Convergence in Distribution and CLT

- "Convergence in distribution":
  $X_n \xrightarrow{\text{d}} X$ iff $\lim_{n\to\infty}|F_n(\cdot) - F(\cdot)| = 0$ at all continuity points of $F(\cdot)$, where F_n and F are the respective cdfs of X_n and X.

  We say $F(\cdot)$ is the **limiting distribution** of X.

  The **limiting mean** and **limiting variance** of a rv are the mean and variance of its limiting distribution.

- Note how the different notions of convergence relate:

$$X_n \xrightarrow{\text{qm}} c \Rightarrow X_n \xrightarrow{\text{P}} c \Rightarrow X_n \xrightarrow{\text{d}} c$$

- Suppose we have a sample estimator $\hat{\theta}_n$ of θ. We want to use the limiting distribution of $\hat{\theta}_n$ to obtain an approximation for the unknown small sample distribution. However,

  1. $\text{plim } \hat{\theta}_n = \theta \Rightarrow \hat{\theta}_n \xrightarrow{\text{d}} \theta$, but this means the distribution of $\hat{\theta}_n$ collapses to a spike.

     To get around this problem we use a stabilizing transformation such as $\sqrt{n}(\hat{\theta}_n - \theta)$.

2. But now what is the distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$? This is given by the **Central Limit Theorem**:

   (a) General statement of CLT: Given sufficient unrelatedness and boundedness,
   $\sqrt{n}$ (sample mean – population mean) $\xrightarrow{\text{d}}$
   $N(0,$ variance of object whose mean is being taken $)$.

   (b) Specific statement: Given $X_i \sim$ iid with $E(X_i) = \mu$ *and* $\text{var}(X_i) = \sigma^2$ then

   $$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{\text{d}} N[0, \sigma^2]$$

   Note: $\sigma^2 \neq \text{var}(\bar{X}_n)$

   (c) To see why stabilizing transformation is important consider how we derive the **asymptotic distribution** of $\bar{X}_n$, where the asymptotic distribution is used to approximate the true finite sample distribution of $\bar{X}_n$.

   (d) To derive the asymptotic distribution we use the limiting distribution of a function of the rv of interest—in this case $\bar{X}_n$.

   (e) From the specific statement of the CLT above we can say

   $$\bar{X}_n \overset{a}{\sim} N\left[\mu, \frac{\sigma^2}{n}\right]$$

   Note why having $\sqrt{n}$ out in front is important: if $\bar{X}_n$ has mean μ and variance $\frac{\sigma^2}{n}$ then $(\bar{X}_n - \mu)$ has mean 0 and variance $\frac{\sigma^2}{n}$.

   (f) Then $k(\bar{X}_n - \mu)$ has mean 0 and variance $\frac{k^2\sigma^2}{n}$.

   We need to pick k such that it gives us a stable variance as $n \to \infty$.
   – If $k > \sqrt{n} \Rightarrow \text{var} \to \infty$.
   – If $k < \sqrt{n} \Rightarrow \text{var} \to 0$.
   – So choose $k = \sqrt{n}$ as our normalizing constant.

- Note: all of the above results extend to the multivariate case. E.g., if we have

$$\bar{\mathbf{X}}_{\mathbf{n}} = \begin{bmatrix} \bar{X}_{1n} \\ \vdots \\ \bar{X}_{kn} \end{bmatrix} \text{ or } \hat{\boldsymbol{\theta}}_{\mathbf{n}} = \begin{bmatrix} \hat{\theta}_{1n} \\ \vdots \\ \hat{\theta}_{kn} \end{bmatrix}$$