Answer Key for POLS W4912 Midterm Exam (Spring 2008)

- Question 1

  (For this problem, you would not have needed to include all of the following steps to receive full credit; you could have relied on equations from the notes, for instance. The complete answer is provided for clarity.)

  If the true equation is

  $$y = \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 x_2 + \epsilon$$

  then we can write this as a partitioned matrix where

  $$\mathbf{y} = \begin{bmatrix} \mathbf{X_M} & \mathbf{X_I} \end{bmatrix} \begin{bmatrix} \beta_\mathbf{M} \\ \beta_\mathbf{I} \end{bmatrix} + \epsilon$$

  Note that the $\mathbf{X_M}$ portion of the partitioned matrix contains the variables representing the two main effects, x_1 and x_2, while the $\mathbf{X_I}$ portion of the partitioned matrix contains the interaction term $x_1 x_2$. The β vector is similar. (Note that you could also solve this problem using a partitioned matrix where one partitioned element contained only one main effect, and the other partitioned element contained the other main effect and the interaction term. You still would be able to demonstrate bias.)

  Our $X'X$ matrix, therefore looks as follows:

  $$\begin{bmatrix} \mathbf{X_M} \\ \mathbf{X_I} \end{bmatrix} \begin{bmatrix} \mathbf{X_M} & \mathbf{X_I} \end{bmatrix} = \begin{bmatrix} \mathbf{X_M'X_M} & \mathbf{X_M'X_I} \\ \mathbf{X_I'X_M} & \mathbf{X_I'X_I} \end{bmatrix}$$

  Our normal equation used to get our estimates of $\hat{\beta}$ in matrix form is $(\mathbf{X'X})\hat{\beta} = \mathbf{X'y}$. Therefore,

  $$\begin{bmatrix} \mathbf{X_M'X_M} & \mathbf{X_M'X_I} \\ \mathbf{X_I'X_M} & \mathbf{X_I'X_I} \end{bmatrix} \begin{bmatrix} \hat{\beta}_\mathbf{M} \\ \hat{\beta}_\mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{X_M'y} \\ \mathbf{X_I'y} \end{bmatrix}$$

  This produces two equations.

  $$(\mathbf{X_M'X_M})\hat{\beta}_\mathbf{M} + (\mathbf{X_M'X_I})\hat{\beta}_\mathbf{I} = \mathbf{X_M'y}$$
  $$(\mathbf{X_I'X_M})\hat{\beta}_\mathbf{M} + (\mathbf{X_I'X_I})\hat{\beta}_\mathbf{I} = \mathbf{X_I'y}$$

  Pre-multiplying the second equation by $(\mathbf{X_I'X_I})^{-1}$ allows us to solve for $\hat{\beta}_\mathbf{I}$:

  $$\hat{\beta}_\mathbf{I} = (\mathbf{X_I'X_I})^{-1}\mathbf{X_I'y} - (\mathbf{X_I'X_I})^{-1}(\mathbf{X_I'X_M})\hat{\beta}_\mathbf{M}$$

  We can see that the estimate of $\hat{\beta}_\mathbf{I}$ depends on $\hat{\beta}_\mathbf{M}$. This equation is found in section 7.1 of the notes on page 74.

  If we incorrectly specify the model omitting the main effects, we would have

  $$y = \beta_3 x_1 x_2 + \epsilon$$

Solving the normal equation here produces

$$\hat{\beta}_{\mathbf{I}} = (\mathbf{X}_{\mathbf{I}}'\mathbf{X}_{\mathbf{I}})^{-1}\mathbf{X}_{\mathbf{I}}'\mathbf{y}$$

Since the latter half of the equation from the correctly specified model is missing, we know that this $\hat{\beta}_{\mathbf{I}}$ is biased *unless* $\hat{\beta}_{\mathbf{M}} = 0$ or $\mathbf{X}_{\mathbf{I}} \perp \mathbf{X}_{\mathbf{M}}$. (Since $\mathbf{X}_{\mathbf{I}}$ is constructed from $\mathbf{X}_{\mathbf{M}}$, however, it will *not* be orthogonal to $\mathbf{X}_{\mathbf{M}}$.)

- Question 2a

The hint instructed you to recall that the regression line passes through the mean of the data. From section 3.5.1 (page 34) of the notes, we have, for a bivariate regression,

$$\begin{aligned} \bar{y} &= \hat{\beta}_0 + \hat{\beta}_1\bar{x} \\ \Rightarrow \hat{\beta}_0 &= \bar{y} - \hat{\beta}_1\bar{x} \end{aligned}$$

We can substitute in the true equation for $\bar{y}$:

$$\begin{aligned} \hat{\beta}_0 &= (\beta_0 + \beta_1\bar{x} + \bar{\epsilon}) - \hat{\beta}_1\bar{x} \\ &= \beta_0 + \beta_1\bar{x} - \hat{\beta}_1\bar{x} + \bar{\epsilon} \\ E[\hat{\beta}_0] &= \beta_0 + \beta_1\bar{x} - \beta_1\bar{x} + 0 \\ E[\hat{\beta}_0] &= \beta_0 \end{aligned}$$

This demonstrates unbiasedness.

A different proof is also presented in Appendix 3A.2 of Gujarati.

- Question 2b

Using the formula for covariance from section 1.3.4 (page 13) of the notes:

$$\begin{aligned} Cov(\hat{\beta}_0, \hat{\beta}_1) &= E[(\hat{\beta}_0 - E[\hat{\beta}_0])(\hat{\beta}_1 - E[\hat{\beta}_1])] \\ &= E[(\hat{\beta}_0 - \beta_0)(\hat{\beta}_1 - \beta_1)] \text{ because of unbiasedness} \\ &= E[\hat{\beta}_0\hat{\beta}_1 - \beta_0\hat{\beta}_1 - \hat{\beta}_0\beta_1 + \beta_0\beta_1] \\ &= E[\hat{\beta}_0\hat{\beta}_1] - \beta_0\beta_1 - \beta_0\beta_1 + \beta_0\beta_1 \\ &= E[\hat{\beta}_0\hat{\beta}_1] - \beta_0\beta_1 \end{aligned}$$

Substituting in the equation from above that describes $\hat{\beta}_0$ in terms of the mean of the data

$$\begin{aligned} &= E[(\bar{y} - \hat{\beta}_1\bar{x})\hat{\beta}_1] - \beta_0\beta_1 \\ &= E[\bar{y}\hat{\beta}_1] - E[\hat{\beta}_1^2\bar{x}] - \beta_0\beta_1 \end{aligned}$$

$$= \bar{y}E[\hat{\beta}_1] - \bar{x}E[\hat{\beta}_1^2] - \beta_0\beta_1$$

This does not equal 0 *except* in the case when $\bar{x} = 0$, as stipulated in the question. When $\bar{x} = 0$, the middle term goes to 0, and we also know that $\bar{y} = \hat{\beta}_0 + \hat{\beta}_1\bar{x} = \hat{\beta}_0$, which in expectation is β_0. Therefore,

$$\beta_0\beta_1 - \beta_0\beta_1 = 0$$

See also Appendix 3A.4 of Gujarati.

- Question 3a

  From section 6.1.1 (page 59) of the notes:

  $$\bar{R}^2 = 1 - \frac{n-1}{n-k}(1 - R^2)$$

  In the first regression

  $$\bar{R}^2 = 1 - \frac{124-1}{124-5}(1 - .16)$$

  And in the second regression

  $$\bar{R}^2 = 1 - \frac{124-1}{124-8}(1 - .27)$$

  (Note that in computing k, we *include* the intercept coefficient.)

  Because $\bar{R}^2$ corrects for the number of estimated parameters, we expect $\bar{R}^2 \leq R^2$.

- Question 3b

  The estimates on the regional fixed effects each indicate that the growth rates for otherwise similar countries in each of the three regions will be higher than in the baseline category. (Since a constant is included in the model, we know that there *is* an omitted category; otherwise, we would fall into the "dummy variable trap," described in section 6.5.2 (page 67) of the notes.) Being a country in the former Soviet Union has the largest substantive effect.

- Question 3c

  As in section 5.3 (page 54) of the notes:

  $$H_0 : \beta_{EA} = \beta_{SA}$$

- Question 3d

  For the first regression, $F_{[4,120]}$. For the second regression, $F_{[7,117]}$.

  Note that the F-distribution, as described in sections 1.4.4 (page 15) and 6.2 (pages 61-3) of the notes, makes use of two degrees of freedom numbers. In this case, these are the number of restrictions imposed if all coefficient estimates are set equal to 0 (J) and the number of degrees of freedom from the estimation of the variance of the regression (n-k). Also see the discussion in section 4.7.4 of Greene.

- Question 3e

  A significant joint hypothesis test on aid and the interaction term might suggest that aid does have an impact on economic growth. (The test would be $H_0 : \beta_{Aid} = \beta_{Aid*Insts} = 0$.)

  Chapter 4 of Kennedy provides intuition for why we would want to do this, and using a joint hypothesis test is suggested at the end of section 7.6 (page 84) of the notes.

- Question 3f

  $$\hat{y}_i = .043 + .004 * \log(10,000) + .013 * 3 + .031 * .05 + .171 * 3 * .05 + .017$$

  The slope of the regression line for a country in the former Soviet Union would be the *same*. It would have a different intercept, but each of the variables would have the same effect, implying the same slope. This distinction is described in section 6.5.3 (pages 68-9) of the notes.

- Question 3g

  First, high collinearity affects standard errors not point estimates. Second, the fact that institutions is included in both the first and second regressions implies that if there was a problem with high collinearity between GDP per capita and institutions, then it would effect the standard errors in *both* regressions. Therefore, we might expect that there is a relationship—and perhaps high collinearity—between GDP per capita and one or more of the regional variables but not between GDP per capita and institutions.

  See the discussion in Chapter 11 of Kennedy and section 7.6 of the notes.