
& + \sum_{\omega_T \in \mathcal{F}_T} \boldsymbol{\theta}(\omega_T)\cdot \big[\big(\boldsymbol{\pi}(\omega_{)} - \boldsymbol{\rho}(\omega_T)\big)\mathbf{S}(\omega_T) + \boldsymbol{\nu}(\omega_T) - \boldsymbol{\mu}(\omega_T) + \boldsymbol{\gamma}(\omega_T) - \boldsymbol{\delta}(\omega_T) \big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{\omega_T \in \mathcal{F}_T} \boldsymbol{\alpha}(\omega_T) \cdot [-\boldsymbol{\gamma}(\omega_T) - \boldsymbol{\delta}(\omega_T) + \lambda \boldsymbol{\rho}(\omega_T)\mathbf{S}(\omega_T)] \\
&+ \mathbf{c}(\omega_0) \cdot [-\boldsymbol{\nu}(\omega_0) - \boldsymbol{\mu}(\omega_0) + \boldsymbol{\rho}(\omega_0)\mathbf{S}(\omega_0)] \\
&+ \sum_{t=1}^{T-1} \sum_{\omega_t \in \mathcal{F}_t} \{\mathbf{c}(\omega_t) \cdot [-\boldsymbol{\nu}(\omega_t) - \boldsymbol{\mu}(\omega_t) + \boldsymbol{\rho}(\omega_t)\mathbf{S}(\omega_t)]\} \\
&+ \mathbf{c}(\omega_T) \cdot [-\boldsymbol{\nu}(\omega_T) - \boldsymbol{\mu}(\omega_T) + \boldsymbol{\rho}(\omega_T)\mathbf{S}(\omega_T)] \Bigg\}. \qquad (43)
\end{aligned}
$$

Arguing as before, we can restrict ourselves to the set of variables for which the value of the Lagrangian $L > -\infty$. Since $(\beta(\omega), \boldsymbol{\theta}(\omega))$, $\mathbf{c}(\omega)$ and $\boldsymbol{\alpha}(\omega)$ are all unrestricted, in order for the Lagrangian to be finite it must be that the terms multiplying the primal variables are all zero, i.e., the dual variables satisfy:

$$
\begin{aligned}
&\text{(a)} \qquad 1 - \boldsymbol{\pi}(\omega_0) = 0 \Rightarrow \boldsymbol{\rho}(\omega_0) = 1, && (44)\\
&\text{(b)} \qquad \boldsymbol{\rho}(\omega_0) - \sum_{\omega \in \mathcal{C}(\omega_0)} \boldsymbol{\pi}(\omega) \Rightarrow 1 = \sum_{\omega \in \mathcal{C}(\omega_0)} \boldsymbol{\pi}(\omega), && (45)\\
&\text{(c)} \qquad \boldsymbol{\pi}(\omega_t) = \sum_{\omega \in \mathcal{C}(\omega_t)} \boldsymbol{\pi}(\omega),\ \forall \omega_t \in \mathcal{F}_t, \forall t = 1, \ldots, T. && (46)
\end{aligned}
$$

(45) and (46) together imply that $\{\boldsymbol{\pi}(\cdot)\}$ is a probability measure on $\mathcal{F}$. Continuing further we have,

$$
\begin{aligned}
&\text{(d)} \qquad \boldsymbol{\rho}(\omega_T) = \boldsymbol{\pi}(\omega_T), && (47)\\
&\text{(e)} \qquad \mathbf{S}(\omega_0) + \boldsymbol{\nu}(\omega_0) - \boldsymbol{\mu}(\omega_0) = \sum_{\omega \in \mathcal{C}(\omega_0)} \left(\boldsymbol{\pi}(\omega)\mathbf{S}(\omega) + \boldsymbol{\nu}(\omega) - \boldsymbol{\mu}(\omega)\right), && (48)\\
&\text{(f)} \qquad \boldsymbol{\pi}(\omega_t)\mathbf{S}(\omega_t) + \boldsymbol{\nu}(\omega_t) - \boldsymbol{\mu}(\omega_t) = \sum_{\omega \in \mathcal{C}(\omega_t)} \left(\boldsymbol{\pi}(\omega)\mathbf{S}(\omega) + \boldsymbol{\nu}(\omega) - \boldsymbol{\mu}(\omega)\right), \forall \omega_t \in \mathcal{F}_t, && (49)\\
&\text{(g)} \qquad \boldsymbol{\nu}(\omega_T) - \boldsymbol{\mu}(\omega_T) = \boldsymbol{\delta}(\omega_T) - \boldsymbol{\gamma}(\omega_T), \forall \omega_T \in \mathcal{F}_T, && (50)\\
&\text{(h)} \qquad \boldsymbol{\gamma}(\omega_T) + \boldsymbol{\delta}(\omega_T) = \lambda \boldsymbol{\pi}(\omega_T)\mathbf{S}(\omega_T), \forall \omega_T \in \mathcal{F}_T, && (51)\\
&\text{(i)} \qquad \boldsymbol{\nu}(\omega_0) + \boldsymbol{\mu}(\omega_0) = \lambda \mathbf{S}(\omega_0), && (52)\\
&\text{(j)} \qquad \boldsymbol{\nu}(\omega_t) + \boldsymbol{\mu}(\omega_t) = \lambda \boldsymbol{\pi}(\omega_t)\mathbf{S}(\omega_t), \forall \omega_t \in \mathcal{F}_t, \forall t = 1, \ldots, T. && (53)
\end{aligned}
$$

Since $\boldsymbol{\nu}(\omega), \boldsymbol{\mu}(\omega) \geq 0$ therefore $\boldsymbol{\pi}(\omega) = 0$ implies $\boldsymbol{\mu}(\omega) = \boldsymbol{\nu}(\omega) = 0$. For all $\omega \in \mathcal{F}$ and $\forall k = 1, \ldots, n$ define new variables:

$$
\tilde{\boldsymbol{\nu}}(\omega)_k = \begin{cases} \frac{\boldsymbol{\nu}(\omega)_k}{\boldsymbol{\pi}(\omega)\mathbf{S}(\omega)_k} & \text{if } \boldsymbol{\pi}(\omega) > 0, \\ 0 & \text{otherwise,} \end{cases} \qquad (54)
$$

$$
\tilde{\boldsymbol{\mu}}(\omega)_k = \begin{cases} \frac{\boldsymbol{\mu}(\omega)_k}{\boldsymbol{\pi}(\omega)\mathbf{S}(\omega)_k} & \text{if } \boldsymbol{\pi}(\omega) > 0, \\ 0 & \text{otherwise.} \end{cases} \qquad (55)
$$

In terms of these new variables one can rewrite condition (49) as follows:

$$
\boldsymbol{\pi}(\omega_t)\left[\mathbf{I} + \mathbf{M}(\omega_t) - \mathbf{N}(\omega_t)\right]\mathbf{S}(\omega_t) = \sum_{\omega \in \mathcal{C}(\omega_t)} \left\{\boldsymbol{\pi}(\omega)\left[\mathbf{I} + \mathbf{M}(\omega) - \mathbf{N}(\omega)\right]\mathbf{S}(\omega)\right\}, \qquad (56)
$$

where

$$
\begin{aligned}
\mathbf{M}(\omega) &= \text{diag}\,[\tilde{\boldsymbol{\mu}}(\omega)_1, \ldots, \tilde{\boldsymbol{\mu}}(\omega)_n], && (57)\\
\mathbf{N}(\omega) &= \text{diag}\,[\tilde{\boldsymbol{\nu}}(\omega)_1, \ldots, \tilde{\boldsymbol{\nu}}(\omega)_n]. && (58)
\end{aligned}
$$

Therefore, $\{\boldsymbol{\pi}(\cdot)\}$ is a martingale measure for the *perturbed* price process

$$\tilde{\mathbf{S}}_t = [\mathbf{I} + \mathbf{M}_t - \mathbf{N}_t]\, \mathbf{S}_t. \tag{59}$$

From condition (52) it follows that

$$0 \leq \tilde{\boldsymbol{\mu}}(\omega)_k + \tilde{\boldsymbol{\nu}}(\omega)_k = \lambda, \tag{60}$$

or

$$-\lambda \leq \tilde{\boldsymbol{\mu}}(\omega)_k + \tilde{\boldsymbol{\nu}}(\omega)_k \leq \lambda. \tag{61}$$

Substituting this into (59) one concludes

$$1 - \lambda \leq \frac{\tilde{\mathbf{S}}(\omega)_k}{\mathbf{S}(\omega)_k} \leq 1 + \lambda. \tag{62}$$

Incorporating the new variables and the conditions (44)-(53) the dual of the linear program (41) is

$$\max \textstyle\sum_{\omega_T \in \mathcal{F}_T} \boldsymbol{\pi}(\omega_T) B(\omega_T) \tag{63}$$

Subject to

(a) $\{\boldsymbol{\pi}(\cdot)\}$ is a martingale measure for the *perturbed* price $\{\tilde{\mathbf{S}}\}$,

(b) $\tilde{\mathbf{S}}_t = [\mathbf{I} + \mathbf{M}_t - \mathbf{N}_t]\, \mathbf{S}_t,$

(c) $\mathbf{M}_t, \mathbf{N}_t$ diagonal $, \mathbf{M}_t \geq 0, \mathbf{N}_t \geq 0, \mathbf{M}_t + \mathbf{N}_t = \lambda \mathbf{I}.$

Thus in a market with transaction costs the maximization is still over martingale measures, but now the investor is allowed to perturb the price process. This dual program also highlights the intimate connection between transaction costs and stochastic volatility. The processes $\{(\mathbf{M}_t, \mathbf{N}_t)\}$ essentially change the volatility of the stock price process $\mathbf{S}_t$. The extent to which the volatility of the price process can be perturbed depends on the transaction cost since $\mathbf{M}_t + \mathbf{N}_t = \lambda \mathbf{I}$. It trivially reduces to the no-transaction cost problem on setting the cost $\lambda = 0$.

Lemma 4 *If for all choices of the perturbing processes $\{(\mathbf{M}_t, \mathbf{N}_t)\}$, $\exists!$ martingale measure $\boldsymbol{\pi}$ such that $\boldsymbol{\pi}(\omega_T) > 0$, $\forall \omega_T \in \mathcal{F}_t$, then the market is complete and all payoffs can be replicated.*

Proof: Let $(\beta^*, \boldsymbol{\theta}^*, \mathbf{c}^*, \boldsymbol{\alpha}^*)$ and $(\boldsymbol{\pi}^*, \boldsymbol{\mu}^*, \boldsymbol{\nu}^*, \boldsymbol{\gamma}^*, \boldsymbol{\delta}^*)$ be the optimal primal and dual solutions respectively. From condition (47) we have

$$\boldsymbol{\rho}(\omega_T) = \boldsymbol{\pi}(\omega_T) > 0,$$

therefore by complementary slackness the constraint (41) (a) will be tight, i.e.,

$$\beta(\omega_T) + \boldsymbol{\theta}(\omega_T) \cdot \mathbf{S}(\omega_T) - \lambda \boldsymbol{\alpha}(\omega_T) \cdot \mathbf{S}(\omega_T) = B(\omega_T),\ \forall \omega_T \in \mathcal{F}_T$$

From (51) we have

$$0 \leq \boldsymbol{\gamma}^*(\omega_T) + \boldsymbol{\delta}^*(\omega_T) = \lambda \boldsymbol{\pi}^*(\omega_T) \mathbf{S}(\omega_T).$$

By assumption, $\boldsymbol{\pi}^*(\omega_T) > 0$ and $\mathbf{S}(\omega_T)_k > 0$ therefore,

$$\max\{\boldsymbol{\gamma}^*(\omega_T)_k, \boldsymbol{\delta}^*(\omega_T)_k\} > 0,\ \forall k = 1, \ldots, n,\ \forall \omega_T \in \mathcal{F}_T.$$

By complementary slackness

$$(\boldsymbol{\alpha}^*(\omega_T)_k - \boldsymbol{\theta}^*(\omega_T)_k)\,\boldsymbol{\gamma}^*(\omega_T)_k = 0,$$

and

$$(\boldsymbol{\alpha}^*(\omega_T)_k + \boldsymbol{\theta}^*(\omega_T)_k)\,\boldsymbol{\delta}^*(\omega_T)_k = 0.$$

Consider the following three cases:

(i) $\boldsymbol{\gamma}^*(\omega_T)_k > 0$ and $\boldsymbol{\delta}^*(\omega_T)_k > 0$
By complementary slackness

$$\left.\begin{array}{r}\boldsymbol{\alpha}^*(\omega_T)_k = \boldsymbol{\theta}^*(\omega_T)_k \\ \boldsymbol{\alpha}^*(\omega_T)_k = -\boldsymbol{\theta}^*(\omega_T)_k\end{array}\right\} \Rightarrow \boldsymbol{\alpha}^*(\omega_T)_k = \mid \boldsymbol{\theta}^*(\omega_T)_k \mid = 0.$$

(ii) $\boldsymbol{\gamma}^*(\omega_T)_k > 0$ and $\boldsymbol{\delta}^*(\omega_T)_k = 0$
By complementary slackness

$$\alpha^*(\omega_T)_k = \boldsymbol{\theta}^*(\omega_T)_k.$$

But $\boldsymbol{\alpha}(\omega_T)_k \geq -\boldsymbol{\theta}^*(o_T)_k$, i.e.,

$$\boldsymbol{\theta}(\omega_T)_k \geq -\boldsymbol{\theta}(o_T)_k \Rightarrow \boldsymbol{\theta}(o_T)_k \geq 0.$$

Therefore,

$$\boldsymbol{\alpha}^*(\omega_T)_k = \mid \boldsymbol{\theta}^*(\omega_T)_k \mid .$$

(ii) $\boldsymbol{\gamma}^*(\omega_T)_k = 0$ and $\boldsymbol{\delta}^*(\omega_T)_k > 0$
By complementary slackness

$$\alpha^*(\omega_T)_k = -\boldsymbol{\theta}^*(\omega_T)_k.$$

But $\boldsymbol{\alpha}(\omega_T)_k \geq \boldsymbol{\theta}^*(o_T)_k$, i.e.,

$$-\boldsymbol{\theta}(\omega_T)_k \geq \boldsymbol{\theta}(\omega_T)_k \Rightarrow \boldsymbol{\theta}(\omega_T)_k \leq 0.$$

Therefore,

$$\boldsymbol{\alpha}^*(\omega_T)_k = \mid \boldsymbol{\theta}^*(\omega_T)_k \mid .$$

Substituting back one gets

$$\beta(\omega_T) + \boldsymbol{\theta}(\omega_T) \cdot \mathbf{S}(\omega_T) - \lambda \mid \boldsymbol{\theta}(\omega_T) \mid \cdot \mathbf{S}(\omega_T) = B(\omega_T),\ \forall \omega_T \in \mathcal{F}_T,$$

i.e., the optimal trading strategy replicates the payoff B. ∎

We now restrict ourselves to a binary market with 1 risky asset and 1 riskless asset. By a binary market we mean that the tree $\mathcal{F}$ is binary, i.e. every state $\omega \in \mathcal{F}$ has two children $\mathcal{C}(\omega) = \{\omega_1, \omega_2\}$.

Lemma 5 *If* $\forall \omega \in \mathcal{F}_t,\ t = 0, \ldots, T-1$

$$\max\left\{\frac{\mathbf{S}(\omega_1)}{\mathbf{S}(\omega)}, \frac{\mathbf{S}(\omega_2)}{\mathbf{S}(\omega)}\right\} > \frac{1+\lambda}{1-\lambda}, \tag{64}$$

and

$$\min\left\{\frac{\mathbf{S}(\omega_1)}{\mathbf{S}(\omega)}, \frac{\mathbf{S}(\omega_2)}{\mathbf{S}(\omega)}\right\} < \frac{1-\lambda}{1+\lambda}, \tag{65}$$

then the market is complete.

Proof: Since the probability assignment in a binary tree is recursive, one can restrict attention to a generic state ω and its *children* $\mathcal{C}(\omega) = \{\omega_1, \omega_2\}$. Let

$$\epsilon_i = \boldsymbol{\mu}_i - \boldsymbol{\nu}_i,\ \ i = 0, 1, 2.$$

Let

$$\alpha_i = \frac{\mathbf{S}(\omega_i)}{\mathbf{S}(\omega)},\ \ i = 1, 2,$$

and without any loss of generality assume that $\alpha_1 > \alpha_2$. The martingale measure $(\boldsymbol{\pi}(\omega_1), \boldsymbol{\pi}(\omega_2)$ satisfies the set of equations:

$$\begin{aligned} \boldsymbol{\pi}(\omega_1)(1+\epsilon_1)\alpha_1\mathbf{S}(\omega) + \boldsymbol{\pi}(\omega_2)(1+\epsilon_2)\alpha_2\mathbf{S}(\omega) &= (1+\epsilon_0)\mathbf{S}(\omega), && (66) \\ \boldsymbol{\pi}(\omega_1) + \boldsymbol{\pi}(\omega_2) &= 1. && (67) \end{aligned}$$

Solving these equations we get

$$\begin{aligned} \boldsymbol{\pi}(\omega_1) &= \frac{(1+\epsilon_0) - \alpha_2(1+\epsilon_2)}{\alpha_1(1+\epsilon_1) - \alpha_2(1+\epsilon_2)}, && (68) \\ \boldsymbol{\pi}(\omega_2) &= \frac{\alpha_1(1+\epsilon_2) - (1+\epsilon_0)}{\alpha_1(1+\epsilon_1) - \alpha_2(1+\epsilon_2)}. && (69) \\ & && (70) \end{aligned}$$

Therefore, for $(\boldsymbol{\pi}(\omega_1), \boldsymbol{\pi}(\omega_2))$ to be a probability measure assigning positive mass to both ω_1 and ω_2 we must have that

$$\boldsymbol{\pi}(\omega_1) = (1+\epsilon_0) - \alpha_2(1+\epsilon_2) > 0 \Rightarrow \alpha_2 < \frac{1+\epsilon_0}{1+\epsilon_2}. \quad (71)$$

Now,

$$\inf_{\{\epsilon_i\}} \left\{ \frac{1+\epsilon_0}{1+\epsilon_2} \right\} = \frac{1-\lambda}{1+\lambda}, \quad (72)$$

therefore from the conditions of the lemma it follows that (71) is satisfied. Similarly one can show that the other condition ensures that $\boldsymbol{\pi}(\omega_2) > 0$. We now appeal to the last lemma to complete the proof. ■