Intuition behind the Law of Iterated Expectations

- Simple version of the law of iterated expectations (from Wooldridge's *Econometric Analysis of Cross Section and Panel Data*, p. 29):

$$E(y) = E_{\mathbf{x}}[E(y|\mathbf{x})].$$

- Think of $\mathbf{x}$ as a discrete vector taking on possible values $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_M$, with probabilities $p_1, p_2, \ldots, p_M$. Then the LIE says:

$$E(y) = p_1 E(y|\mathbf{x} = \mathbf{c}_1) + p_2 E(y|\mathbf{x} = \mathbf{c}_2) + \cdots + p_M E(y|\mathbf{x} = \mathbf{c}_M)$$

- That is, $E(y)$ is simply a weighted average of the $E(y|\mathbf{x} = \mathbf{c}_j)$, where the weight p_j is the probability that $\mathbf{x}$ takes on the value of $\mathbf{c}_j$. In other words, a weighted average of averages.
- E.g., suppose we are interested in average IQ generally, but we have measures of average IQ by gender. We could figure out the quantity of interest by weighting average IQ by the relative proportions of men and women.

Decomposition of variance (Wooldridge, p. 31)

- Proof that $\text{var}(y) = \text{var}_{\mathbf{x}}[E(y|\mathbf{x})] + E_{\mathbf{x}}[\text{var}(y|\mathbf{x})]$ (i.e., the variance of y decomposes into the variance of the conditional mean plus the expected variance around the conditional mean).

$$\begin{aligned}\text{var}(y) = E[(y - E(y))^2] &= E[(y - E(y|\mathbf{x}) + E(y|\mathbf{x}) + E(y))^2] \\ &= E[(y - E(y|\mathbf{x}))^2] + E[(E(y|\mathbf{x}) - E(y))^2] \\ &+ 2E[(y - E(y|\mathbf{x}))(E(y|\mathbf{x}) - E(y))]\end{aligned}$$

- The last term drops out b/c $E[(y - E(y|\mathbf{x}))(E(y|\mathbf{x}) - E(y))] = 0$ b/c $E(u|\mathbf{x}) = E[(y - E(y|\mathbf{x}))|\mathbf{x}] = E(y|\mathbf{x}) - E(y|\mathbf{x}) = 0$.
- We can then use the LIE to rewrite what is left as

$$E_{\mathbf{x}}\{E[(y - E(y|\mathbf{x}))^2|\mathbf{x}]\} + E[(E(y|\mathbf{x}) - E_{\mathbf{x}}[E(y|\mathbf{x})])^2]$$

  which is just $E_{\mathbf{x}}[\text{var}(y|\mathbf{x})] + \text{var}_{\mathbf{x}}[E(y|\mathbf{x})]$.