IEOR 4000: Production Management

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Solution to HW 4, Fall 2004, Prepared by Lin Li

10.1 In this problem, the parameters are, unit cost $c = 150$, selling price $p = 210$, salvage value $s = 150$. Demand is uniform distributed between 75 and 125, $D \sim U(75, 125)$. Q is the order quantity.

(a) The total profit is

$$\pi(Q) = E[p \cdot \min(Q, D) + (1 - 0.15)s \cdot (Q - D)^+] - cQ$$

The optimal order quantity Q^* satisfies

$$Q^* = \min\{Q : Pr(D < Q) \geq \beta = \frac{c - p}{0.85s - p} = \frac{150 - 210}{0.85 \times 150 - 210} \approx 0.73\}$$

which implies that $\frac{Q^* - 75}{125 - 75} = 0.73$ (since D follows uniform distribution). A straightforward calculation gives

$$Q_1^* = 50 \times 0.73 + 75 = 111.5 \approx 112$$

(b) If the demand is Normally distributed, then $\Phi(Q) = Pr(D < Q) = \beta = 0.73$. Check the table of standard normal distribution, we get $z_\beta \approx 0.61$. So the optimal order quantity is

$$Q_2^* = \mu + \sigma z_\beta = 110 + 15 \times 0.61 = 119$$

The total profit with order quantity $Q_2^* = 119$ is

$$\begin{aligned}
\pi(Q_2^*) &= (p - c)\mu_1 - G(Q_2^*) \\
&= (p - c)\mu - (p - 0.85s)\sigma\phi(z_\beta) \\
&= (210 - 150) \times 110 - (210 - 0.85 \times 150) \times 15 \times \phi(0.6) \\
&= \$6187.63
\end{aligned}$$

We know that the distribution is normal. If we use the order quantity $Q_1^* = 112$, then $z_\beta = (112 - 110)/15 = 0.133$ and $\beta = \Phi(0.133) = 0.553$. Let $h = c - 0.85s$, $b = p - c$. So the total profit is:

$$\begin{aligned}
\pi(Q_1^*) &= (p - c)\mu_1 - G(Q_1^*) \\
&= (p - c)\mu - [h\sigma z_\beta + (h + b)\sigma(\phi(z_\beta - (1 - \beta)z_\beta))] \\
&= (210 - 150) \times 110 - 460.65 \\
&= \$6139.35
\end{aligned}$$

This inside information worth $\pi(Q_2^*) - \pi(Q_1^*) = 6187.63 - 6139.35 = \$48.28$

10.3 Selling price is $p = 100$, order cost is $c = 60$ and salvage value is $s = 51$. Let $h = c - s$ and $b = p - c$

(a) The expected demand is $\mu = E(D) = 100\sum_{j=3}^{8}(jp(j)) = 540$

(b) The variance of demand is

$$Var(D) = 100^2 Var(d) = 100^2[E(d^2) - (E(d))^2] = 100^2[31 - 5.4^2]$$

So the standard deviation is $\sigma_D = \sqrt{Var(D)} = 100 \times \sqrt{31 - 5.4^2} \approx 135.6$

(c) $\beta = \frac{p-c}{p-s} = \frac{100-60}{100-51} = 0.816$ The optimal order quantity Q^* should be

$$Q^* = \min\{Q : Pr(D < Q) \geq \beta\}$$

Calculate the $c.d.f$ of demand D, it is easy to see that the optimal quantity is $Q^* = 700$

(d) The expected profit is,

$$\begin{aligned}
\pi(Q^*) &= (p-c)\mu - G(Q^*) \\
&= (100-60)540 - [(60-51) \times 100 \times \sum_{j=3}^{6}(7-j)p(j) + (100-60) \times (800-700) \times 0.1] \\
&= 21600 - 1930 = \$19670
\end{aligned}$$

(e) If Smith decides to use a normal distribution, then the optimal order quantity is

$$Q^* = \mu + \sigma z_\beta = 540 + 135.6 \times 0.9 \approx 662$$

Rounded to the nearest hundred units, we have $Q = 700$.

(f) Since the order quantity are the same under these two models, there is no penalty for using normal distribution.

10.11 (a) Using the formula $\hat{x} = E(S) = \sum_{x_0 > Q}(x_0 - Q)p_x(x_0)$, we have

x_0	$E(S)if(Q = x_0)$
1	$0.4 \times 1 + 0.35 \times 2 = 1.1$
2	0.35
3	0

The mean of demand is $\mu = 2.1$. Use equation (10.3), we get

$$E[P(Q)] = (p-g)\mu - (v-g)Q - (p-g+B)E(S) = 9 \times 2.1 - 4Q - 10E(S)$$

Thus,

x_0	$E(S)if(Q = x_0)$	$E[P(Q)]$
1	1.1	3.9
2	0.35	7.4
3	0	6.9

The best order quantity is $Q = 2$.

Use equation (10.10), we need smallest Q such that

$$Pr(D < Q) \geq \frac{p - v + B}{p - g + B} = 0.6$$

Note that $Pr(D \leq 1) = 0.25$ and $Pr(D \leq 2) = 0.65 > 0.6$. So the optimal $Q = 2$ which is the same as above.

(b)

$$
\begin{aligned}
E[P(Q)] &= \sum_{x_0} P(Q,x_0)p_x(x_0) \\
&= \sum_{x_0 \le Q} (px_0 + g(Q-x_0) - Qv)p_x(x_0) + \sum_{x_0 \ge Q} (pQ - B(x_0-Q) - Qv)p_x(x_0) \\
&= -Qv + gQ \sum_{x_0 \le Q} (p_x(x_0)) + (p-g) \sum_{x_0 \le Q} x_0 p_x(x_0) + pQ \sum_{x_0 > Q} p_x(x_0) \\
&\quad -B \sum_{x_0 > Q} (x_0 - Q)p_x(x_0) \\
&= -Qv + gQ \sum_{x_0 \le Q} (p_x(x_0)) + (p-g) \left[\sum_{x_0} x_0 p_x(x_0) - \sum_{x_0 > Q} (x_0 - Q)p_x(x_0) \right. \\
&\quad \left. -Q \sum_{x_0 > Q} p_x(x_0) \right] + pQ \sum_{x_0 > Q} p_x(x_0) - B \sum_{x_0 > Q} (x_0 - Q)p_x(x_0) \\
&= -Qv + gQ \sum_{x_0 \le Q} (p_x(x_0)) + (p-g) \left[\hat{x} - E(S) - Q \sum_{x_0 > Q} p_x(x_0) \right] \\
&\quad +pQ \sum_{x_0 > Q} p_x(x_0) - BE(S) \\
&= (p-g)\hat{x} - (v-g)Q - (p-g+B)E(S) \qquad Q,E,D.
\end{aligned}
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10.15 Selling price is $p = 425$, order cost is $c = 300$ and salvage value is $s = 250$. Let $h = c - s$ and $b = p - c$ The optimal order quantity should be

$$Q^* = \min\{Q : Pr(D < Q) \ge \beta\}$$

where $\beta = \frac{p-c}{p-s} = \frac{125}{175} = 0.714$ Use the probability distribution, the optimal is $Q^* = 3$.